THE FIRST STEPS WITH ALEXIA, THE AVERAGE LEXICOGRAPHIC VALUE

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Discussion Paper

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Abstract

The new value AL for balanced games is discussed, which is based on averaging of lexicographic maxima of the core. Exactifications of games play a special role to find interesting relations of AL with other solution concepts for various classes of games as convex games, big boss games, simplex games etc. Also exactifications are helpful to associate fully defined games to partially defined games and to develop solution concepts there.

1 Introduction

Two solution concepts are dominant in game theory: the Nash equilibrium set (NE-set) and the core. The importance of the first concept was recognized immediately, the history of the core is more complex. It turns out that there are remarkable similarities if one looks at the role of the NE-set in non-cooperative game theory and the role of the core in cooperative game theory. Non-cooperative games without Nash equilibria as well as cooperative games with an empty core are not very attractive for a game theorist and also not in practice. In such cases one can still hope for the existence of approximate Nash equilibria or approximate core elements. In case of a determined non-cooperative game an agreement of the players on some NE gives a certain stability because in a play none of the players can profit in unilaterally deviating from the agreed equilibrium. Similarly, if in a game a core element is proposed no subgroup of players can perform better in splitting off. In case the NE-set is large or the core is large there is room for a selection theory or a refinement theory. To finish with the similarities, for both solution concepts, there are axiomatizations using consistency and converse consistency.

In this paper we concentrate on balanced cooperative games, which are games with a non-empty core [4], and introduce for such games a new core selection, the AL-value. Just as in the definition of the well-known Shapley value [8] an averaging of $n!$ vectors takes place which correspond to the $n!$ possible orders of the players in an n-person game. For the Shapley value the vectors are the marginal vectors of the game, for the AL-value the vectors are the lexicographical optimal points in the core.

The outline of the paper is as follows. In section 2 I introduce the AL-value, discuss some interesting properties and treat some examples. It turns out that on the cone of convex games [9] the AL-value and the Shapley value coincide. Section 3 deals with exact games and the exactification operator on games. The AL-value is an additive function on the cone of exact games. Characteristic for the AL-value is the INVEX-property (the invariance w.r.t. exactification). Further, for simplex games [1, 11] and also for dual simplex games or 1-convex games [1, 3] it is shown that the AL-value of such a game coincides with the Shapley value of the exactification of the game. In section 4 the AL-value is studied for the cone of big boss games [5] and it coincides there with the τ-value [10] and the nucleolus [6]. In section 5 the exactification operator is adapted to treat a family of partially defined games, which gives a possibility to define for such games also the AL-value and other values. Section 6 indicates topics for further research.

2 The average lexiographic value AL

Given a balanced n-person $< N, v >$ and given an ordering $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$ of the players in N, the lexicographic maximum of the core $C(v)$ with respect to σ is denoted by $S^\sigma(v)$. It is the unique point of the core $C(v)$ with the properties: $(S^\sigma(v))_{\sigma(1)} = \max\{x_{\sigma(1)} | x \in C(v)\}$, $(S^\sigma(v))_{\sigma(2)} = \max\{x_{\sigma(2)} | x \in C(v)$ with $x_{\sigma(1)} = (S^\sigma(v))_{\sigma(1)}\}, \ldots, (S^\sigma(v))_{\sigma(n)} = \max\{x_{\sigma(n)} | x \in C(v)$ with $(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n-1)}) = ((S^\sigma(v))_{\sigma(1)}, (S^\sigma(v))_{\sigma(2)}, \ldots, (S^\sigma(v)_{\sigma(n-1)}))\}$.

Note that $S^\sigma(v)$ is an extreme point of the core for each σ.

The average lexicographic value of $< N, v >$ is the average over all $S^\sigma(v)$ i.e. $\mathrm{AL}(v) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} S^\sigma(v)$, where $\Pi(N)$ denotes the set of $n!$ orderings of N.

Example 2.1. Let $< N, v >$ be a 2-person balanced game with $N = \{1, 2\}$. Then $v(1, 2) \geq v(1) + v(2)$ and $C(v) = \mathrm{conv}(\{f^1, f^2\})$ with $f^1 = (v(N) - v(2), v(2))$, $f^2 = (v(1), v(N) - v(1))$. Further $\Pi(N) = \{(1, 2), (2, 1)\}$, $S^{(1,2)}(v) = f^1$, $S^{(2,1)}(v) = f^2$, So, $\mathrm{AL}(v) = \frac{1}{2}(f^1 + f^2) = (v(1) + \frac{1}{2}(v(1, 2) - v(1) - v(2)), v(2) + \frac{1}{2}(v(1, 2) - v(1) - v(2))$, the standard solution for the 2-person game $< N, v >$.

Example 2.2. Let $< N, v >$ be the 3-person convex game with $N = \{1, 2, 3\}$, $v(i) = 0$ for each $i \in N$, $v(S) = 10$ if $|S| = 2$ and $v(N) = 30$. Then $S^{(1,2,3)}(v) = (20, 10, 0) = m^{(3,2,1)}(v)$, $S^{(1,3,2)}(v) = (20, 0, 10) = m^{(2,3,1)}(v), \ldots, S^{(3,2,1)}(v) = (0, 10, 20) = m^{(1,2,3)}(v)$. Here $m^\sigma(v)$ is the marginal vector w.r.t. σ with

$$m^\sigma_{\sigma(k)}(v) = v(\sigma(1), \ldots, \sigma(k)) - v(\sigma(1), \sigma(2), \ldots, \sigma(k-1)) \text{ for each } k \in N.$$

So, $\mathrm{AL}(v) = (10, 10, 10) = \frac{1}{3!} \sum_{\sigma \in \Pi(N)} S^\sigma(v) = \frac{1}{3!} \sum_{\sigma \in \Pi(N)} m^{\bar{\sigma}}(v) = \phi(v)$ where $\bar{\sigma} = (\sigma(3), \sigma(2), \sigma(1))$, the reverse order of σ, and $\phi(v)$ is the Shapley value of $< N, v >$.

Theorem 2.3. For each convex game $< N, v >$: $\mathrm{AL}(v) = \phi(v)$.

Proof. Note that for each $\sigma \in \Pi(N)$: $S^\sigma(v) = m^{\bar{\sigma}}(v)$, where $\bar{\sigma} = (\sigma(n), \sigma(n-1), \ldots \sigma(2), \sigma(1))$. $\square$

Theorem 2.4. Let $< N, v >$ be a balanced simplex game [1,11] i.e. a game where $C(v)$ is equal to the non-empty imputation set $I(v) = \{x \in \mathbb{R}^n | \sum_{i=1}^{n} x_i = v(N), x_i \geq v(\{i\})$ for each $i \in N\}$. Then $\mathrm{AL}(v) = CIS(v)$, the center of the imputation set.

Proof. Note that $I(v) = \mathrm{conv}\{f^1(v), f^2(v), \ldots, f^n(v)\}$ and $CIS(v) = \frac{1}{n} \sum_{k=1}^{n} f^k(v)$ where $(f^k(v))_i = v(i)$ for $i \in N \setminus \{k\}$ and $(f^k(v))_k = v(N) - \sum_{i \in N \setminus \{k\}} v(i)$. Because $S^\sigma(v) =$

$f^{\sigma(1)}(v)$ for each $\sigma \in \Pi(N)$ we obtain $\mathrm{AL}(v) = \frac{1}{n!} \sum\limits_{\sigma \in \Pi(N)} f^{\sigma(1)}(v) = \frac{1}{n} \sum\limits_{k=1}^{n} f^k(v) = CIS(v)$.

$\square$

For dual simplex games (also called 1-convex games) (see [3],[11]) we give without proof the following results.

Theorem 2.5. Let $< N, v >$ be a balanced n-person game with $C(v) = I^*(v) = \mathrm{conv}(\{g^1(v), g^2(v), \ldots, g^n(v)\})$, where

$$(g^k(v))_i = v^*(k) = v(N) - v(N \backslash \{k\}) \text{ for } i \neq k \text{ and}$$

$$(g^k(v))_k = v(N) - \sum_{i \in N \backslash \{k\}} v^*(i).$$

Then $\mathrm{AL}(v) = ENSR(v)$, where $ENSR$ is the rule which splits equally the non-separable rewards. $\mathrm{AL}(v)$ is also equal to the nucleolus [6] and the τ-value [10] of (N, v).

It will be clear that AL satisfies the following properties: IR (Individual rationality), EFF (efficiency), S-equivalence, CS (core selection) and SYM (symmetry). Also DUM (the dummy property) holds for AL because for each balanced game $< N, v >$ the AL-value $\mathrm{AL}(v)$ is an element of the core and for each $x \in C(v)$:

$$v(i) \leq x_i = \sum_{k=1}^{n} x_k - \sum_{k \in N \backslash \{i\}} x_k \leq v(N) - v(N \backslash \{i\})$$

So, if i is a dummy player, then $x_i = v(i)$ for each core element and, especially $\mathrm{AL}_i(v) = v(i)$ for a dummy player. In the next section we consider two other properties of AL : INVEX, ADD^E.

3 Exact games

Exact games are introduced by Schmeidler [7] and they play an interesting role in this section. Recall that a game is an exact game if for each coalition $S \in 2^N \backslash \{\phi\}$ there is an element $x^S \in C(v)$ such that $\sum\limits_{i \in S} x_i^S = v(S)$. Let us denote by EX^N the set of exact games with player set N. In fact EX^N is a cone of games and one easily sees that AL : $EX^N \to \mathbb{R}^n$ is additive. We call this interesting property ADD^E : $\mathrm{AL}(v+w) = \mathrm{AL}(v) + \mathrm{AL}(w)$ for each $v, w \in EX^N$.

Note that for each balanced game $< N, v >$ there is a unique exact game $< N, v^E >$ with the same core as the original game. This exactification $< N, v^E >$ of $< N, v >$ is defined by $v^E(\phi) = 0$ and

$$v^E(S) = \min\{\sum_{i \in S} x_i | x \in C(v)\} \text{ for each } S \in 2^N \backslash \{\phi\}$$

So, $C(v^E) = C(v)$ for each balanced game $< N, v >$ and $v^E = v$ iff $< N, v >$ is exact. Note that an interesting property for AL is: if for $< N, v >, < N, w >$ we have $C(v) = C(w) \neq \phi$, then $AL(v) = AL(w)$. This property is equivalent with the property INVEX : $AL(v) = AL(v^E)$ for each balanced game $< N, v >$, where INVEX stands for 'invariant w.r.t. exactification'.

In view of theorem 2.3 this INVEX-property of AL gives the possibility to prove that for some games $< N, v >$ the AL-value of $< N, v >$ coincides with the Shapley value $\phi(v^E)$ of the exactification $< N, v^E >$ of $< N, v >$. This is the case for those game $< N, v >$ for which the exactification is convex. This holds e.g. for simplex games, dual simplex games and also for 2- and 3-person balanced game. So we obtain

Theorem 3.1.

(i) If $< N, v >$ is a balanced 2-person game or a 3-person game, then $AL(v) = \phi(v^E)$.

(ii) For each simplex game $< N, v >$ we have $AL(v) = \phi(v^E)$.

(iii) For each dual simplex game $< N, v >$ we have $AL(v) = \phi(v^E)$.

Proof of (ii) only. Let $< N, v >$ be a simplex game. Then $C(v) = I(v) = \text{conv}\{f^1(v), f^2(v), \dots, f^n(v)\}$. So $v^E(N) = v(N)$ and for each $S \in 2^N \backslash \{\phi, N\}$:
$v^E(S) = \min\{\sum_{i \in S} x_i | x \in C(v)\} = \min\{\sum_{i \in S} f_i^k | k \in \{1, 2, \dots, n\}\} = \min\{\sum_{i \in S} v(i), v(N) - \sum_{i \in N \backslash S} v(i)\} = \sum_{i \in S} v(i)$. This implies that v^E is a sum of convex games namely $v^E = \sum_{i=1}^{n} v(i) u_{\{i\}} + (v(N) - \sum_{k=1}^{n} v(k)) u_N$ (where u_S denotes the unanimity game with $u_S(T) = 1$ if $S \subset T$ and $u_S(T) = 0$ otherwise). So, $< N, v^E >$ is a convex game and $AL(v) = AL(v^E) = \phi(v^E)$. $\square$

Now we give a 4-person exact game $< N, v >$, where $\phi(v) = \phi(v^E) \neq AL(v)$. This game is a slight variant of an example in [2] on p. 91.

Example 3.2. Let $\varepsilon \in (0,1]$ and let $< N, v >$ be the game with $N = \{1, 2, 3, 4\}$,

$v(S) = 7, 12, 22$ if $|S| = 2, 3, 4$ respectively and

$v(1) = \varepsilon, \; v(2) = v(3) = v(4) = 0.$

Note that $< N, v >$ is not convex because

$v(1, 2, 3) - v(1, 2) = 5 < v(1, 3) - v(1) = 7 - \varepsilon.$

Note further that $\mathrm{Ext}(C(v))$ has the maximum number of 24 extreme points:

(i) 12 extreme points which are permutations of (10,5,5,2),

(ii) 9 extreme points which are permutations of (7,7,8,0) but with first coordinate unequal to 0,

(iii) $(\varepsilon, 7 - \varepsilon, 7 - \varepsilon, 8 + \varepsilon), (\varepsilon, 7 - \varepsilon, 8 - \varepsilon, 7 - \varepsilon)$ and $(\varepsilon, 8 - \varepsilon, 7 - \varepsilon, 7 - \varepsilon).$

From this follows that $< N, v >$ is an exact game, and that each lexicographic maximum $S^{\sigma}(v)$ is equal to a permutation of the vector (10,5,5,2), where each such permutation corresponds to two orders.
So, $\mathrm{AL}(v) = (5\frac{1}{2}, 5\frac{1}{2}, 5\frac{1}{2}, 5\frac{1}{2})$ and is unequal to

$\phi(v^E) = \phi(v) = (5\frac{1}{2} + \frac{1}{4}\varepsilon, 5\frac{1}{2} - \frac{1}{12}\varepsilon, 5\frac{1}{2} - \frac{1}{12}\varepsilon, 5\frac{1}{2} - \frac{1}{12}\varepsilon).$

4 Big boss games and the average lexicographic value

Big boss games are introduced in [5] and further discussed in [1] and [11]. Recall that an n-person game $< N, v >$ is a big boss game with n as big boss if the following three conditions hold:

1. Big boss property: $v(S) = 0$ for all S with $n \notin S$.

2. Monotonicity property: $v(S) \le v(T)$ for all $S, T \in 2^N$ with $S \subset T$.

3. Union property: $v(N) - v(S) \ge \sum\limits_{i \in N \setminus S} M_i(v)$ for each $S \in 2^N$ with $n \in S$.

It is well-known that the core of a big boss game with n as big boss is given by

$$C(v) = \{x \in \mathbb{R}^n | 0 \le x_i \le M_i(v) \text{ for each } i \in N\backslash\{n\}, \sum_{i=1}^{n} x_i = v(N)\}$$

and the τ-value by $\tau(v) = (\frac{1}{2}M_1(v), \frac{1}{2}M_2(v), \ldots, \frac{1}{2}M_{n-1}(v), v(N) - \frac{1}{2}\sum_{i=1}^{n-1} M_i(v))$.

The extreme points of a big boss game $< N, v >$ with n as big boss are of the form P^T where $T \subset N\backslash\{n\}$ and $P_i^T = M_i(v)$ if $i \in T$, $P_i^T = 0$ if $i \in N\backslash T \cup \{n\}$ and $P_n^T = v(N) - \sum_{i \in T} M_i(v)$. For each $\sigma \in \Pi(N)$ the lexiocographic maximum $S^\sigma(v)$ equals $P^{T(\sigma)}$, where $T(\sigma) = \{i \in N\backslash\{n\} | \sigma(i) < \sigma(n)\}$.

Theorem 4.1. Let $< N, v >$ be a big boss game with n as big boss. Then $\text{AL}(v) = \tau(v)$.

Proof. For each $i \in N\backslash\{n\}$: $\text{AL}_i(v) = \frac{1}{n!}\sum_{\sigma \in \Pi(N)}(S^\sigma(v))_i = \frac{1}{n!}\sum_{\sigma \in \Pi(N)}(P^{T(\sigma)})_i = \frac{1}{n!}M_i(v)|\{\sigma \in \Pi(N)|\sigma(i) < \sigma(n)\}| = \frac{1}{2}M_i(v) = \tau_i(v)$

By EFF of τ and AL then also $\text{AL}_n(v) = \tau_n(v)$. $\qquad\square$

Let us look at the exactification $< N, v^E >$ of the big boss game $< N, v >$ with n as a big boss.

(i) For $S \subset N\backslash\{n\}$ we have

$$v^E(S) = \min_{T \subset N\backslash\{n\}}\sum_{i \in S} P_i^T = \sum_{i \in S} P_i^\phi = 0$$

(ii) For S with $n \in S$ we have

$$
\begin{aligned}
v^E(S) &= \min_{T \subset N\backslash\{n\}}\sum_{i \in S} P_i^T = \min_{T \subset N\backslash\{n\}}(v(N) - \sum_{i \in T\backslash S} M_i(v)) \\
&= \sum_{i \in S} P_i^{N\backslash\{n\}} = (v(N) - \sum_{i=1}^{n-1} M_i(v)) + \sum_{i \in S} M_i(v).
\end{aligned}
$$

This implies that v^E is a non-negative combination of convex unanimity games:

$$v^E = (v(N) - \sum_{i=1}^{n-1} M_i(v))u_{\{n\}} + \sum_{i \in N\backslash\{n\}} M_i(v)u_{\{i,n\}}$$

So, v^E is a convex game (and also a big boss game) and the extreme points of $C(v)$ and of $C(v^E)$ coincide. So we obtain $\tau(v) = \text{AL}(v) = \text{AL}(v^E) = \phi(v^E)$.

Theorem 4.2. The AL-value of a big boss game equals the Shapley value of the exactification of the big boss game.

5 An approach to handle partially defined games

Cases where a player set N is confronted with the problem of dividing $v(N)$, where not for each subcoalition of N the worth is given, are discussed extensively in the literature. I will consider special balanced partially defined games. These are games $< N, v, \mathcal{F} >$, where N is the player set, $\mathcal{F}$ is a subset of 2^N, containing N and ϕ and $v : \mathcal{F} \to \mathbb{R}$ has the properties $v(\phi) = 0$ and

$$C_{\mathcal{F}}(v) := \{x \in \mathbb{R}^n | \sum_{i=1}^{n} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } \mathcal{F} \in 2^N\}$$

is a non-empty and bounded set.

For such a balanced $\mathcal{F}$-game v one can study the exact 'extension' $\bar{v} : 2^N \to \mathbb{R}$ where

$$\bar{v}(S) = \min\{\sum_{i \in S} x_i | x \in C_{\mathcal{F}}(v)\}$$

where we have a real extension if $< N, v, \mathcal{F} >$ has the exactness property: $v(S) = \bar{v}(S)$ for $S \in \mathcal{F}$. Given a solution Ψ for games $< N, v >$ one can define a solution $\bar{\Psi}$ for balanced partially defined game by

$$\bar{\Psi}(N, v, \mathcal{F}) = \Psi(N, \bar{v})$$

It is interesting to study $\overline{AL}$ in such situations.

6 Concluding remarks

Further research on the average lexicographic value will include

(i) monotonicity properties of AL,

(ii) continuity properties of AL,

(iii) consistency properties of AL,

(iv) axiomatizations,

(v) numerical aspects,

(vi) cones with a perfect kernel system and AL,

(vii) relations with other core selections,

(viii) extensions of the AL-value for non-balanced games,

(ix) more relations with other solution concepts.

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