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**ON THE INTUITION OF RANK-DEPENDENT
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Utility

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Abstract

Among the most popular models for decision under risk and uncertainty are the rank-dependent models, introduced by Quiggin and Schmeidler. Central concepts in these models are rank-dependence and comonotonicity. It has been suggested in the literature that these concepts are technical tools that have no intuitive or empirical content. This paper describes such contents. As

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a result, rank-dependence and comonotonicity become natural concepts upon which preference conditions, empirical tests, and improvements for utility measurement can be based. Further, a new derivation of the rank-dependent models is obtained. It is not based on observable preference axioms or on empirical data, but naturally follows from the intuitive perspective assumed. We think that the popularity of the rank-dependent theories is mainly due to the natural concepts adopted in these theories.

In the last two decades, many models for decision under risk and uncertainty have been proposed that deviate from classical expected utility. Among the most popular are the rank-dependent models. They were introduced by Quiggin (1981) for decision under risk (known probabilities) and by Schmeidler (1989) for decision under uncertainty (unknown probabilities), and have been incorporated in prospect theory (Tversky & Kahneman 1992). The present paper proposes an intuitive justification of rank-dependence, building on Lopes (1984), Yaari (1987), and Weber (1994). A new derivation of rank-dependent utility is presented that naturally follows from the intuitive conditions.

In order to generate fruitful applications, a decision model should satisfy three requirements. First, it should be mathematically sound. For instance, it should not exhibit behavioral anomalies such as implausible violations of stochastic dominance (Fishburn 1978). This first requirement can be guaranteed by preference axiomatizations. For the rank-dependent models, such axiomatizations were given by Quiggin (1982), Schmeidler (1989), and many others (see Karni & Schmeidler 1991, Schmidt 1998, and Starmer 1999 for surveys).

The second requirement for a decision model concerns its empirical performance. It has been found that rank-dependent utility can accommodate several empirical violations of expected utility. The study of its empirical potential is still going on today (Harless & Camerer 1994, Tversky & Fox 1995, Birnbaum & McIntosh 1996, Bleichrodt & Pinto 2000, Abdellaoui & Munier 1999, Gonzalez & Wu 1999).

The third requirement is that the model should be intuitively plausible. Its concepts should provide new insights and be economically meaningful. Future connections with concepts from other fields should be conceivable ("nomological validity"). Only few authors have provided intuitive arguments for rank-dependence. The arguments are scattered around over various papers from different fields. It has been pointed out recently that a complete intuitive foundation of rank-dependence is still lacking (Luce 1996a p. 85, Luce 1996b p. 304, Safra & Segal 1998 p. 28). Providing such a foundation is the purpose of this paper. We will argue, using the terminology of Backhouse (1998, p. 1857), that rank-dependence relates to "real-world" (psychological) concepts. As suggested by Backhouse, such arguments are, "in the last resort, informal."

The paper is structured as follows. Section 1 presents the first attempt to model nonadditive probabilities, commonly used before the 1980s. Section 2 describes the intuition of rank-dependence for decision under risk. The rank-dependent utility formula follows from this intuition in a natural and elementary manner (Section 3). The intuition also leads to natural ways for modeling pessimism and optimism, two important attitudes with respect to probabilistic risk (Section 4). Section 5 extends the foundation to uncertainty. It shows that Quiggin's (1981) contribution for risk and Schmeidler's (1989) contribution for uncertainty are based on the same intuition. Using the intuitive foundation of the preceding sections, Section 6 argues that preference conditions and measurement procedures based on "comonotonicity" are not

only valid under the rank-dependent *theory* but also have merits in the "real world."

Conclusions and comments are given in Section 7. Appendix A discusses intuitive arguments for rank-dependence presented before in the literature and Appendix B gives proofs.

1   The First Attempt

Consider a *lottery* $(p_1, x_1; \cdots; p_n, x_n)$, yielding outcome x_j with probability p_j, $j = 1, \ldots, n$. The probabilities $p_1, \ldots, p_n$ are nonnegative and sum to one. In this paper, *outcomes* are real numbers designating money. It will be assumed throughout that the lottery is evaluated by the following formula, called the *general weighting model*:

$$\sum_{j=1}^{n} \pi_j U(x_j) \ . \tag{1}$$

U is the *utility function* and the π_js are called *decision weights*. The decision weights are nonnegative and sum to one, and will be discussed later. The general weighting model is not intended to immediately provide operational predictions but serves as a general point of departure. Intuitive arguments will be formulated in terms of the model, and operational implications then follow.

Stochastic dominance is assumed throughout the paper. It means that moving positive probability mass from an outcome to a strictly higher outcome leads to a strictly higher evaluation. This assumption implies that the utility function is strictly increasing (as follows from considering riskless lotteries). We do not yet make any

further assumption about the decision weights, and they may depend on the entire lottery for now.[1] In a descriptive context, π_j can be interpreted as the attention given to outcome x_j, possibly due to misperception of probability. In a normative context, π_j can be interpreted as an importance weight for outcome x_j that may deliberately have been chosen different than the probability p_j.

It may be possible to relate decision weights to psychological notions such as the time span during which the decision maker looks at outcomes (Johnson & Schkade 1989). An empirical operationalization of decision weights is, however, not our purpose at this stage. When further assumptions have been added, the decision weights will become operational.

Utility is assumed independent of the lottery under consideration. Like decision weights, utility is not operational at this stage but will become so later when further assumptions have been added. Utility can be operationalized if it is interpreted in the riskless sense of Allais (1953). *Expected utility* is the special case of the general weighting model where π_j agrees with p_j for all j.

As a preparation for what follows, and for historical reasons, we start with the following assumption. It will turn out to be too restrictive for our purposes and will be relaxed later on.

ASSUMPTION 1 [independence of beliefs from tastes]. The decision weight π_j of receiving outcome x_j depends only on the probability p_j. $\square$

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[1]This dependence is not expressed in the notation.

The assumption requires that the decision weight π_j is independent of the outcomes and the other probabilities of the lottery. We can now write $w(p)$ for the decision weight generated by a probability p, thus defining a function w. The general weighting model thus becomes

$$\sum_{j=1}^{n} w(p_j)U(x_j). \tag{2}$$

As we will see next, the requirement that the $w(p_j)$s sum to one implies expected utility. The proof is given in Appendix B.

THEOREM 2 *Under Assumption 1, the general weighting model (1) reduces to expected utility, i.e. $w(p) = p$.* $\square$

To obtain Eq. (2) with a nonlinear w function, the requirement that decision weights sum to one will have to be relaxed. This was indeed the approach originally taken in the literature (Preston & Baratta 1948, Edwards 1955). Then, however, the model leads to violations of stochastic dominance (Fishburn 1978). We conclude that a transformation of probabilities, independently of outcomes, is not well possible. To obtain a decision theory with transformed probabilities, an additional relaxation of the expected utility principles is required. Such a relaxation, rank-dependence, was introduced by Quiggin (1981). Its intuition is explained in the next section and the rest of the paper elaborates on this intuition.

2 The Intuition and Definition of Rank-Dependence for Decision under Risk

The intuition of rank-dependence entails that the attention given to an outcome depends not only on the probability of the outcome, it also depends on how good the outcome is in comparison to the other possible outcomes. To illustrate this intuition, assume that the decision maker is a pessimist and evaluates the lottery $\left(\frac{1}{3}, 30; \frac{1}{3}, 20; \frac{1}{3}, 10\right)$. Then he will pay more than $\frac{1}{3}$ of his attention to outcome 10, the reason being that 10 is the worst outcome. Say that π_3, the decision weight for outcome 10, is $\frac{1}{2}$. The decision maker, accordingly, pays relatively less attention to each of the other outcomes ($\pi_1 + \pi_2 = \frac{1}{2}$ if $\pi_3 = \frac{1}{2}$). Being a pessimist, he will pay more than half of the remaining attention to outcome 20, hence $\pi_2 > \frac{1}{4}$; say $\pi_2 = \frac{1}{3}$. The remainder of the attention, devoted to outcome 30, is small ($\pi_1 = \frac{1}{6}$). Next consider the lottery with outcome 20 changed into 0, i.e. $\left(\frac{1}{3}, 30; \frac{1}{3}, 0; \frac{1}{3}, 10\right)$. The outcome 10 is no longer the worst outcome and a pessimist will therefore pay less attention to it than in the first lottery. In human behavior, such attitudes are commonly observed in every-day life. Rank-dependence is a psychologically realistic phenomenon. Savage (1954, end of Chapter 4) already pointed out that there is no room for expressing pessimism or optimism in traditional expected utility.

Descriptively, a pessimistic attitude can be due to an irrational belief that unfavorable events tend to happen more often, leading to an unrealistic overweighting

of unfavorable likelihoods (Murphy's law). If rank-dependence is taken normatively, then a pessimistic attitude can result from a conscious and deliberate decision. The decision maker may decide that unfavorable outcomes are especially important in decision making and therefore should receive more attention than equally likely favorable outcomes (Fellner 1961 p. 681, Weber 1994 p. 236, Lopes & Oden 1999 p. 310).

Empirically, another kind of rank-dependence is often found, where subjects not only pay much attention to the worst outcomes but also to the best outcomes. Less attention is paid to the intermediate outcomes. This phenomenon may be due to extreme outcomes being more noticeable and provides another illustration of the realistic nature of rank-dependence. A discussion of the phenomenon is given in Section 4.

Further generalizations of expected utility could obviously be considered. To some degree, the decision weight of an outcome will depend not only on whether it is better than some other outcome but also on how much better it is. Such generalizations may be considered in future developments. It should, however, be kept in mind that a theory should not be too general. The theory should be sufficiently restrictive to allow for specific predictions. In that sense rank-dependence can be considered a pragmatic compromise between generality and parsimony. Rank-dependence incorporates some major deviations from expected utility but at the same time provides analytical tractability and specific empirical predictions.
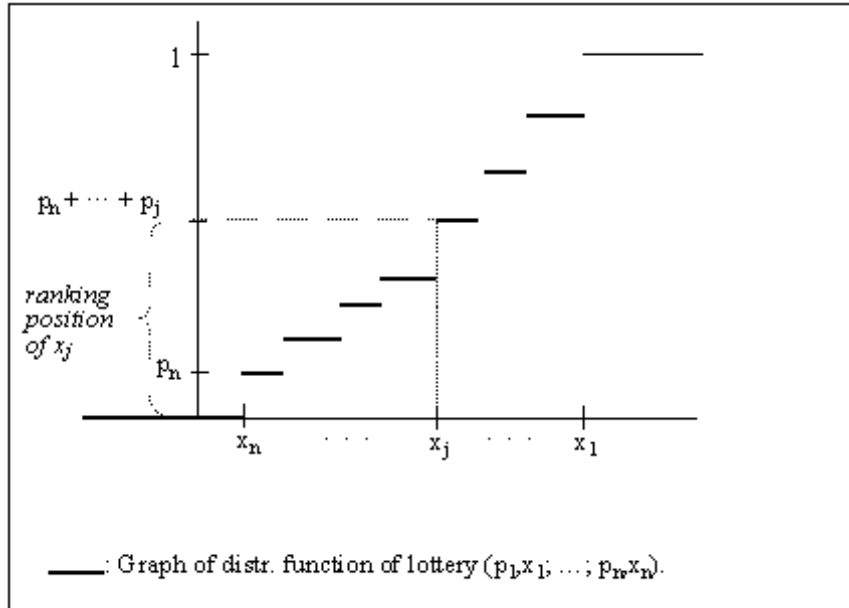
Figure 1: Ranking position of outcome x_j

For the following analysis, we consider rank-ordered lotteries $(p_1, x_1; \cdots; p_n, x_n)$ with $x_1 > \cdots > x_n$. Every lottery can obviously be written in this manner by coalescing identical outcomes and then reordering the outcomes.

The distribution function of the lottery (see Figure 1) will be used for the formal definition of ranking positions. The distribution function assigns to each outcome the probability of receiving that outcome or anything worse. It therefore orders the outcomes from best to worst, with value zero assigned to anything below the worst outcome, value one to the best outcome, and value p to the outcome for which a p part of the other outcomes is worst and a $1 - p$ part is better. Therefore the *ranking position* of any outcome x_j is defined as its distribution function, i.e. it is $p_j + \cdots + p_n$.

ASSUMPTION 3 [rank-dependence]. The decision weight π_j of receiving outcome x_j depends only on its probability p_j and its ranking position. $\square$

The assumption has relaxed Assumption 1 by also permitting rank-dependence. To illustrate the assumption, consider the lottery $\left(\frac{1}{3}, 30; \frac{1}{3}, 20; \frac{1}{3}, 10\right)$. The ranking position of outcome 10 is $\frac{1}{3}$. For the lottery $\left(\frac{2}{3}, 25; \frac{1}{3}, 12\right)$, the ranking position of outcome 12 is also $\frac{1}{3}$. The two outcomes also have the same probability, hence, by Assumption 3, they must have the same decision weight.

3 Operational Implications: Rank-Dependent Utility for Risk

With Assumption 3 added, the decision weights become operational and empirical predictions can be derived from the decision weights. For example, with $\sim$ denoting equivalence, assume that

$$(p_1, 10; p_2, \mathbf{2}; p_3, 1) \sim (q_1, 12; q_2, \mathbf{2}; q_3, 0).$$

Then the decision weight of outcome 2 in the left lottery exceeds the corresponding decision weight in the right lottery if and only if, with $\succcurlyeq$ denoting preference,

$$(p_1, 10; p_2, \mathbf{3}; p_3, 1) \succcurlyeq (q_1, 12; q_2, \mathbf{3}; q_3, 0).$$

The claim follows because, under Assumption 3, the middle outcomes of the left lotteries (2 in the upper lottery and 3 in the lower) have the same decision weight

π_2, and the middle outcomes of the right lotteries (2 in the upper lottery and 3 in the lower) have the same decision weight π_2'. The increase in evaluation of the left lottery, $\pi_2(U(3) - U(2))$ apparently exceeds the increase in evaluation of the right lottery, $\pi_2'(U(3) - U(2))$. This implies that $\pi_2 \geq \pi_2'$. That is, the decision weights show "where to put your money" (see Sarin & Wakker 1998, using an idea of Gilboa 1987).

We will now demonstrate that rank-dependent utility follows from the general weighting model and Assumption 3. Assumption 3 implies in particular that the decision weight of a maximal outcome of a lottery depends only on its probability p, its ranking position always being one. The function $w(p)$ can be defined as this decision weight. $w(p)$ is therefore the decision weight generated by the probability *p when associated with the best outcome.* Obviously, $w(0) = 0$, $w(1) = 1$, and w is strictly increasing because of stochastic dominance.

It is next demonstrated that the general rank-dependent formula for the lottery $(p_1, x_1; \cdots; p_n, x_n)$ with $x_1 > \cdots > x_n$ can be expressed in terms of the function w. The decision weight π_1 is by definition equal to $w(p_1)$. Next we turn to the decision weight of outcome x_i for some general i. The following observation serves as a preparation.

Observation. The total decision weight assigned to outcomes $x_1, \ldots, x_i$, i.e. $\pi_1 + \cdots + \pi_i$, is $w(p_1 + \cdots + p_i)$.

Explanation. Consider the lotteries $(p_1, x_1; \cdots; p_i, x_i; p_{i+1}, x_{i+1}; \cdots; p_n, x_n)$ and $((p_1 + \cdots + p_i), z; p_{i+1}, x_{i+1}; \cdots; p_n, x_n)$ for any outcome z exceeding x_{i+1}, e.g., $z = x_1$.

Because decision weights must sum to one, $\pi_1 + \cdots + \pi_i = 1 - \pi_{i+1} - \cdots - \pi_n = w(p_1 + \cdots + p_i)$, where the second equality is inferred from inspecting the second lottery. This reasoning is based on the fact that, by Assumption 3, the outcomes $x_{i+1}, \ldots, x_n$ all have the same ranking position in the two lotteries and therefore the same decision weights denoted $\pi_{i+1}, \ldots, \pi_n$. $\square$

The decision weight π_i of outcome x_i is $\pi_1 + \cdots + \pi_i - (\pi_1 + \cdots + \pi_{i-1})$. By the preceding observation, this is equal to $w(p_1 + \cdots + p_i) - w(p_1 + \cdots + p_{i-1})$. Hence every decision weight can be expressed in terms of w. In agreement with the rank-dependence Assumption 3, the decision weight of x_i depends only on its probability p_i and its ranking position $q = p_i + \cdots + p_n$, because it can be written as $w(p_i + 1 - q) - w(1 - q)$.

Let us summarize. The model that has been derived is called *rank-dependent utility* (*RDU*). If $x_1 > \cdots > x_n$ then

$$RDU(p_1, x_1; \cdots ; p_n, x_n) = \sum_{j=1}^{n} \pi_j U(x_j) \tag{3}$$

where, for each j,

$$\pi_j = w(p_1 + \cdots + p_j) - w(p_1 + \cdots + p_{j-1}).$$

In particular, $\pi_1 = w(p_1)$.

CONCLUSION 4 Under the general weighting model (1), stochastic dominance and Assumption 3 imply rank-dependent utility. $\square$

The preceding analysis has used the function $w(p)$, the decision weight generated by probability p when associated with the best outcome. An equivalent analysis could have been presented in terms of a dual function $w^*(p)$, describing the decision weight generated by probability p when associated with the worst outcome. The two functions are dual in the sense that $w^*(p) = 1 - w(1 - p)$ for all p; this follows from the requirement that decision weights should sum to one for any lottery $(p, M; 1 - p, m)$ with outcomes $M > m$. Both w or w^* can be used as the basis of the analysis, as long as it is kept in mind whether the function describes decision weights of best outcomes or of worst outcomes. In (3), π_j can as well be expressed in terms of w^*,

$\pi_j = w^*(p_j + \cdots + p_n) - w^*(p_{j+1} + \cdots + p_n)$ for each j. w can be called the *goodnews weighting function* and w^* the *badnews weighting function*.

The decision weights are now uniquely determined and can be derived from observable choice. Most empirical studies of decision weights have used simultaneous parametric fittings for U and w. Non-parametric fittings still involving utility estimation were provided by three independent simultaneous studies: Abdellaoui (2000), Bleichrodt & Pinto (2000), and Gonzalez & Wu (1999). Abdellaoui (1999) introduced a parameter-free method for measuring decision weights without the need to estimate utilities.

Other nonexpected utility models than the rank-dependent ones can be derived from the general weighting model. For example, if the decision weights do not depend on the rank-ordering of outcomes but instead on the equivalence class that a lottery

is contained in, then "betweenness" models result (Chew 1989). These models are outside the scope of this paper.

We hope that the preceding explanation has demonstrated that RDU is not solely a mathematical device for deriving decisions from nonlinear probabilities. The theory is based on two intuitive assumptions regarding decision making. First, people may process probabilities in a nonlinear manner. Second, the attention people pay to outcomes may depend on how good or bad these outcomes are. The RDU formula naturally follows from these two intuitive assumptions.

4   Pessimism and Optimism

This section shows how rank-dependence can describe phenomena outside the domain of expected utility. Let us first consider pessimism. Assume that a lottery yields outcome x with probability p. Let q denote the ranking position of x, i.e. the probability of receiving a lower or equal outcome. The decision weight of x then is $w(p+(1-q))-w(1-q)$. Under pessimism, improving the ranking position (increasing the probability q of receiving something nonpreferred) decreases the decision weight of x. It is well-known that $w(p + (1 - q)) - w(1 - q)$ is decreasing in q if and only if w is convex. Hence, pessimism is characterized by a convex weighting function.

Similarly, optimism corresponds to a decision weight $w(p+(1-q))-w(1-q)$ that is increasing in q, and thus to a concave weighting function. This rank-dependent way of modeling pessimism and optimism was already suggested by Quiggin (1982, p.

335). It was described in full by Yaari (1987, p. 108) and, subsequently, by many other authors. It is in full agreement with the intuition advanced in this paper. Similar effects have been observed in other contexts. For instance, when aggregating different sources of information on risk, people assign disproportionately high weights to the worst risk assessments (Viscusi 1997, p. 1667). Viscusi uses the term "informationally risk-averse" to indicate independence from the shape of utility for wealth.

In empirical investigations, many observed weighting functions are not completely convex or concave but exhibit a mixed pattern. They are concave for small probabilities and convex for moderate and high probabilities. This pattern is called *inverse-S*. The pattern implies that subjects pay much attention to best and worst outcomes, and little attention to intermediate outcomes (Quiggin 1982, Weber 1994). Empirical support has been found (Yaari 1965, Allais 1988, Karni & Safra 1990, Birnbaum et al. 1992, Tversky & Kahneman 1992, Kachelmeier & Shetata 1992, Camerer & Ho 1994, Tversky & Fox 1995, Wu & Gonzalez 1996, Abdellaoui 2000, Bleichrodt & Pinto 2000, Gonzalez & Wu 1999). Counterevidence has been provided by Birnbaum & McIntosh (1996) and Birnbaum & Navarrete (1998). A psychological theory for the attention to low outcomes ("security") and high outcomes ("potential") has been developed by Lopes (Lopes & Oden 1999). The pattern suggests that people are overly sensitive to changes from impossible to possible and from possible to certain but are insufficiently sensitive to probabilistic information otherwise (Karmarkar 1978, Tversky & Wakker 1995).

The inverse-S shape predicts that people are optimistic and hence risk seeking for gambles that yield gains with small probabilities such as in public lotteries. People are pessimistic and hence risk averse for gambles that yield losses with small probabilities, a phenomenon relevant for insurance. Hence, the simultaneous existence of lotteries and insurance, a classical paradox in economics, can be explained by the inverse-S pattern (Quiggin 1982).

5   The Intuition for Decision under Uncertainty

The analysis of uncertainty, presented in this section, is parallel to the analysis of risk. Uncertainty is, however, more interesting because subjective degrees of belief can play a role. Risk is the special case of uncertainty where probabilities are unambiguously known. We briefly describe the uncertainty framework. A set of *states (of nature)* S is given. This set is considered to be an exhaustive list of mutually exclusive states: one and only one state will be the true state, but the decision maker is uncertain about which that will be. Subsets of S are called *events*. As in Section 1, the *outcome* set is assumed to be $\rm I\!R$. *Acts* are finite-valued functions from S to $\rm I\!R$. The generic notation for an act is $(E_1, x_1; \cdots; E_n, x_n)$. This act yields outcome x_j if the true state belongs to event E_j. It is implicitly understood in this notation that the events $(E_1, \ldots, E_m)$ partition the state space.

We assume that the act $(E_1, x_1; \cdots; E_n, x_n)$ is evaluated by the following formula,

17

the *general weighting model*:

$$\sum_{j=1}^{n} \pi_j U(x_j). \tag{4}$$

U denotes the *utility function* and the π_js are *decision weights*. Decision weights are nonnegative and sum to 1. We assume *monotonicity*, i.e. if for some states of nature the outcomes of an act are replaced by better outcomes then the resulting act is weakly preferred to the original act. This implies that the utility function is nondecreasing. The utility function is assumed to be non-constant so as to avoid triviality. No assumption is yet made about the π_js and they are permitted to depend on the act in any possible manner. *Subjective expected utility (SEU)* is the special case where the π_js are *subjective probabilities*, i.e. the following two assumptions hold.

ASSUMPTION 5 [independence of beliefs from tastes]. The decision weight π_j of an event E_j depends only on the event itself. $\square$

With Assumption 5 satisfied, the following assumption can be formulated:

ASSUMPTION 6 [additivity]. The decision weight $\pi_{A \cup B}$ of a disjoint union $A \cup B$ is the sum $\pi_A + \pi_B$ of the decision weights of the separate events A and B. $\square$

There is much interest in relaxations of Assumption 6. First, it is psychologically plausible that people perceive likelihood in a nonlinear manner, a phenomenon which is usually more pronounced under uncertainty than under risk (Keynes 1921, Fellner 1961 p. 684, Weber 1994, Currim & Sarin 1989, Tversky & Wakker 1998). Nonlinear

sensitivity towards probabilities seems as plausible as towards outcomes, and therefore probability transformation seems to be as useful for descriptive purposes as utility. Second, nonadditive measures of belief, such as Dempster-Shafer belief functions, are extensively used in artificial intelligence (Dempster 1967, Shafer 1976). Unfortunately, a relaxation of only Assumption 6 while maintaining full independence of beliefs from tastes turns out to be impossible.

THEOREM 7 *Eq. (4) and Assumption 5 imply subjective expected utility (thus Assumption 6).* $\square$

Theorem 7 can be interpreted as a negative result. Nonadditive measures cannot be implemented in decisions if Assumption 5 is to be maintained. We therefore turn to a weakening of Assumption 5. The weakening could be interpreted as giving up independence of beliefs from tastes. However, once Assumption 5 is given up, the interpretation of decision weights as indexes of belief, already questionable under expected utility, becomes highly problematic. Obviously, the interpretation of nonadditive measures, which are simply the decision weights of good- or badnews events, as indexes of belief is similarly problematic. Another, more plausible, interpretation of decision weights is therefore that they are not pure indexes of belief. They may also comprise a component of decision attitude, in addition to the belief component. Under such an interpretation, a decomposition of decision weights into the belief and decision component can be conjectured (Epstein 1999, Wu & Gonzalez 1999, Tversky & Wakker 1998). For consistency with traditional terminology, the

name of Assumption 5 is maintained.

A relaxation of Assumption 5 that permits nonadditive measures is provided by Choquet expected utility (CEU), introduced by Schmeidler (1989). His model can be based on the intuition of rank-dependence. That is, the attention paid to an event depends not only on the event but also on how good the outcome yielded by the event is in comparison to the outcomes yielded by the other events. This is the way in which subjective expected utility is generalized.

For the following analysis, consider rank-ordered acts $(E_1, x_1; \cdots; E_n, x_n)$, with $x_1 > \cdots > x_n$. For event E_j the *ranking position* is identified with the event of receiving a worse or equivalent outcome, i.e. it is $E_j \cup \cdots \cup E_n$. Sarin & Wakker (1998) used the term dominating event for the complement of the ranking position. The following analysis is similar to the analysis under risk. It is presented concisely but in full because it demonstrates the similarity of RDU under risk and CEU under uncertainty, thus the similarity of Quiggin's (1981) and Schmeidler's (1989) ideas.

ASSUMPTION 8 [rank-dependence]. The decision weight π_j of an event E_j depends only on the event and its ranking position. $\square$

Next Choquet expected utility is derived from Assumption 8. The assumption implies in particular that the decision weight of a maximal outcome of a lottery depends only on the belonging event E, the ranking position always being the universal event. $W(E)$ can now be defined as this decision weight. $W(E)$ is therefore the deci-

20

sion weight generated by the event E *when associated with the best outcome.* W is a *capacity,* i.e. (1) $W(\emptyset) = 0$, (2) $W(S) = 1$, and (3) W is nondecreasing with respect to set inclusion. (Condition (3) follows from consideration of acts $(A, x; B \cup C, y)$ and $(A \cup B, x; C, y)$ with $U(x) > U(y)$. Monotonicity implies preference for the first act, which implies that $W(A \cup B) \geq W(A)$.)

We express the general weighting model in terms of the capacity W. Consider the act $(E_1, x_1; \cdots; E_n, x_n)$. We assume that the events have been rank-ordered such that $x_1 > \cdots > x_n$. The decision weight π_1 is by definition equal to $W(E_1)$. Next consider a general i.

Observation. The total decision weight assigned to outcomes $x_1, \ldots, x_i$, i.e. $\pi_1 + \cdots + \pi_i$, is $W(E_1 \cup \cdots \cup E_i)$.

Explanation. Consider the acts $(E_1, x_1; \cdots; E_i, x_i; E_{i+1}, x_{i+1}; \cdots; E_n, x_n)$ and $((E_1 \cup \cdots \cup E_i), z; E_{i+1}, x_{i+1}; \cdots; E_n, x_n)$ for any outcome z exceeding x_{i+1}, e.g., $z = x_1$. Because decision weights must sum to one, $\pi_1 + \cdots + \pi_i = 1 - \pi_{i+1} - \cdots - \pi_n = W(E_1 \cup \cdots \cup E_i)$. Note that, by Assumption 8, the outcomes $x_{i+1}, \ldots, x_n$ all have the same ranking position in the two acts and therefore the same decision weights. $\square$

The observation implies that the decision weight π_i of event E_i is the difference $W(E_1 \cup \cdots \cup E_i) - W(E_1 \cup \cdots \cup E_{i-1})$. It is common convention that for $i = 1$ such a difference is $\pi_1 = W(E_1)$. The rank-ordering of the events was crucial in this derivation. Let us summarize and give the formal definition of *Choquet expected*

utility (CEU). For $x_1 > \cdots > x_n$,

$$CEU(E_1, x_1; \cdots; E_n, x_n) = \sum_{j=1}^{n} \pi_j U(x_j) \tag{5}$$

where

$$\pi_j = W(E_1 \cup \cdots \cup E_j) - W(E_1 \cup \cdots \cup E_{j-1}).$$

CONCLUSION 9 Eq. (4), monotonicity, and Assumption 8 imply CEU. $\square$

Empirical measurements of decision weights have been described by Fox & Tversky (1995), Fox, Rogers, & Tversky (1996), Wu & Gonzalez (1999), and Kilka & Weber (1999). We hope that the preceding explanation clarifies that the intuitive basis of CEU is the same as of RDU. Thus, a psychological background has also resulted for Schmeidler's (1989) Choquet expected utility. It will be argued in the next section that, given this intuition, the "comonotonicity" condition is not just a mathematical tool but is a natural concept. Let us now turn to a discussion of pessimism.

Pessimism means again that the attention paid to an event gets higher as the event gets rank-ordered worse. That is, assume that event E yields outcome x and D is the ranking position of E. Then the decision weight of E is $W(D^c \cup E) - W(D^c)$. Under pessimism, worsening the ranking position (i.e. decreasing the event D of receiving something worse) increases the decision weight of E. That is, if $C \subset D$, then

$$W(C^c \cup E) - W(C^c) \geq W(D^c \cup E) - W(D^c). \tag{6}$$

Similar to risk, a capacity W satisfying (6) is called *convex*. (6) can be rewritten as

$W(A \cup B) + W(A \cap B) \geq W(A) + W(B)$ after appropriate substitution of symbols

(left to the reader.) Optimism is similarly characterized by *concavity* of the capacity,

i.e. (6) with $\leq$ instead of $\geq$.

6 Coalescing and Comonotonicity

Both in risk and in uncertainty, the rank-dependent formulas have been given for

distinct outcomes $x_1 > \cdots > x_n$. Eqs. (3) and (5) can also be used if the inequalities

are weak, i.e. $x_1 \geq \cdots \geq x_n$. These claims follow from substitution and are left to the

reader. For an act $(E_1, x_1; \cdots; E_n, x_n)$ with $x_i = x_{i+1}$, the decision weight and the

ranking position of event E_i depend on the chosen rank-ordering between x_i and x_{i+1}.

This choice can be made arbitrarily and is immaterial for all empirical purposes.

We next discuss "comonotonicity," introduced by Schmeidler (1989, first ver-

sion 1982). The condition has sometimes been criticized, hence an explanation of

its intuition seems to be useful. For simplicity, assume a finite state space $S = $

$\{s_1, \ldots, s_n\}$. For a permutation $(\rho_1, \ldots, \rho_n)$ of $(1, \ldots, n)$, consider the set $C^\rho = $

$\{f \in I\!R^n : f_{\rho_1} \geq \cdots \geq f_{\rho_n}\}$. It can be seen that C^ρ is a convex cone. For all acts in

the cone C^ρ, we can use the same decision weights π_{ρ_j} determined by

$$\pi_{\rho_j} := W(s_{\rho_1}, \ldots, s_{\rho_j}) - W(s_{\rho_1}, \ldots, s_{\rho_{j-1}})$$

in the computation of CEU. If acts are in the same cone, then $f_i > f_j$ and $g_j > g_i$ for

no states s_i and s_j. Acts that belong to the same cone are called *comonotonic.*

Within comonotonic sets, CEU coincides with an SEU functional. This SEU functional is defined by taking the CEU utility function and taking as probabilities the decision weights π_{ρ_j} belonging to the comonotonic set. Therefore, CEU exhibits many characteristics of SEU within comonotonic sets. In particular, it satisfies the same preference axioms.

The "comonotonic" agreement of CEU with SEU is implied by the theory but is also empirically interesting. Consider acts belonging to different comonotonic sets. The states of nature are rank-ordered differently for such acts. This difference will enhance variations in the psychological attention paid to the states. Subjects will exhibit more pronounced violations of SEU, due to pessimism, optimism, etc. When only acts are considered from one comonotonic set, fewer violations of SEU can be expected. According to CEU *theory,* the effects of pessimism and optimism will then be kept constant. In reality, they can be expected to be smaller than when the rank-ordering of the acts varies.

Comonotonicity is extensively used in preference axiomatizations of CEU. Most axiomatizations consist of restricting the SEU axioms to comonotonic acts. For a continuum of outcomes, CEU holds as soon as SEU holds within every comonotonic set (this is easily derived from Wakker & Tversky 1993, Proposition 8.2). An empirical application of comonotonicity can be found in utility measurement. Wakker & Deneffe (1996) demonstrated that utility can be measured under CEU by restricting SEU

techniques to comonotonic sets of acts. Such a restriction has the empirical advantage of avoiding the biases generated by rank-dependence, and therefore seems desirable.

Some authors have pointed out that rank-dependence and comonotonicity are often used as technical tools and that there is still need for an intuitive foundation (Luce 1996a p. 85, Luce 1996b p. 304, Safra & Segal 1998 p. 28). Our paper has provided such a foundation, building on ideas provided before in the literature. We have argued that rank-dependence and comonotonicity do have intuitive and empirical merit. Yaari (1987, p. 104) already emphasized the intuitive importance of comonotonicity when discussing his central axiom (dual independence): "The foregoing proposition makes it clear that the economic interpretation of dual independence lies in the intuitive meaning of comonotonicity."

Obviously, alternative derivations of CEU and RDU that do not use rank-dependence or comonotonicity in their axioms are also interesting. Such derivations were provided by Luce (1998) and Safra & Segal (1998). In these derivations, rank-dependence follows from other conditions.

7   Conclusion

This paper has argued that RDU is not just a mathematical device but that it is based on intuition and has "real-world" merits. The intuition of rank-dependence was described in terms of decision weights. The RDU formula naturally followed as well as empirically meaningful preference conditions. Optimism and pessimism were

explained in terms of the intuitive foundation. An analogous reasoning was applied to the uncertainty case and a psychological background for Schmeidler's (1989) Choquet expected utility resulted. Once the intuition understood, comonotonicity conditions and rank-dependence are no longer mere theoretical tools. They become natural concepts upon which preference conditions, empirical tests, and improvements for utility measurement can be based.

Our preference for RDU, and we believe also its general popularity, depends not only on its mathematical or empirical performance but also on an intuitive aspect of the model: Nonlinear sensitivity towards chance, and nonadditive measures of belief, have the potential of becoming useful concepts, not only in economics but also in other areas such as psychology and artificial intelligence.

Appendix A. Related Literature on The Intuition of Rank-Dependence

This appendix presents some intuitive arguments for rank-dependence that have been presented in the literature. A first example from the psychological literature is Birnbaum's (1974) study of the formation of personality impressions. For example, Birnbaum studies the likableness of a person on the basis of intellectuality, shyness, loyalty, etc. He finds empirical violations of additive aggregation and proposes a

"configural weighting model" that better describes how intellectuality etc. are aggregated into likableness of a person. Under configural weighting, "... the weight of a stimulus depends upon its rank within the set to be judged" (page 559). Although this model is formally different from RDU, it does already contain an intuition of rank-dependence. Configural weighting theory was later extended to risky choices (Birnbaum & Navarrete 1998 and the references therein).

A remarkable study is Lopes (1984) who argues for the intuitive value of rank-dependence in risk theory as an extension of the "Gini index" of inequality. The rank-dependent aspect of such measures of inequality is formulated by her as " ... embody distributional objectives in terms of the relative weight given to inequality at different points on the income scale ... The central psychological premise in this article is that people's intuitions about risks are functionally similar to intuitions about distributional inequality. ... representation that captures psychologically salient features of risky distributions" (p. 468). She then explains that people, well aware of the objective probabilities, still "may wish to weight outcomes differently at different points in the distribution" and discusses human ways of reasoning reflecting this procedure (p. 469). Experiments are presented to test for the role of rank-dependence. Lopes concludes that rank-dependence (called the distributional model) "... seems to offer the potential of capturing in a psychologically meaningful way many interesting and important features of people's processing of and preference for risks" (p. 484).

Let us emphasize that Lopes (1984) derived her ideas solely from intuition and

psychological principles. No preference axioms were considered. Her work was developed independently of Quiggin (1981, 1982) or other presentations of RDU. Lopes (1987) presents experiments where subjects were asked to speak aloud on their motives for choices between multiple outcomes gambles. It turned out that subjects pay much attention to "good-news events" (receiving at least as much as ...) and, similarly, bad-news events. This attention is formalized through the probability weighting function and its dual in rank-dependent theories. Rank-dependent decision weights then result from difference-taking. The great attention to good- and bad-news events also supports the inverse-S shapes of the weighting functions.

A third example from the psychological literature is Weber (1994). She uses a somewhat different approach than this paper, invoking an analogy with estimation theory and asymmetric loss functions, and concludes "these processes need not necessarily be perceptual in origin. Instead, in this article, I argued that configural or rank-dependent weighting could be interpreted as strategic or motivational (i.e. a reasonable response that takes into consideration existing constraints that are ignored by the expected utility model)" (p. 236). On p. 237 she discusses perceptual origins ("attentional salience"): "... and more extreme outcomes may get greater weight than outcomes in the middle of the distribution, simply because they are more noticeable."

Models that pay special attention to highest or lowest outcomes can be considered to be special cases of rank-dependence. An example is Rawls' (1971) proposal for welfare evaluation, where all importance weight is allocated to the poorest person

in society. Rank-dependent models for welfare were developed by Weymark (1981) and Ebert (1988). Similar models, with the importance weight divided over the highest and lowest outcomes, were proposed by Hurwicz (1951) and Arrow and Hurwicz (1972). Models that deviate from expected utility only by overweighting highest and/or lowest outcomes were proposed by Bell (1985), Jaffray (1988), Gilboa (1988), and Cohen (1992). In time preference, rank-dependence arises when people are especially sensitive to decreases in salary. This is a special case of rank-dependence, related to the immediately preceding period (Gilboa 1989, Shalev 1997).

Yaari (1987) relates the intuitive meaning of comonotonicity to the phenomenon of "hedging." This phenomenon considers combinations of outcomes and therefore requires a linear structure on the outcome set. For example, consider two gambles for money on the same toss of a coin. The first gamble is $(H, 30; T, 10)$, yielding \$30 if heads comes up and \$10 if tails comes up. The second gamble is $(H, 10; T, 30)$. These two gambles are equivalent but their "fifty-fifty-outcomes-mixture" $(H, 20; T, 20)$ is usually preferred. In the mixture, a reduction of risk has resulted. Hedging occurs because the good outcome of one lottery neutralizes the bad outcome of the other lottery and vice versa. The lotteries serve as complementary goods. The described neutralization can only occur because the gambles are not comonotonic. Hence, Yaari argues that an independence condition (his Axiom A5) is only natural in the absence of hedging, i.e. for comonotonic gambles. A same argument is presented by Röell (1987). Hedging is central in the portfolio selection of assets.

Schmeidler (1989) uses a similar framework that generalizes Yaari's model in two respects. First, Yaari considers real-valued outcomes (interpreted as money), whereas Schmeidler deals with general convex outcome sets (interpreted as probability distributions over prizes). Second, Schmeidler assumes states of nature for which no probabilities need to be given. Yaari's model can be considered the special case of Schmeidler's model where probabilities of the states of nature are given and outcomes are one-dimensional.

Next, we discuss Quiggin (1982). He first discusses the "primitive approach" (our Eq. 2), transforming only probabilities of fixed outcomes, and points out: "the fundamental problem in these theories is that any two outcomes with the same probability must have the same decision weight. This fails to take account of the fact that while individuals may distort the probability of an extreme outcome in some way, they need not treat intermediate outcomes with the same probability in the same fashion." In order to formalize this observation, Quiggin proposes to order the possible outcomes x_i and the corresponding probabilities p_i in each prospect and denotes the rank-ordered probability vector $p_1, \ldots, p_n$ by p. Quoting again from his paper: "The anticipated utility function is defined to be $V = h(p)U(x) = \sum_i h_i(p)U(x_i)$ where U is a utility function with properties similar to that of von Neumann-Morgenstern, while $h(p)$ is a vector of decision weights satisfying $\sum_i h_i(p) = 1$. In general, $h_i(p)$ depends on all the p_js and not just on p_i. Thus, for example, the fact that $p_j = p_k$ would not imply that $h_j(p) = h_k(p)$."

Quiggin's formula is a special case of the general weighting model where the decision weights are independent of the outcomes given the rank-ordered probability vector p. Quiggin gives preference conditions to characterize his formula. He then shows that RDU follows from a continuity condition. The purpose of our analysis was different. We did not invoke technical conditions such as continuity in the derivation but derived rank-dependence from intuitive arguments.

Appendix B. Proofs

Proof of Theorem 2. Decision weights always sum to one, hence $w(p_1 + p_2) = 1 - w(1 - p_1 - p_2) = w(p_1) + w(p_2)$. Therefore, w satisfies Cauchy's equation. By Aczél (1966), w must be linear. Note here that w is bounded by 0 and 1 so that no nonlinear solutions of the Cauchy equation are possible. w is the identity function because $w(0) = 0$ and $w(1) = 1$. $\square$

Proof of Theorem 7. For each event E, define $W(E)$ as the decision weight of that event. $W(E)$ is nonnegative, $W(\emptyset) = 0$, and $W(S) = 1$. Decision weights of partitions always sum to one, hence $W(E_1 \cup E_2) = 1 - W((E_1 \cup E_2)^c) = W(E_1) + W(E_2)$. W is a probability measure and SEU follows. $\square$

References

Abdellaoui, Mohammed (1999), "A Genuine Rank-Dependent Generalization of von Neumann-Morgenstern Expected Utility Theorem," working paper, GRID, ENS, Cachan, France.

Abdellaoui, Mohammed (2000), "Parameter-Free Eliciting of Utilities and Probability Weighting Functions," *Management Science*, forthcoming.

Abdellaoui, Mohammed & Bertrand Munier (1999), in preparation, GRID, ENS, Cachan, France.

Aczél, János, (1966), "*Lectures on Functional Equations and Their Applications*." Academic Press, New York.

Allais, Maurice (1953), "Fondements d'une Théorie Positive des Choix Comportant un Risque et Critique des Postulats et Axiomes de l'Ecole Américaine," *Colloques Internationaux du Centre National de la Recherche Scientifique* 40, Econométrie, 257–332. Paris: Centre National de la Recherche Scientifique. Translated into English, with additions, as "The Foundations of a Positive Theory of Choice Involving Risk and a Criticism of the Postulates and Axioms of the American School," *in* Maurice Allais & O. Hagen (1979, Eds.), *Expected Utility Hypotheses and the Allais Paradox*, 27–145, Reidel, Dordrecht, The Netherlands.

Allais, Maurice (1988), "The General Theory of Random Choices in Relation to the Invariant Cardinal Utility Function and the Specific Probability Function," *In*

Bertrand R. Munier (Ed.), *Risk, Decision and Rationality*, 233–289, Reidel, Dordrecht.

Arrow, Kenneth J. & Leonid Hurwicz (1972), "An Optimality Criterion for Decision Making under Ignorance." *In* C.F. Carter & James L. Ford (1972), *Uncertainty and Expectations in Economics*, Basil Blackwell & Mott Ltd., Oxford, UK.

Backhouse, Roger E. (1998), "If Mathematics Is Informal, then perhaps We Should Accept that Economics Must Be Informal too," *The Economic Journal* 108, 1848–1858.

Bell, David E. (1985), "Disappointment in Decision Making under Uncertainty," *Operations Research* 33, 1–27.

Birnbaum, Michael H. (1974), "The Nonadditivity of Personality Impressions," *Journal of Experimental Psychology* 102, 543–561.

Birnbaum, Michael H. & Juan B. Navarrete (1998), "Testing Descriptive Utility Theories: Violations of Stochastic Dominance and Cumulative Independence," *Journal of Risk and Uncertainty* 17, 49–78.

Birnbaum, Michael H., Gregory Coffey, Barbara A. Mellers, & Robin Weiss (1992), "Utility Measurement: Configural-Weight Theory and the Judge's Point of View," *Journal of Experimental Psychology: Human Perception and Performance* 18, 331–346.

Birnbaum, Michael H. & William R. McIntosh (1996), "Violations of Branch Independence in Choices between Gambles," *Organizational Behavior and Human*

Decision Processes 67, 91–110.

Bleichrodt, Han & Jose Luis Pinto (2000), "A Parameter-Free Elicitation of the Probability Weighting Function in Medical Decision Making," *Management Science*, forthcoming.

Camerer, Colin F. & Teck-Hua Ho (1994), "Violations of the Betweenness Axiom and Nonlinearity in Probability," *Journal of Risk and Uncertainty* 8, 167–196.

Chew, Soo Hong (1989), "Axiomatic Utility Theories with the Betweenness Property," *Annals of Operations Research* 19, 273–298.

Cohen, Michèle (1992), "Security Level, Potential Level, Expected Utility: A Three-Criteria Decision Model under Risk," *Theory and Decision* 33, 101–134.

Currim, Imran S. & Rakesh K. Sarin (1989), "Prospect versus Utility," *Management Science* 35, 22–41.

Dempster, Arthur P. (1967), "Upper and Lower Probabilities Induced by a Multivalued Mapping," *Annals of Mathematical Statistics* 38, 325–339.

Ebert, Udo (1988), "Measurement of Inequality: An Attempt at Unification and Generalization," *Social Choice and Welfare* 5, 147–169.

Edwards, Ward (1955), "The Prediction of Decisions Among Bets," *Journal of Experimental Psychology* 50, 201–214.

Epstein, Larry G. (1999), " A Definition of Uncertainty Aversion," *Review of Economic Studies* 66, 579–608.

Fellner, William (1961), "Distortion of Subjective Probabilities as a Reaction to Un-

certainty," *Quarterly Journal of Economics* 75, 670–689.

Fishburn, Peter C. (1978), "On Handa's "New Theory of Cardinal Utility" and the Maximization of Expected Return," *Journal of Political Economy* 86, 321–324.

Fox, Craig R., Brett A. Rogers, & Amos Tversky (1996), "Options Traders Exhibit Subadditive Decision Weights," *Journal of Risk and Uncertainty* 13, 5–17.

Fox, Craig R. & Amos Tversky (1995), "Ambiguity Aversion and Comparative Ignorance," *Quarterly Journal of Economics* 110, 585–603.

Gilboa, Itzhak (1987), "Expected Utility with Purely Subjective Non-Additive Probabilities," *Journal of Mathematical Economics* 16, 65–88.

Gilboa, Itzhak (1988), "A Combination of Expected Utility Theory and Maxmin Decision Criteria," *Journal of Mathematical Psychology* 32, 405–420.

Gilboa, Itzhak (1989), "Expectations and Variations in Multi-Period Decisions," *Econometrica* 57, 1153–1169.

Gonzalez, Richard & George Wu (1999), "On the Shape of the Probability Weighting Function," *Cognitive Psychology* 38, 129–166.

Harless, David W. & Colin F. Camerer (1994), "The Predictive Utility of Generalized Expected Utility Theories," *Econometrica* 62, 1251–1289.

Hurwicz, Leonid (1951), "Optimality Criteria for Decision Making under Ignorance," Cowles Commission Discussion Paper, Statistics, No. 370, mimeographed.

Jaffray, Jean-Yves (1988), "Choice under Risk and the Security Factor," *Theory and Decision* 24, 169–200.

Johnson, Eric J. & David A. Schkade (1989), "Bias in Utility Assessments: Further Evidence and Explanations," *Management Science* 35, 406–424.

Kachelmeier, Steven J. & Mohammed Shehata (1992), "Examining Risk Preferences under High Monetary Incentives: Experimental Evidence from the People's Republic of China," *American Economic Review* 82, 1120–1141; comment see AER 84, 1994, 1104–1106.

Karmarkar, Uday S. (1978), "Subjectively Weighted Utility: A Descriptive Extension of the Expected Utility Model," *Organizational Behavior and Human Performance* 21, 61–72.

Karni, Edi & Zvi Safra (1990), "Rank-Dependent Probabilities," *The Economic Journal* 100, 487–495.

Karni, Edi & David Schmeidler (1991), "Utility Theory with Uncertainty." *In* Werner Hildenbrand & Hugo F. Sonnenschein (Eds.), *Handbook of Mathematical Economics* 4, Chapter 33, 1763–1831, North-Holland, Amsterdam.

Keynes, John Maynard (1921), "*A Treatise on Probability.*" McMillan, London. Second edition 1948.

Kilka, Michael & Martin Weber (1998), "What Determines the Shape of the Probability Weighting Function under Uncertainty," University of Mannheim, Mannheim, Germany.

Lopes, Lola L. (1984), "Risk and Distributional Inequality," *Journal of Experimental Psychology: Human Perception and Performance* 10, 465–485.

Lopes, Lola L. (1987), "Between Hope and Fear: The Psychology of Risk," *Advances in Experimental Psychology* 20, 255–295.

Lopes, Lola L. & Gregg C. Oden (1999), "The Role of Aspiration Level in Risky Choice: A Comparison of Cumulative Prospect Theory and SP/A Theory," *Journal of Mathematical Psychology* 43, 286–313.

Luce, R. Duncan (1996a), "The Ongoing Dialog between Empirical Science and Measurement Theory," *Journal of Mathematical Psychology* 40, 78–98.

Luce, R. Duncan (1996b), " When Four Distinct Ways to Measure Utility Are the Same," *Journal of Mathematical Psychology* 40, 297–317.

Luce, R. Duncan (1998), "Coalescing, Event Commutativity, and Theories of Utility," *Journal of Risk and Uncertainty* 16, 87–114.

Preston, Malcolm G. & Philip Baratta (1948), "An Experimental Study of the Auction Value of an Uncertain Outcome," *American Journal of Psychology* 61, 183–193.

Quiggin, John (1981), "Risk Perception and Risk Aversion among Australian Farmers," *Australian Journal of Agricultural Economics* 25, 160–169.

Quiggin, John (1982), "A Theory of Anticipated Utility," *Journal of Economic Behaviour and Organization* 3, 323–343.

Rawls, John (1971), "*A Theory of Justice*." Harvard University Press, Cambridge MA.

Röell, Ailsa (1987), "Risk Aversion in Quiggin and Yaari's Rank-Order Model of

Choice under Uncertainty," (Supplement to the) *The Economic Journal* 97, 143–160.

Safra, Zvi & Uzi Segal (1998), "Constant Risk Aversion," *Journal of Economic Theory* 83, 19–42.

Sarin, Rakesh K. & Peter P. Wakker (1998), "Revealed Likelihood and Knightian Uncertainty," *Journal of Risk and Uncertainty* 16, 223–250.

Schmeidler, David (1989), "Subjective Probability and Expected Utility without Additivity," *Econometrica* 57, 571–587.

Schmidt, Ulrich (1998), "Alternatives to Expected Utility: Some Formal Theories." *In* Barberà, Salvador, Peter J. Hammond, & Christian Seidl, *Handbook of Utility Theory*, Kluwer Academic Publishers, Dordrecht, in press.

Shafer, Glenn (1976), "*A Mathematical Theory of Evidence.*" Princeton University Press, Princeton NJ.

Shalev, Jonathan (1997), "Loss Aversion in a Multi-Period Model," *Mathematical Social Sciences* 33, 203–226.

Starmer, Chris (1999), "Developments in Non-Expected Utility Theory: The Hunt for a Descriptive Theory of Choice under Risk," *Journal of Economic Literature*, forthcoming.

Tversky, Amos & Craig R. Fox (1995), "Weighing Risk and Uncertainty," *Psychological Review* 102, 269–283.

Tversky, Amos & Daniel Kahneman (1992), "Advances in Prospect Theory: Cumula-

tive Representation of Uncertainty," *Journal of Risk and Uncertainty* 5, 297–323.

Tversky, Amos & Peter P. Wakker (1995), "Risk Attitudes and Decision Weights," *Econometrica* 63, 1255–1280.

Tversky, Amos, & Peter P. Wakker (1998), "On the Composition of Risk Preference and Belief," Medical Decision Making Unit, Leiden University, The Netherlands.

Viscusi, W. Kip (1997), "Alarmist Decisions with Divergent Risk Information," *The Economic Journal* 107, 1657–1670.

Wakker, Peter P. (1994), "Separating Marginal Utility and Probabilistic Risk Aversion," *Theory and Decision* 36, 1–44.

Wakker, Peter P. & Daniel Deneffe (1996), "Eliciting von Neumann-Morgenstern Utilities when Probabilities Are Distorted or Unknown," *Management Science* 42, 1131–1150.

Wakker, Peter P. & Amos Tversky (1993), "An Axiomatization of Cumulative Prospect Theory," *Journal of Risk and Uncertainty* 7, 147–176.

Weber, Elke U. (1994), "From Subjective Probabilities to Decision Weights: The Effects of Asymmetric Loss Functions on the Evaluation of Uncertain Outcomes and Events," *Psychological Bulletin* 115, 228–242.

Weymark, John A. (1981), "Generalized Gini Inequality Indices," *Mathematical Social Sciences* 1, 409–430.

Wu, George & Richard Gonzalez (1996), "Curvature of the Probability Weighting Function," *Management Science* 42, 1676–1690.

Wu, George & Richard Gonzalez (1999), "Nonlinear Decision Weights in Choice under Uncertainty," *Management Science* 45, 74–85.

Yaari, Menachim E. (1965), "Convexity in the Theory of Choice under Risk," *Quarterly Journal of Economics* 79, 278–290.

Yaari, Menachim E. (1987), "The Dual Theory of Choice under Risk," *Econometrica* 55, 95–115.