# A CONCEDE-AND-DIVIDE RULE FOR BANKRUPTCY PROBLEMS

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## Abstract

The concede-and-divide rule is a basic solution for bankruptcy problems with two claimants. An extension of the concede-and-divide rule to bankruptcy problems with more than two claimants is provided. This extension not only uses the concede-and-divide principle in its procedural definition, but also preserves the main properties of the concede-and-divide rule.

**Keywords:** Bankruptcy problems, concede-and-divide rule.

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# 1 Introduction

Bankruptcy problems are first treated within the framework of interactive allocation problems in O'Neill (1982). In a bankruptcy problem a certain amount of money, the estate, has to be divided among a group of claimants. The amount claimed is larger than the estate, which gives rise to the problem of a fair division of the estate, satisfactory to all claimants. Many solutions for bankruptcy problems have been proposed in literature. An overview of bankruptcy rules and their properties can be found in Thomson (2003).

In a bankruptcy situation with only two claimants, e.g. the run-to-the-bank rule (cf. O'Neill (1982)), the Talmud rule (cf. Aumann and Maschler (1985)) and the adjusted proportional rule (cf. Curiel, Maschler, and Tijs (1988)) coincide. Thomson (2003) refers to this basic or standard rule as the concede-and-divide rule. It is based on the idea that each claimant concedes the amount of the estate that is not claimed by himself to the other claimant. This amount can be seen as a minimal right of a claimant. Subsequently, the amount left of the estate after giving both claimants their minimal rights is divided equally. Dagan (1996) provides several characterisations of the concede-and-divide rule using the properties self-duality, minimal right first and invariance under claim truncation. Another characterization can be found in Moreno-Ternero and Villar (2004) using among others the property of lower securement. The idea behind the concede-and-divide rule is appealing, but is it has never been extended to bankruptcy problems with more than two claimants.

In this paper we provide such an extension of the concede-and-divide rule. This extension is inspired by the extension of the standard solution of two person cooperative games to a solution of cooperative games with an arbitrary finite set of players provided by Ju, Borm, and Ruys (2004). The underlying idea is that claimants leave the group one by one in a specific (but random) order. At each moment a claimant leaves the (remaining) group, he receives a part of the estate that is left at that stage. The amount he gets is based on the concede-and-divide rule for a two person bankruptcy situation in which he is seen as one claimant and the rest of the group together as the other. In this way the concede-and-divide principle is applied recursively. Taking the average over all possible orders, one then obtains the concede-and-divide allocation. Alternatively the concede-and divide allocation can be seen as the result of the following procedure. Suppose there is a bank that manages the estate, which knows the total amount of the claims. The bank however is ignorant about the exact number of claimants. Now the claimants arrive in a specific (but random) order at the bank. So at the time a specific claimant arrives, the bank only knows the remaining estate, the claim of the present claimant and the total claims of the possible claimants that are to arrive in future. Therefore, at that instance, an application of the concede-and-divide principle seems natural.

In section 2 the formal definition of the extended concede-and-divide rule for bankruptcy problems with more than two claimants is introduced. Furthermore a recursive formula is provided. In section 3 it is shown that this extension preserves the main properties of the concede-and divide rule.

# 2 The concede-and-divide rule

This section formally introduces a concede-and-divide rule for arbitrary bankruptcy problems.

A bankruptcy problem consists of a triple $(N, E, c)$, where $E$ is the estate that has to be divided among a finite set of claimants $N$, and $c \in \mathbb{R}^N$, $c \geq 0$ is a vector of claims. By the nature of a

bankruptcy problem we assume that $E \leq \sum_{i \in N} c_i$. The class of all bankruptcy problems is denoted by $\mathcal{C}$. Let $(N, E, c) \in \mathcal{C}$ and $S \subset N$. Then the part of $E$ that is not claimed by $N \backslash S$ can be seen as a right of coalition $S$ and is denoted by

$$r_S(N, E, c) = \max\{E - \sum_{j \in N \backslash S} c_j, 0\}.$$

A function $f$ on $\mathcal{C}$ is a bankruptcy rule if for all bankruptcy problems $(N, E, c)$, $f(N, E, c) \in \mathbb{R}^N$, $0 \leq f_i(N, E, c) \leq c_i$ for all $i \in N$ and $\sum_{i \in N} f_i(N, E, c) = E$. An overview of bankruptcy rules proposed in the literature is found in Thomson (2003).

The concede-and-divide rule (cf. Thomson (2003)), denoted by CD, is only defined for bankruptcy problems with two claimants. Let $|N| = 2$, $N = \{i, j\}$ and $(N, E, c) \in \mathcal{C}$. Then

$$CD_i(N, E, c) = \max\{E - c_j, 0\} + \frac{E - \max\{E - c_j, 0\} - \max\{E - c_i, 0\}}{2},$$

or alternatively

$$CD_i(N, E, c) = r_{\{i\}}(N, E, c) + \frac{E - r_{\{i\}}(N, E, c) - r_{\{j\}}(N, E, c)}{2}.$$

The concede-and-divide rule first gives each claimant that part of the estate that is not claimed by the other. Subsequently the remaining estate is equally divided among both claimants. In the case that $|N| = 2$, the concede-and-divide rule can be seen as the standard solution.

We now introduce an extension of the concede-and-divide rule. This extended rule is applicable to arbitrary bankruptcy problems and respects the concede-and-divide principle in its procedural definition. The underlying idea is that claimants leave the group one by one in a specific (but random) order. If a claimant leaves the group he receives a part of the estate. The amount given to him is based on the concede-and-divide rule for bankruptcy problems with two claimants. He himself is seen as one claimant, while the rest of the group together is viewed as the other. The part that is distributed to the the rest of the group acts as the new estate in the following step, when the next claimant is leaving. Taking the average over all possible orders, one obtains the concede-and-divide allocation.

An order of $N$ is a bijective function $\sigma : \{1, \ldots, |N|\} \to N$. The claimant at position $k$ in the order $\sigma$ is denoted by $\sigma(k)$. The set of all orders of $N$ is denoted by $\Pi(N)$.

Let $\sigma \in \Pi(N)$. This order is interpreted as the order in which claimants are leaving with a certain amount of the estate. This results in a vector $s^\sigma(N, E, c)$. Formally the vector $s^\sigma(N, E, c)$ is defined as follows: let $k \in \{1, \ldots, |N|\}$. Then

$$s^\sigma_{\sigma(k)}(N, E, c) = r_{\sigma(k)}(M^k, E^k, d^k) + \frac{E - r_{\sigma(k)}(M^k, E^k, d^k) - r_{M^k \backslash \{\sigma(k)\}}(M^k, E^k, d^k)}{2},$$

here $M^k = \{\sigma(k), \ldots, \sigma(|N|)\}$, $d^k \in \mathbb{R}^{M^k}$ is the restriction of the vector $c$ to the claimants in $M^k$ and $E^1 = E$, $E^k = E^{k-1} - CD^\sigma_{\sigma(k-1)}(E, c)$ ($2 \leq k \leq |N|$).

The concede-and-divide rule for general bankruptcy problems averages between all possible orders.

**Definition** *The concede-and-divide rule is defined by*

$$CD_i(N, E, c) = \sum_{\sigma \in \Pi(N)} s^\sigma_i(E, c)$$

*for all $i \in N$ and all $(N, E, c) \in \mathcal{C}$.*

3

Note that in case $|N| = 2$, both orders $\sigma^1 = (12)$ and $\sigma^2 = (21)$ give rise to the same allocation. Consequently, one readily verifies that the concede-and-divide rule, as defined above, is indeed an extension of the concede-and-divide rule for two claimant bankruptcy situations.

**Example 2.1** *Let $N = \{1, 2, 3\}$, $E = 12$ and $c = (4, 6, 8)$. Consider the order $\sigma = (123)$. Then claimant 1 is the first claimant who leaves the group. Claimant 2 and 3 together have a total claim of 14. The minimal right of claimant 1 equals 0, and claimant 2 and 3 have together a minimal right of $E - c_1 = 8$. Claimant 1 receives $0 + \frac{E-0-8}{2} = 2$. Now claimant 2 is leaving and the remaining estate equals $12 - 2 = 10$. The minimal right of claimant 2 and 3 equal $r_{\{2\}}(\{2, 3\}, 10, (6, 8)) = 2$ and $r_{\{3\}}(\{2, 3\}, 10, (6, 8)) = 4$, respectively. Hence claimant 2 leaves with $2 + \frac{10-2-4}{2} = 4$ and claimant 3 receives $4 + \frac{10-4-2}{2} = 6$. This yields that $s^{\sigma}(N, E, c) = (2, 4, 6)$.*

*All vectors $s^{\sigma}(N, E, c)$, $\sigma \in \Pi(N)$ are given in the table below.*

| $\sigma$ | $s^{\sigma}(N, E, c)$ |
|---|---|
| $(123)$ | $(2, 4, 6)$ |
| $(132)$ | $(2, 4, 6)$ |
| $(213)$ | $(2\frac{1}{2}, 3, 6\frac{1}{2})$ |
| $(231)$ | $(2\frac{1}{2}, 3, 6\frac{1}{2})$ |
| $(312)$ | $(2\frac{1}{2}, 4\frac{1}{2}, 5)$ |
| $(321)$ | $(2\frac{1}{2}, 4\frac{1}{2}, 5)$ |

*And consequently*

$$CD(N, E, c) = \frac{1}{6} \sum_{\sigma \in \Pi(N)} s^{\sigma}(N, E, c) = \frac{1}{6}(14, 23, 39).$$

The approach used to define the concede-and-divide rule for arbitrary situations is inspired by Ju et al. (2004), where the standard solution for a two-person game is extended to a solution (the consensus value) for arbitrary cooperative games[1]. The following lemma provides a recursive formula for the concede-and-divide rule.

**Lemma 2.1** *Let $(N, E, c) \in \mathcal{C}$. Then for all $i \in N$*

$$CD_i(N, E, c) = \frac{1}{|N|}\left(r_{\{i\}}(N, E, c) + \frac{E - r_{\{i\}}(N, e, c) - r_{N\setminus\{i\}}(N, E, c)}{2}\right) +$$
$$\frac{1}{|N|} \sum_{j \in N\setminus\{i\}} CD_i(N\setminus\{j\}, E_{-j}, c_{-j}))$$

*where $E_{-j}$ is the amount that is left of the estate if claimant $j$ is the first one to leave, i.e.,*

$$E_{-j} = E - r_{\{j\}}(N, E, c) - \frac{E - r_{\{j\}}(N, e, c) - r_{N\setminus\{j\}}(N, E, c)}{2}$$

*and $c_{-j} \in \mathbb{R}^{N\setminus\{j\}}$ denotes the claim vector $c$, in which the claim of $j$ is omitted.*

By means of Lemma 2.1 it readily follows (by induction) that $0 \leq CD(N, E, c) \leq c$.

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[1]The concede-and-divide allocation of a bankruptcy problem generally does not equal the consensus value of the corresponding bankruptcy game. In fact the last approach does not determine a bankruptcy rule. Consider the bankruptcy situation $(N, E, c) = (\{1, 2, 3\}, 100, (60, 60, 1))$. Then the consensus value of the corresponding bankruptcy game equals $(48, 48, 2)$. For comparison, $CD(N, E, c) = (49\frac{3}{4}, 49\frac{3}{4}, \frac{1}{2})$.

# 3  Properties of the concede-and-divide rule

In this section we analyze some general properties of the concede-and-divide rule.

First we derive an explicit formula for the concede-and-divide rule if the bankruptcy situation is a small claims situation. Let $(N, E, c) \in \mathcal{C}$. Then $(N, E, c)$ is a small claims situation if for all $i \in N$, $\sum_{j \in N \setminus \{i\}} c_j \leq E$. In a small claims situation $(N, E, c)$ the Talmud rule, the run-to-the-bank rule and the adjusted proportional rule coincide and claimant $i \in N$ will be allocated $\frac{1}{|N|}(E - \sum_{j \in N} c_j) + c_i$. The following theorem shows that in a small claims situation the concede-and-divide rule prescribes the same allocation.

**Theorem 3.1** *Let $(N, E, c) \in \mathcal{C}$ be a small claims situation. Then*

$$CD_i(N, E, c) = \frac{1}{|N|}(E - \sum_{j \in N} c_j) + c_i \tag{1}$$

*for all $i \in N$.*

**Proof**: The proof is given by an induction argument on the number of claimants, using the recursive formula of Lemma 2.1.

Let $|N| = 2$. Then $(N, E, c)$ is a small claims situation if $c_1 \leq E$ and $c_2 \leq E$. It follows that

$$CD_1(N, E, c) = E - c_2 + \tfrac{E - (E - c_2) - (E - c_1)}{2} = \tfrac{1}{2}(E - c_1 - c_2) + c_1$$

and

$$CD_2(N, E, c) = E - c_1 + \tfrac{E - (E - c_1) - (E - c_2)}{2} = \tfrac{1}{2}(E - c_2 - c_1) + c_2.$$

Let $k \in \mathbb{N}$. Assume that for all small claims situations with $|N| \leq k$, equation (1) is valid for all $i \in N$. Denote $\sum_{j \in N} c_j$ by $C$. Let $(N, E, c) \in \mathcal{C}$ be a small claims situation such that $|N| = k + 1$ and let $i \in N$. According to Lemma 2.1, combined with the fact that we have a small claims situation, we have

$$CD_i(N, E, c) \;=\; \tfrac{1}{|N|}\left(\tfrac{1}{2}(E - C) + c_i\right) + \tfrac{1}{|N|} \sum_{j \in N \setminus \{i\}} CD_i(N \setminus \{j\}, E_{-j}, c_{-j}),$$

where

$$E_{-j} = E - (E - \sum_{\ell \in N \setminus \{j\}} c_\ell) - \frac{1}{2}(E - (E - \sum_{\ell \in N \setminus \{j\}} c_\ell) - (E - c_j)) = \frac{1}{2}E + \frac{1}{2}C - c_j$$

for all $j \in N \setminus \{i\}$.

Now consider $j \in N$ and $m \in N \setminus \{j\}$. Then

$$
\begin{aligned}
E_{-j} - \sum_{\ell \in N \setminus \{j,m\}} c_\ell \;&=\; \tfrac{1}{2}E + \tfrac{1}{2}C - c_j - \sum_{\ell \in N \setminus \{j,m\}} c_\ell \\
&=\; \tfrac{1}{2}(E - \sum_{\ell \in N \setminus \{m\}} c_\ell) + \tfrac{1}{2}c_m \geq 0.
\end{aligned}
$$

Hence $(N\backslash\{j\}, E_{-j}, c_{-j})$ is a small claims situation. Since $|N\backslash\{j\}| = k$, the induction hypothesis implies

$$
\begin{aligned}
CD_i(N, E, c) &= \tfrac{1}{|N|}\Big(\tfrac{1}{2}(E - C) + c_i\Big) + \tfrac{1}{|N|}\sum_{j \in N\backslash\{i\}} \Big(\tfrac{1}{|N|-1}(E_{-j} - \sum_{\ell \in N\backslash\{j\}} c_\ell) + c_i\Big) \\
&= \tfrac{1}{|N|}\Big(\tfrac{1}{2}(E - C) + c_i\Big) + \tfrac{1}{|N|}\sum_{j \in N\backslash\{i\}} \Big(\tfrac{1}{|N|-1}\tfrac{1}{2}(E - C) + c_i\Big) \\
&= \tfrac{1}{|N|}\Big(\tfrac{1}{2}(E - C) + c_i\Big) + \tfrac{1}{|N|}\big(\tfrac{1}{2}(E - C) + (|N| - 1)c_i\big) \\
&= \tfrac{1}{|N|}(E - C) + c_i.
\end{aligned}
$$

$\square$

The concede-and-divide rule satisfies the properties of homogeneity, resource monotonicity, claims monotonicity and equal treatment of equals. The proof is rather straightforward (by induction) and is left to the reader. We now recall the definitions of three properties that are used to characterize the concede-and-divide rule in the two claimant case. Let $f$ be a bankruptcy rule. The rule $f$ is *invariant under claims truncation* if for all $(N, E, c) \in \mathcal{C}$, we have $f(N, E, c) = f(N, E, \bar{c})$, where $\bar{c} \in \mathbb{R}^N$ is the truncated claim vector, i.e., $\bar{c}_i = \min\{E, c_i\}$ for all $i \in N$. The bankruptcy rule $f$ is *self-dual* if for all $(N, E, c) \in \mathcal{C}$, we have $f(N, E, c) = c - f(N, \sum_{j \in N} c_j - E, c)$. The rule $f$ satisfies *minimal rights first* if for all $(N, E, c) \in \mathcal{C}$, we have $f(N, E, c) = r + f(N, E - \sum_{j \in N} r_j, c - r)$, where $r \in \mathbb{R}^N$ is the minimal right vector, i.e., $r_i = r_{\{i\}}(N, E, c)$ for all $i \in N$.

**Theorem 3.2 (cf. Dagan (1996))** *The concede-and-divide rule is the only rule on bankruptcy situations with two claimants satisfying*

  *(i) invariance under claims truncation and self-duality;*

*or, alternatively,*

  *(ii) minimal rights first and self-duality.*

The next theorem proves that the extended concede-and-divide rule still satisfies these three properties on the domain of all bankruptcy situations.

**Theorem 3.3** *The concede-and-divide rule satisfies invariance under claims truncation, self-duality and minimal rights first.*

**Proof**: (1) Invariance under claims truncation. The proof is given by an induction argument. Let $|N| = 2$. Since the extension of the concede-and-divide rule coincides with the original concede-and-divide rule in the two claimants case, it follows by Theorem 3.2 that the concede-and-divide rule satisfies truncation.

Let $k \in \mathbb{N}$, $k \geq 2$. Assume that for all bankruptcy situations $(N, E, c)$ such that $|N| \leq k$ the concede-and-divide rule satisfies invariance under claims truncation. Let $N$ be such that $|N| = k+1$ and let $(N, E, c)$ be a bankruptcy situation. Then for all $S \subset N$,

$$
r_S(N, E, c) = \max\{E - \sum_{j \in N\backslash S} c_j, 0\} = \max\{E - \sum_{j \in N\backslash S} \bar{c}_j, 0\} = r_S(N, E, \bar{c}).
$$

Let $j \in N$ and let $\tilde{c} \in \mathbb{R}^{N\setminus\{j\}}$ be defined as $\tilde{c}_i = \min\{E_{-j}, c_i\}$ for all $i \in N\setminus\{j\}$. Then $\tilde{c}_i = \min\{E_{-j}, \bar{c}_i\}$. Using the induction hypothesis twice we find that

$$CD(N\setminus\{j\}, E_{-j}, c_{-j}) = CD(N\setminus\{j\}, E_{-j}, \tilde{c}_{-j}) = CD(N\setminus\{j\}, E_{-j}, \bar{c}_{-j}).$$

Using the above observations, the recursive formula of Lemma 2.1 directly implies that $CD(N, E, c) = CD(N, E, \bar{c})$.

(2) Self-duality. First note that it is sufficient to prove that for all $\sigma \in \Pi(N)$ and all $(N, E, c)$ we have that

$$s^\sigma(N, E, c) = c - s^\sigma(N, \sum_{j\in N} c_j - E, c). \tag{2}$$

For $|N| = 2$, this is obvious by Theorem 3.2. Let $k \in \mathbb{N}$, $k \geq 2$. Assume that for all bankruptcy situations $(N, E, c)$ such that $|N| \leq k$ and all $\sigma \in \Pi(N)$, formula (2) is satisfied. Let $N$ be such that $|N| = k + 1$. Take $(N, E, c) \in \mathcal{C}$ and $\sigma \in \Pi(N)$. Denote $\sum_{j\in N} c_j$ by $C$. We first prove that

$$s^\sigma_{\sigma(1)}(N, E, c) + s^\sigma_{\sigma(1)}(N, C - E, c) = c_{\sigma(1)}. \tag{3}$$

Since $\sigma(1)$ is the first claimant that leaves the group

$$s^\sigma_{\sigma(1)}(N, E, c) = \tfrac{1}{2}\max\{E - \sum_{j\in N\setminus\{\sigma(1)\}} c_j, 0\} + \tfrac{1}{2}(E - \max\{E - c_{\sigma(1)}, 0\}), \tag{4}$$

and

$$\begin{aligned}
s^\sigma_{\sigma(1)}(N, C - E, c) &= \tfrac{1}{2}\max\{C - E - \sum_{j\in N\setminus\{\sigma(1)\}} c_j, 0\} + \tfrac{1}{2}(C - E - \max\{C - E - c_{\sigma(1)}, 0\} \\
&= \tfrac{1}{2}\max\{c_{\sigma(1)} - E, 0\} + \tfrac{1}{2}(C - E - \max\{\sum_{j\in N\setminus\{\sigma(1)\}} c_j - E, 0\}).
\end{aligned} \tag{5}$$

Note that

$$\max\{E - \sum_{j\in N\setminus\{\sigma(1)\}} c_j, 0\} - \max\{\sum_{j\in N\setminus\{\sigma(1)\}} c_j - E, 0\} = E - \sum_{j\in N\setminus\{\sigma(1)\}} c_j,$$

and

$$\max\{c_{\sigma(1)} - E, 0\} - \max\{E - c_{\sigma(1)}, 0\} = c_{\sigma(1)} - E.$$

Adding equations (4) and (5) yields

$$s^\sigma_{\sigma(1)}(N, E, c) + s^\sigma_{\sigma(1)}(N, C - E, c) = \tfrac{1}{2}(c_{\sigma(1)} - E) - \tfrac{1}{2}(\sum_{j\in N\setminus\{\sigma(1)\}} c_j - E) + \tfrac{1}{2}C = c_{\sigma(1)}.$$

$E_{-\sigma(1)}$ is the amount that is left of $E$, if $\sigma(1)$ has first left the group in the bankruptcy situation $(N, E, c)$ and similarly $(C - E)_{-\sigma(1)}$ is the amount that is left of $E$, if $\sigma(1)$ has first left the group in the bankruptcy situation $(N, C - E, c)$. From equation (3) it follows that

$$E_{-\sigma(1)} + (C - E)_{-\sigma(1)} = C - c_{\sigma(1)}. \tag{6}$$

Let $\bar{\sigma} \in \Pi(N\backslash\{\sigma(1)\})$ be the order of all other claimants that is prescribed by $\sigma$. Let $i \in N\backslash\{\sigma(1)\}$. Then

$$
\begin{aligned}
s_i^\sigma(N, E, c) &= s_i^{\bar{\sigma}}(N\backslash\{\sigma(1)\}, E_{-\sigma(1)}, c_{-\sigma(1)}) \\
&\overset{\text{ind.}}{=} c_i - s_i^{\bar{\sigma}}(N\backslash\{\sigma(1)\}, \sum_{j \in N\backslash\{\sigma(1)\}} c_j - E_{-\sigma(1)}, c_{-\sigma(1)}) \\
&\overset{(6)}{=} c_i - s_i^{\bar{\sigma}}(N\backslash\{\sigma(1)\}, (C - E)_{-\sigma(1)}, c_{-\sigma(1)}) \\
&= c_i - s_i^\sigma(N, C - E, c).
\end{aligned}
$$

(3) Minimal roghts first. Note that it is sufficient to prove that for all $(N, E, c)$ and all $\sigma \in \Pi(N)$ we have that

$$
s^\sigma(N, E, c) = r + s^\sigma(N, E - \sum_{j \in N} r_j, c - r). \tag{7}
$$

For $|N| = 2$, this is obvious by Theorem 3.2. Let $k \in \mathbb{N}$, $k \geq 2$. Assume that for all bankruptcy situations $(N, E, c)$ such that $|N| \leq k$ and for all $\sigma \in \Pi(N)$, formula (7) is satisfied. Let $N$ be such that $|N| = k + 1$ and let $(N, E, c) \in \mathcal{C}$, $\sigma \in \Pi(N)$. Recall that $r_i = r_{\{i\}}(N, E, c)$ and define $R = \sum_{j \in N} r_j$. We first prove that

$$
s_{\sigma(1)}^\sigma(N, E, c) = r_{\sigma(1)} + s_{\sigma(1)}^\sigma(N, E - R, c - r). \tag{8}
$$

Now,

$$
\begin{aligned}
s_{\sigma(1)}^\sigma(N, E, c) &= r_{\sigma(1)} + \tfrac{1}{2}(E - r_{\sigma(1)} - \max\{E - c_{\sigma(1)}, 0\}) \\
&= r_{\sigma(1)} + \tfrac{1}{2}\min\{c_{\sigma(1)} - r_{\sigma(1)}, E - r_{\sigma(1)}\}, \tag{9}
\end{aligned}
$$

and, since $r_{\{\ell\}}(N, E - R, c - r) = 0$ for all $\ell \in N$,

$$
\begin{aligned}
r_{\sigma(1)} + s_{\sigma(1)}^\sigma(N, E - R, c - r) &= r_{\sigma(1)} + \tfrac{1}{2}(E - R - \max\{E - R - (c_{\sigma(1)} - r_{\sigma(1)}), 0\}) \\
&= r_{\sigma(1)} + \tfrac{1}{2}\min\{c_{\sigma(1)} - r_{\sigma(1)}, E - R\}. \tag{10}
\end{aligned}
$$

There are two cases. First if $r_j = 0$ for all $j \in N\backslash\{\sigma(1)\}$, then trivially (9) and (10) are equal. Secondly suppose there is a $j \in N\backslash\{\sigma(1)\}$ such that $r_j > 0$. Then $c_{\sigma(1)} < E$ and $r_j = E - \sum_{\ell \in N\backslash\{j\}} c_\ell \leq E - \sum_{\ell \in N\backslash\{\sigma(1),j\}} r_\ell - c_{\sigma(1)}$ indicating that $c_{\sigma(1)} - r_{\sigma(1)} \leq E - R$ and hence $c_{\sigma(1)} - r_{\sigma(1)} \leq E - r_{\sigma(1)}$. So also in this case (9) and (10) are equal, and (8) is established.

From (8) and the definition of $E_{-\sigma(1)}$ and $(E - R)_{-\sigma(1)}$, it can be concluded that

$$
E_{-\sigma(1)} - R = (E - R)_{-\sigma(1)} - r_{\sigma(1)} \tag{11}
$$

Next define $\tilde{r} \in \mathbb{R}^{N\backslash\{\sigma(1)\}}$ by $\tilde{r}_\ell = r_{\{\ell\}}(N\backslash\{\sigma(1)\}, E_{-\sigma(1)}, c_{-\sigma(1)})$ for all $\ell \in N\backslash\{\sigma(1)\}$ and denote $\tilde{R} = \sum_{\ell \in N\backslash\{\sigma(1)\}} \tilde{r}_\ell$. Note that $\tilde{r}_i \geq r_i$, since $E_{-\sigma(1)} \geq E - c_{\sigma(1)}$. Let $\bar{\sigma} \in \Pi(N\backslash\sigma(1))$ be the order of all other claimants that is prescribed by $\sigma$. Let $i \in N\backslash\{\sigma(1)\}$. Using the induction hypothesis

$$
\begin{aligned}
s_i^\sigma(N, E, c) &= s_i^{\bar{\sigma}}(N\backslash\{\sigma(1)\}, E_{-\sigma(1)}, c_{-\sigma(1)}) \\
&= \tilde{r}_i + s_i^{\bar{\sigma}}(N\backslash\{\sigma(1)\}, E_{-\sigma(1)} - \tilde{R}, c_{-\sigma(1)} - \tilde{r}).
\end{aligned}
$$

8

On the other hand

$$r_i + s_i^\sigma(N, E - R, c - r) = r_i + s_i^{\bar\sigma}(N\backslash\{\sigma(1)\}, (E - R)_{-\sigma(1)}, (c - r)_{-\sigma(1)})$$

$$\overset{(11)}{=} r_i + s_i^{\bar\sigma}(N\backslash\{\sigma(1)\}, E_{-\sigma(1)} - R + r_{\sigma(1)}, (c - r)_{-\sigma(1)}). \quad (12)$$

Since $|N\backslash\{\sigma(1)\}| = k$, we can apply the induction hypothesis to the bankruptcy situation in (12). For this we first calculate the minimal right $\bar r_\ell$ of a claimant $\ell \in N\backslash\{\sigma(1)\}$ in the bankruptcy situation $(N\backslash\{\sigma(1)\}, E_{-\sigma(1)} - R + r_{\sigma(1)}, (c - r)_{-\sigma(1)})$.

$$
\begin{aligned}
\bar r_\ell &= \max\{E_{-\sigma(1)} - R + r_{\sigma(1)} - \sum_{j \in N\backslash\{\ell, \sigma(1)\}} (c_j - r_j), 0\} \\
&= \max\{E_{-\sigma(1)} - r_\ell - \sum_{j \in N\backslash\{\ell, \sigma(1)\}} c_j, 0\} \\
&= \max\{E_{-\sigma(1)} - \sum_{j \in N\backslash\{\ell, \sigma(1)\}} c_j, r_\ell\} - r_\ell \\
&= \tilde r_\ell - r_\ell
\end{aligned}
$$

The last equality follows since $\tilde r_\ell \geq r_\ell$. It follows from (12) and the induction hypothesis that

$$
\begin{aligned}
& r_i + s_i^\sigma(N, E - R, c - r) \\
=\ & r_i + \bar r_i + s_i^{\bar\sigma}(N\backslash\{\sigma(1)\}, E_{-\sigma(1)} - R + r_{\sigma(1)} - \sum_{\ell \in N\backslash\{\sigma(1)\}} \bar r_\ell, (c - r)_{-\sigma(1)} - \bar r) \\
=\ & \tilde r_i + s_i^{\bar\sigma}(N\backslash\{\sigma(1)\}, E_{-\sigma(1)} - \tilde R, c_{-\sigma(1)} - \tilde r).
\end{aligned}
$$

$\square$

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