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Good and bad objects: cardinality-based rules*

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Abstract

We consider the problem of ranking sets of objects, the members of which are mutually compatible. Assuming that each object is either good or bad, we axiomatically characterize three cardinality-based rules which arise naturally in this dichotomous setting. They are what we call the symmetric difference rule, the lexicographic good-bad rule, and the lexicographic bad-good rule. Each of these rules induces a unique additive separable preference relation over the set of all groups of objects.

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1 Introduction

In this paper we consider the problem of ranking sets of objects the members of which are mutually compatible. Situations for which such a ranking can

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be of use include matching, choice of assemblies, election of new members of a committee, group identification and coalition formation. Very often in these situations the a priori information one has about the decision maker's opinion is simple: it takes the form of a partition on the set of all objects into "good" and "bad" items.

For example, if the set of objects is a set of individuals, the set of good objects can consist of all individuals who share a given property or who are qualified for a job (cf. Kasher and Rubinstein (1997), Samet and Schmeidler (2002)). Accordingly, in this case the set of bad objects will include all individuals who do not have this property or who are not qualified. Or, in a slightly different context, the set of good objects can contain all candidates who deserve to become new society members according to the opinion of some society founder (cf. Berga et al. (2002)), while the set of bad objects will consist of all candidates who do not deserve this grace. Finally, one can think of the assignment of software to workers when a given software program can be compatible or not with a worker's own machine (cf. Bogomolnaia and Moulin (2003)). Notice that the specification of good and bad objects in all these contexts implies homogeneity and full substitutability of the objects within a particular group.

Given such kind of dichotomous setting, the question we ask in the present paper is the following: how can one extend in a meaningful way this rudimentary information about one's opinion over the single objects to an ordering on the power set of the set of objects? We answer this question by presenting axiomatic characterizations of three cardinality-based rules. Each of them takes into account the number of good objects and the number of bad objects in the sets under comparison. The rules differ in the way in which these two numbers are combined. Under our first rule, the *symmetric difference* rule, a set C is considered to be better than another set D if and only if the cardinality of the symmetric difference between C and the set of all good objects is smaller than the cardinality of the symmetric difference between D and the set of all good objects. Our other two rules are lexicographic. According to the *good-bad* rule, the priority is given to the good objects in the corresponding sets, and if these sets have the same number of good objects, then the number of bad objects is decisive for the comparison. In the *bad-good* rule the priority is given to the bad objects and it is the "dual" version of the good-bad rule.

It is not difficult to see that each of these rules induces a unique additive separable preference relation over the set of all groups of objects, the set of all good objects being its top and the set of all bad objects being its bottom. Having this in mind, our results can be interpreted as characterizations of three different subclasses of the class of separable preferences, the latter being commonly used as a primitive in the analysis of voting situations (cf. Barberà et al. (1991)) and coalition formation games (cf. Burani and Zwicker (2003)). On the other hand, this paper contributes to the problem of ranking sets of objects in general. Especially, it can be seen as a translation of the purely cardinality-based freedom ranking of opportunity sets (cf. Pattanaik and Xu (1990)) into the context of ranking sets of objects the members of which are mutually compatible (cf. Barberà et al. (2003)).

The plan of the paper is as follows. Section 2 is devoted to our notation and basic definitions. In Section 3 we introduce and discuss the axioms used in this paper. Our characterization results are collected in Section 4. We conclude in Section 5.

2 Notation and definitions

Throughout the paper, let X be a nonempty finite set of objects. These objects may be for example candidates considered for membership in a club, possible coalition partners, bills considered for adoption by a legislature, etc. Following Fishburn (1992, p. 4), we assume that the information one has about the decision maker's opinion on the different objects consist only of a partition $\{G, B\}$ of X into good objects (collected in G) and bad objects (collected in $B = X \setminus G$). Furthermore, we assume that there is at least one good object and at least one bad object.

In what follows, the set of all subsets of X will be denoted by $\mathcal{X}$. The elements of $\mathcal{X}$ are the (alternative) groups of objects an agent may be confronted with. The question arises how this agent ranks different sets consisting on different number of good and bad items based on his partition $\{G, B\}$ of X. Consequently, the problem to be analyzed in this paper is how to establish a *quasi-ordering* (that is, a reflexive and transitive, but not necessarily complete, binary relation) $\succeq$ on $\mathcal{X}$. For all $C, D \in \mathcal{X}$, $C \succeq D$ is to be interpreted as "C is at least as good as D". The asymmetric and symmetric factors of $\succeq$ will be denoted by $\succ$ ("is better than") and $\sim$ ("is as good as"), respectively.

The first ranking we are interested in is the *symmetric difference* ranking $\succeq_\triangle$ defined by letting, for all $C, D \in \mathcal{X}$,

$$C \succeq_\triangle D \Leftrightarrow |C \triangle G| \leq |D \triangle G| \, .$$

According to this rule, if C and D contain the same number of good objects, C will be at least as good as D if and only if the number of bad objects in C is less than the corresponding number for D. Analogously, if C and D contain the same number of bad objects, C will be at least as good as D if and only if the number of good objects in C is greater than the corresponding number for D. Finally, if C and D have different numbers of good and bad objects, the rule adds the number of bad objects in the corresponding set to the number of good objects which are not in that set in order to produce a comparison.

In a voting context for example, one can think of a voter caring not only about the elected candidates he likes, but also paying attention to the elected candidates he does not like. Assuming that the set of candidates consists only of "candidates, who deserve to be elected" and "candidates, who do not deserve to be elected", the two sets being homogeneous, the symmetric difference rule presents a natural way of extracting voter's preferences over the set of different committees.

Next, we introduce a rule which gives priority to the good elements in the two sets under comparison.

The *good-bad* rule $\succeq_{GB}$ is defined by letting, for all $C, D \in \mathcal{X}$,

$$C \succeq_{GB} D \Leftrightarrow$$
$$(|C \cap G| > |D \cap G| \text{ or } [|C \cap G| = |D \cap G| \text{ and } |C \cap B| \leq |D \cap B|]).$$

This rule looks first at the good elements in the sets C and D. If the number for C is greater than the number for D, then C is declared as better than D. If these two numbers are equal, C is better than D if and only if the number of bad elements in C is smaller than the corresponding number for D.

Our last rule, the *bad-good* rule $\succeq_{BG}$ is "dual" to the good-bad rule, and it is defined by letting, for all $C, D \in \mathcal{X}$,

$$C \succeq_{BG} D \Leftrightarrow$$
$$(|C \cap B| < |D \cap B| \text{ or } [|C \cap B| = |D \cap B| \text{ and } |C \cap G| \geq |D \cap G|]).$$

As already mentioned in the Introduction and exemplified below, each of these rules induces a unique additive separable preference relation over the set of all groups of objects. Recall, that a binary relation $\succeq$ on $\mathcal{X}$ is *additive separable* if there exists a real-valued function $u : X \rightarrow \Re$ such that $C \succeq D \Leftrightarrow \sum_{w \in C} u(w) \geq \sum_{w \in D} u(w)$ for all $C, D \in \mathcal{X}$.

Example 1 *Let $X = \{x, y, z\}$ and $(G, B) = (\{x, y\}, \{z\})$. Then, the rankings over $\mathcal{X}$ induced by the three rules are the following:*
(i) $\{x, y\} \succ_{\triangle} \{x\} \sim_{\triangle} \{y\} \sim_{\triangle} \{x, y, z\} \succ_{\triangle} \emptyset \sim_{\triangle} \{x, z\} \sim_{\triangle} \{y, z\} \succ_{\triangle} \{z\}$*;*
(ii) $\{x, y\} \succ_{GB} \{x, y, z\} \succ_{GB} \{x\} \sim_{GB} \{y\} \succ_{GB} \{x, z\} \sim_{GB} \{y, z\} \succ_{GB} \emptyset \succ_{GB} \{z\}$*;*
(iii) $\{x, y\} \succ_{BG} \{x\} \sim_{BG} \{y\} \succ_{BG} \emptyset \succ_{BG} \{x, y, z\} \succ_{BG} \{x, z\} \sim_{BG} \{y, z\} \succ_{BG} \{z\}$*.*
To see that these binary relations are additive separable, take $u(x) = u(y) = 1$ and $u(z) = -1$ (for $\succeq_{\triangle}$), $u(x) = u(y) = 2$ and $u(z) = -1$ (for $\succeq_{GB}$), $u(x) = u(y) = 1$ and $u(z) = -3$ (for $\succeq_{BG}$).

3 Axioms

In this section we first present several properties a binary relation $\succeq$ on $\mathcal{X}$ may satisfy. Then we discuss their contents and some of their implications, paying attention to our specific setting.
We will say that $\succeq \in \mathcal{X} \times \mathcal{X}$ satisfies

- *Independence of good objects* (IGO) iff, for all $C, D \in \mathcal{X}$, all $x \in G \setminus C$, and all $y \in G \setminus D$,
  $C \sim D \Leftrightarrow C \cup \{x\} \sim D \cup \{y\}$.

- *Independence of bad objects* (IBO) iff, for all $C, D \in \mathcal{X}$, all $x \in B \setminus C$, and all $y \in B \setminus D$,
  $C \sim D \Leftrightarrow C \cup \{x\} \sim D \cup \{y\}$.

- *Perfect dichotomy* (PD) iff, for all $C, D \in \mathcal{X}$, all $x \in C \cap G$, and all $y \in B \setminus D$,
  $C \sim D \Rightarrow C \setminus \{x\} \sim D \cup \{y\}$.

- *Simple addition of good objects* (SAGO) iff, for all $C, D \in \mathcal{X}$, all $y \in G$,
  $\{x, y\} \succ \{x\}$.

- *Good influence* (GINF) iff, for all $C, D \in \mathcal{X}$, all $x \in D \cap G$,
  $C \succeq D \Rightarrow C \succ D \setminus \{x\}$.

- *Simple addition of bad objects* (SABO) iff, for all $C, D \in \mathcal{X}$, all $y \in B$,
  $\{x\} \succ \{x, y\}$.

- *Bad influence* (BINF) iff, for all $C, D \in \mathcal{X}$, all $x \in B \setminus D$,
  $C \succeq D \Rightarrow C \succ D \cup \{x\}$.

- *Strict independence* (SI) iff, for all $C, D \in \mathcal{X}$, and all $x \in X \setminus (C \cup D)$,
  $C \succ D \Rightarrow C \cup \{x\} \succ D \cup \{x\}$.

In independence of good objects we express the fact that all good objects are perfect *substitutes*. If a set C is as good as another set D, then adding or deleting a simple good object to or from each set preserves the indifference regardless of the identity of the other objects in these sets. Together with reflexivity, this axiom implies that every two singletons consisting of (possibly different) good objects will be indifferent (cf. Jones and Sugden (1982), Pattanaik and Xu (1990)). Note in addition that application of the independence of good objects will produce indifference between any two sets of objects having the same number of elements and containing good objects only. The interpretation of independence of bad objects is similar.

The next axiom, perfect dichotomy, states that "being good" and "not being bad" have the same meaning for the decision maker, i.e. the "marginal rate of substitution" between good and bad objects is 1. This axiom says that if two sets of objects are indifferent, adding a bad element to the one set does not change the things if it is combined with a subtraction of a good object from the other set.

Simple addition of good objects requires that a doubleton, consisting of any object x and a good object, is always better than the singleton consisting of x. Given that $\succeq$ is reflexive and satisfies independence of good objects, it is implied by the good influence axiom. The latter requires that if two sets of objects are indifferent, deleting a good object from one of them makes that set worse. In addition, the axiom says that the strict preference is preserved if one deletes a good element from the worse set. This axiom was introduced by Barberà et al. (1991) and, more explicitly, by Fishburn (1992). The interpretation of the next two axioms, simple addition of bad objects and bad influence, is analogous.

Finally, strict independence shares the same spirit as the independence axiom (labeled as monotonicity) by Kannai and Peleg (1984) and the independence requirement used by Pattanaik and Xu (1990). Essentially, this axiom says that adding the same element (good or bad) to two sets under comparison respects the strict preference between these sets. Note also that strict independence is stronger than Kannai and Peleg's corresponding axiom and weaker than Pattanaik and Xu's independence condition.

4 Characterization results

In this section we characterize the three cardinality-based rules introduced in Section 2.

Theorem 1 $\succeq$ *satisfies IGO, IBO, PD, GINF, and BINF if and only if* $\succeq = \succeq_{\triangle}$.

Proof. It is straightforward to verify that $\succeq_{\triangle}$ satisfies the corresponding axioms. Suppose that $\succeq$ satisfies IGO, IBO, PD, GINF, and BINF. It is sufficient to prove that, for all $C, D \in \mathcal{X}$,

$$|C \triangle G| = |D \triangle G| \Rightarrow C \sim D, \tag{1}$$

and

$$|C \triangle G| > |D \triangle G| \Rightarrow C \prec D. \tag{2}$$

For (1), let $C, D \in \mathcal{X}$ be such that $|C \triangle G| = |D \triangle G|$. We prove $C \sim D$ by induction on $k = |C \triangle G|$. If $k = 0$, then $C = G = D$, and $C \sim D$ follows from reflexivity of $\succeq$. If $k > 0$, assume that $C \sim D$ holds for $k-1$. Construct $C' \in \mathcal{X}$ as follows: if $G \subset C$, then take C' by deleting a bad object from C (which exists since $k > 0$), otherwise take C' by adding a good object to C. Construct D' analogously. Then $|C' \triangle G| = |D' \triangle G| = k - 1$ and, by induction, $C' \sim D'$. If $C' \subset C$ and $D' \subset D$, then $C \sim D$ follows by IBO. If $C' \supset C$ and $D' \supset D$, then $C \sim D$ follows by IGO. Otherwise, $C \sim D$ follows by PD.

For (2), let $C, D \in \mathcal{X}$ be such that $|C \triangle G| > |D \triangle G|$. Then there exist $\alpha \subseteq C \cap B$ and $\beta \subseteq G \setminus C$ such that $|((C \setminus \alpha) \cup \beta) \triangle G| = |D \triangle G|$. By (1), we then have $(C \setminus \alpha) \cup \beta \sim D$. Applying GINF the necessary times we obtain $(C \setminus \alpha) \prec D$. By the repeated use of BINF, we finally have $C \prec D$. ■

Theorem 2 $\succeq$ *satisfies IGO, IBO, SAGO, BINF, and SI if and only if* $\succeq=\succeq_{GB}$.

Proof. It is straightforward to verify that $\succeq_{GB}$ satisfies the corresponding axioms. Suppose that $\succeq$ satisfies IGO, IBO, SAGO, BINF, and SI. It is sufficient to prove that, for all $C, D \in \mathcal{X}$,

$$C \sim_{GB} D \Rightarrow C \sim D, \tag{3}$$

and

$$C \succ_{GB} D \Rightarrow C \succ D. \tag{4}$$

For (3), suppose that $C \sim_{GB} D$ for some $C, D \in \mathcal{X}$. By definition, this is equivalent to $|C \cap G| = |D \cap G|$ and $|C \cap B| = |D \cap B|$. If $C = D$, then $C \sim D$ follows from reflexivity of $\succeq$. Now suppose $C \neq D$. By the repeated use of IGO (starting from $\emptyset \sim \emptyset$), $C \cap G \sim D \cap G$. Repeating IBO the necessary times, we obtain $C \sim D$.

For (4), we can distinguish between two cases:

(i) $|C \cap G| > |D \cap G|$. Then there exists a $D' \in \mathcal{X}$ such that $|C \cap G| = |D' \cap G|$ and $D' = D \cup D''$ with $D'' \subset G$. By (3), we then have

$$C \sim D'. \tag{5}$$

Let $d \in D$ and $D'' = \{d''_1, \ldots, d''_t\}$. Then, by SAGO, $\{d, d''_1\} \succ \{d\}$. Hence, by the repeated use of SI, we have $D \cup \{d''_1\} \succ D$. By the same construction, we have $D \cup \{d''_1, d''_2\} \succ D \cup \{d''_1\}$, $D \cup \{d''_1, d''_2, d''_3\} \succ D \cup \{d''_1, d''_2\}$, ..., $D' = D \cup D'' \succ D \cup (D'' \setminus d''_t)$. By transitivity of $\succeq$, we have $D' \succ D$ and, together with (5), we obtain $C \succ D$.

(ii) $|C \cap G| = |D \cap G|$ and $|C \cap B| < |D \cap B|$. Let $C = \{c_1, \ldots, c_s\}$, $D = \{d_1, \ldots, d_t\}$, $C \cap G = \{c_1, \ldots, c_p\}$, $D \cap G = \{d_1, \ldots, d_p\}$, $p < s < t$. By the repeated use of IGO, $C \cap G \sim D \cap G$. Repeating IBO $s-p$ times, we obtain $C = (C \cap G) \cup \{c_{p+1}, \ldots, c_s\} \sim (D \cap G) \cup \{d_{p+1}, \ldots, d_s\}$. Finally, by the repeated use of BINF we have $C \succ (D \cap G) \cup \{d_{p+1}, \ldots, d_s, \ldots, d_t\}$, i.e. $C \succ D$. ■

Theorem 3 *$\succeq$ satisfies IGO, IBO, SABO, GINF, and SI if and only if $\succeq = \succeq_{BG}$.*

Proof. Similar to the proof of Theorem 2. ■

Note that in the three characterizations it is sufficient to assume that $\succeq$ is reflexive and transitive - completeness follows as a consequence of this assumption and the axioms.

5 Conclusion

In this paper, we have characterized three cardinality-based rules for ranking sets of objects, the members of which are mutually compatible, namely, the symmetric difference rule, the good-bad rule and the bad-good rule. All these rules share two important features. First, each of them induces a unique additive separable preference relation over the set of all groups of objects and, second, each rule makes use of both good and bad objects in the sets under comparison.

A related issue that may be addressed in the future concerns the introduction of a partition on each particular group of objects. For example, the simple information structure we have in this paper can be enriched by embedding a similarity relation (cf. Pattanaik and Xu (2000)) on the group of good and bad objects. This will allow one to discriminate among different subgroups of similar good (bad) objects and to consider extensions of the rules characterized here.

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