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Abstract

A path scheme for a simple game is composed of a *path*, i.e., a sequence of coalitions that is formed during the coalition formation process and a *scheme*, i.e., a payoff vector for each coalition in the path. A path scheme is called population monotonic if a player's payoff does not decrease as the path coalition grows. In this study, we focus on Shapley path schemes of simple games in which for every path coalition the Shapley value of the associated subgame provides the allocation at hand. We show that a simple game allows for population monotonic Shapley path schemes if and only if the game is balanced. Moreover, the Shapley path scheme of a specific path is population monotonic if and only if the first winning coalition that is formed along the path contains every minimal winning coalition. Extensions of these results to other probabilistic values are discussed.

Keywords: cooperative games; simple games; population monotonic path schemes; coalition formation; probabilistic values.
JEL classification: C71, D72

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1 Introduction

In many real life contexts, ranging from the formation of pre/post-electoral coalitions of parties to the formation of mergers and partnerships between firms, coalitions form through a sequence of binding bilateral agreements. From among the numerous examples of such coalition formation processes, we may single out the recent mergers between the banks and between the consultancy firms that are observed in many countries and the Oslo agreements between Israel and its neighbors. An important characteristic of such coalition formation processes is the effect of the sequence of agreements on the future potential agreements. For a coalition formed through bilateral agreements may grow larger because the synergy/commitment obtained by a coalition may create new agreement opportunities which are profitable both for the members of the coalition and the agent which will join the coalition. Hence, the determination of the sequences of binding bilateral agreements which will result in the exploitation of the greatest possible amount of synergy is of both theoretical and practical importance.

The coalition formation processes which end up with the formation of the grand coalition deserve particular interest. Because, first of all, in many situations (e.g., situations of increasing returns to size), the grand coalition is the unique efficient coalition structure. Secondly, the formation of the grand coalition among agents which have common properties (e.g., the formation of the grand coalition among leftist parties) has been the focal point of many branches of social sciences.

In this study, we will focus on the formation of the grand coalition through binding bilateral agreements in voting/government formation situations. We aim to address two important questions in this context.

(i) Which voting situations allow for the formation of the grand coalition through binding bilateral agreements?

(ii) In these situations, which agreement sequences must be followed to form the grand coalition?

We will address these questions by modeling voting situations by simple transferable utility cooperative games. In voting situations, the voters' incentive to form coalitions arises from their will to increase their power to affect the outcome of the voting process. Modelling of these situations as simple transferable utility games allows us to predict the voters' power to affect the result of voting by using appropriate values for transferable utility games. Many values have been offered for simple games as appropriate measures of voting power and the two most widely used ones are the Shapley-Shubik (1954) and Banzhaf (1965) power indices. If we assume that each voter's voting power is predicted by such an appropriate index, then the sequences of binding bilateral agreements which result in the formation of the grand coalition boils down to the notion of *population monotonic path schemes*. Postponing a precise definition to the next section, a population monotonic path scheme for a simple game is composed of a *path*, i.e., a sequence of coalitions that is formed during the coalition formation process and a *scheme*, i.e., a power index vector for each coalition in the path such that each player's index does not decrease as the path coalition grows. In this study, we focus on the Shapley-Shubik power index as an appropriate measure of voting power. Hence, the two questions that we address can be rephrased as

(i) Which simple games allow for population monotonic Shapley path schemes?

(ii) In these simple games, which paths have a population monotonic Shapley path scheme?

It turns out that existence of veto players, i.e., a subgroup of voters whose unanimous agreement is necessary to pass a decision, is required for the existence of population monotonic Shapley path schemes and vice versa. Moreover, a Shapley path scheme is population monotonic if and only if the first winning coalition that is formed along the path contains every minimal winning coalition of the game. We further show how to extend these results to probabilistic values, generalizations of the Shapley value introduced by Weber (1988).

The notion of population monotonic (Shapley) path schemes is introduced by Cruijssen, Borm, Fleuren and Hamers (2005). This study analyzes *insinking* (the antonym of outsourcing) situations in logistics and the transportation sector. In these sectors, shippers often outsource their transportation activities to a logistics service provider of their choice. Cruijssen et al. (2005) proposes an insinking procedure in which the logistics service provider initiates the shift of logistics activities instead of waiting for the shippers to outsource their activities. This procedure has the advantage that the logistics service provider can proactively select a group of shippers with a strong synergy potential (like ordering dynamics, locations etc.). Moreover, the gains resulting from this kind of cooperation can be allocated to the participating shippers in a fair and sustainable way by means of customized tariffs. Naturally, the attainment of gains in such situations requires the consent of the shippers. Hence, to obtain the greatest possible amount of gains, the service provider has to find an effective way of proposing offers to shippers through which it can acquire the involvement of each shipper. At this point, Cruijssen et al.(2005) proposes a sequence of binding bilateral agreements arguing that compared to the simultaneous comprehensive agreements, by following an appropriate sequence of binding bilateral agreements, the service provider can attract new customers to the project by using the level of synergy and commitment already attained in the sequence.

The notion of population monotonic path schemes (PMPS) shares the same spirit with the notion of population monotonic allocation schemes (PMAS) introduced and characterized by Sprumont (1990). A PMAS is an efficient allocation scheme such that the payoff of any player does not decrease as the coalition he belongs grows larger. Hence, if the gains associated with coalition formation are allocated with respect to a PMAS, the formation of the grand coalition is guaranteed. In fact, the existence of a PMPS is a weaker condition for a TU-game than the existence of a PMAS since every path scheme of a PMAS is population monotonic. Nevertheless, if the coalitions are formed through binding bilateral agreements, the grand coalition may still form when the gains associated with coalition formation are allocated with respect to an allocation scheme which has a PMPS.

Our study in particular provides an alternative prediction of what kind of coalitions form in voting situations which differs from the mainstream prediction of Riker (1962). Riker (1962) predicts that only minimal winning coalitions will form in equilibrium. This idea has been the conclusion of many studies in the general coalition formation literature based on the seminal noncooperative bargaining approach of Baron and Ferejohn (1989) and also the studies which analyze coalition formation in voting situations that are modeled by simple TU-games like Shenoy (1979). However, the emprical data on government/coalition formation shows that among all coalitions formed after the second world war in European democracies only a third of them is minimal winning (Laver and Schofield, 1990). Our current study shows that a wide spectrum of coalitions including the minimal winning ones can form as a result of binding bilateral agreements providing an alternative point of view for the analysis and the explanation of the data.

In a companion paper (Çiftçi and Dimitrov, 2006), we study the stability of coalition structures in hedonic coalition formation games in which players' preferences over coalitions are induced by the Shapley value of a simple game with veto control. It is shown that the coalition structures which contain the union of minimal winning coalitions (or one of its supersets) are strictly core stable in these hedonic coalition formation games. Hence, from this aspect, the notion of population monotonic Shapley path schemes can be regarded as a natural and decentralized coalition formation procedure which ensures the formation of a stable coalition structure in simple games with veto control.

The outline of the paper is as follows. In Section 2, we will begin by introducing the preliminaries about TU-games with particular attention to simple games and then we will continue with a brief review of the seminal notion of the Shapley value. Section 2 also formally introduces population monotonic path schemes. Section 3 presents the main results regarding the characterization of population monotonic Shapley path schemes of simple games. Section 4 discusses extensions of the results to other probabilistic values.

2 Preliminaries

Given a nonempty, finite set of players N, a transferable utility game (TU-game) with player set N is a function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. A coalition is a set of players $S \subset N$ and N is called the grand coalition. For any coalition $S \subset N$, $v(S)$ is called the worth of coalition S. We denote the set of TU-games with player set N by $\mathcal{G}^N$. A TU-game $v \in \mathcal{G}^N$ is monotonic if $v(S) \geq v(T)$ for every $S, T \in 2^N$ with $T \subset S$. A player $i \in N$ is a null player in v if $v(S \cup \{i\}) = v(S)$ for every $S \subset N \backslash \{i\}$. Given $v \in \mathcal{G}^N$ and $S \in 2^N$, the restriction of v to S (a subgame of v) is denoted by $v_{|S}$ and is defined by $v_{|S}(T) = v(T)$ for every $T \subset S$. The core of a TU-game $v \in \mathcal{G}^N$ is denoted by $C(v)$ and is defined as the set of efficient payoff vectors for which no coalition has an incentive to split off from the grand coalition, i.e., $C(v) = \{x \in R^N | \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N\}$. A TU-game which has a nonempty core is called a balanced game.

A function $F : \mathcal{G}^N \to \mathbb{R}^N$ is called a value. A value F is efficient if for all $v \in \mathcal{G}^N$, $\sum_{i \in N} F_i(v) = v(N)$. Let $\Pi(N)$ denote the set of permutations on the player set N. F is called anonymous if for all $v \in \mathcal{G}^N$ and for any permutation $\pi \in \Pi(N)$, $F(\pi(v)) = \pi(F(v))$. Here, with $v \in \mathcal{G}^N$ and $\pi \in \Pi(N)$, $\pi(v) \in \mathcal{G}^N$ is defined by $\pi(v)(S) = v(\pi(S))$ for each $S \in 2^N$. F is said to satisfy the null player property if for any $v \in \mathcal{G}^N$ and any null player $i \in N$ in v, $F_i(v) = 0$.

A TU-game $v \in \mathcal{G}^N$ is called simple if v is monotonic, $v(S) \in \{0, 1\}$ for every $S \in 2^N$ and $v(N) = 1$. We denote the set of simple TU-games with player set N by $\mathcal{S}^N$. Given $v \in \mathcal{S}^N$, a coalition $S \in 2^N$ is called a winning coalition if $v(S) = 1$ and is called a losing coalition if $v(S) = 0$. A winning coalition S is called minimal winning if there does not exist a coalition $T \subsetneq S$ which is winning. Every simple game v is characterized by its set of minimal winning coalitions, $MWC(v)$. A player $i \in N$ is a veto player in $v \in \mathcal{S}^N$ if $S \subset N$, $v(S) = 1$ implies that $i \in S$. The set of veto players of v is denoted by $veto(v)$. A simple game v is balanced if and only if $veto(v) \neq \emptyset$ (Curiel, 1997, Theorem 1.10.6).

Voting or decision making situations in committees like parliaments can easily be modeled into the framework of simple games by representing the coalitions which possesses the necessary power to pass a decision as the winning coalitions of the game. This model enables the employment of values for simple games to measure the parties' power to effect the outcome of the voting situations at hand. Many values have been offered for simple games and studied in the literature as appropriate measures of decisional power, i.e., as *power* *indices.* We will shortly review the Shapley-Shubik (1954) power index that arises from the Shapley value.

The Shapley value (Shapley, 1953) is one of the most important solution concepts in cooperative game theory and has been studied extensively. Given $v \in \mathcal{G}^N$, the Shapley value Φ assigns to player $i \in N$

$$\Phi_i(v) = \sum_{S \subset N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \left(v(S \cup \{i\}) - v(S) \right).$$

Shapley and Shubik (1954) proposed to use the Shapley value as a power index for voting situations in committees. For a simple game $v \in \mathcal{S}^N$ the Shapley-Shubik index for $i \in N$ boils down to

$$\Phi_i(v) = \sum_{\{S \subset N \setminus \{i\} | v(S)=0, v(S \cup \{i\})=1\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!}. \tag{1}$$

The value assigned to each voter can be interpreted by using the sequential probabilistic interpretation of the Shapley value which stems from a procedure to form the grand coalition (which is described also by Shapley (1953)) that yields the Shapley value of the game as an expected payoff of each player. In this procedure, the grand coalition N is formed by introducing the players one by one and each player is assigned the marginal contribution to the worth of the coalition formed when she joins the set of her predecessors. Hence, the value assigned by Shapley-Shubik index is the probability of turning the coalition of predecessors from losing to winning when the order of arrival of players is random and all orders are equally likely. For further discussion of the importance of the Shapley value as an estimator of political power and several examples of its applications, the reader is referred to Straffin (1994) and Winter (2002).

We are now ready to introduce the notion of path schemes for TU-games.

Definition 2.1 *Let $v \in \mathcal{G}^N$. A path consists of a sequence $\mathbb{S} = \{S_1, S_2, ...S_{|N|}\}$ of coalitions such that $|S_k| = k$ for all $k \in \{1, ..., |N|\}$ and $S_m \subset S_{m+1}$ for all $m \in \{1, ..., |N| - 1\}$. A path scheme $(\mathbb{S}, (x^S)_{S \in \mathbb{S}})$ for v consists of a path $\mathbb{S}$ and an allocation vector $x^S \in \mathbb{R}^S$ for every coalition $S \in \mathbb{S}$.*

A path scheme $(\mathbb{S}, (x^S)_{S \in \mathbb{S}})$ for $v \in \mathcal{G}^N$ is called *population monotonic* if it satisfies the following conditions:

- $x_i^S \geq v(\{i\})$ for all $S \in \mathbb{S}$ and $i \in S$. (individual rationality)

- $x_i^S \geq x_i^T$ for every $S, T \in \mathbb{S}$ such that $T \subset S$ and $i \in T$. (monotonicity)

Naturally, every value for TU-games defines a path scheme where the allocation for every path coalition is obtained by applying the value to the restriction of the game to the path coalition. A path scheme in which the Shapley value is used as allocation vector is called a *Shapley path scheme*.

We will illustrate the notion of Shapley path schemes and their properties in the following example.

Example 2.1 Let $N = \{1, 2, 3\}$ and $v \in \mathcal{S}^N$ be such that $MWC(v) = \{\{1, 2\}, \{2, 3\}\}$. The Shapley value of v and its subgames are provided in Table 1 below.

Coalition	Player 1	Player 2	Player 3
{1}	0	-	-
{2}	-	0	-
{3}	-	-	0
{1,2}	$\frac{1}{2}$	$\frac{1}{2}$	-
{1,3}	0	-	0
{2,3}	-	$\frac{1}{2}$	$\frac{1}{2}$
N	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$

Table 1: The Shapley value of v and its subgames

It can easily be observed that this game has exactly two population monotonic Shapley path schemes on paths $\{\{1\}, \{1, 3\}, N\}$ and $\{\{3\}, \{1, 3\}, N\}$. ⋄

3 Population Monotonic Shapley Path Schemes

We will begin with presenting a preliminary result which is useful in understanding the structure of population monotonic Shapley path schemes of simple games.

Lemma 3.1 *Given a simple game $v \in \mathcal{S}^N$, let $\mathbb{S} = \{S_1, S_2, ..., S_{|N|}\}$ be a path of coalitions such that $S_m = \{i_1, ..., i_m\}$ for every $m \in \{1, ..., |N|\}$. Assume that the first winning coalition along the path $\mathbb{S}$ is S_k. If the Shapley path scheme of $\mathbb{S}$ is population monotonic, then the following must hold:*

(R1) $\Phi_{i_m}(v_{|S_p}) = 0$, for all $m \in \{k + 1, ..., |N|\}$ and for all $p \in \{m, ..., |N|\}$.

(R2) $\Phi_i(v_{|S_k}) = \Phi_i(v_{|S_p})$, for all $p \in \{k + 1, ..., |N|\}$ and for all $i \in S_k$.

(R1) and *(R2)* are direct consequences of the efficiency of the Shapley value and the monotonicity property of the Shapley path scheme. Efficiency of the Shapley value implies that the Shapley-Shubik power indices of the members of any winning coalition along a path must add up to 1, the worth of the winning coalitions. Hence, once a winning coalition is formed along a path, the population monotonicity requires two conditions to be met by the Shapley path scheme. Firstly, the total decision power of one must continue to be shared among the members of the first winning coalition, i.e., all the subsequent players must have a zero Shapley-Shubik index at every coalition that they are involved along the path (R1). Secondly, the composition of the power among the members of the first winning coalition must stay constant along the rest of the path since an increase in one member's power index directly implies a decrease in one other member's index (R2).

We are now ready to provide a characterization of the family of simple games which allow population monotonic Shapley path schemes.

Theorem 3.1 *Let $v \in \mathcal{S}^N$. Then v has a population monotonic Shapley path scheme if and only if v is balanced.*

Proof. Let $v \in \mathcal{S}^N$ be a simple game which has a population monotonic Shapley path scheme. Let $(\mathbb{S}, (\Phi(v_{|S}))_{S\in\mathbb{S}})$ be a population monotonic Shapley path scheme for v such that $\mathbb{S} = \{S_1, S_2, ..., S_{|N|}\}$ and $S_m = \{i_1, ..., i_m\}$ for every $m \in \{1, ..., |N|\}$. Assume that the first winning coalition along the path $\mathbb{S}$ is S_k; i.e., $v(S_1) = ... = v(S_{k-1}) = 0$ and $v(S_k) = 1$. Then, obviously $i_k \in veto(v_{|S_k})$. Now, consider $v_{|S_k}$ and $v_{|S_{k+1}}$. Since v is monotonic, for every winning coalition $S \subset S_k$, $S \cup \{i_{k+1}\}$ is also winning. Then, if for a losing coalition $S \subset S_k$, $S \cup \{i_{k+1}\}$ is winning, it can easily be observed that $\Phi_{i_{k+1}}(v_{|S_{k+1}})$ is strictly positive which contradicts with (R1). Hence, i_k is also a veto player in $v_{|S_{k+1}}$. Applying the same reasoning recursively one finds that i_k is a veto player of v, i.e., v is balanced.

Now, assume that v is balanced. Then, $veto(v) \neq \emptyset$. Let $i \in veto(v)$. We will show that the Shapley path scheme of any path $\mathbb{S} = \{S_1, S_2, ..., S_{|N|}\}$ with $S_{|N|-1} = N\backslash\{i\}$ is a population monotonic Shapley path scheme. We know that $S_{|N|-1} = N\backslash\{i\}$ is a losing coalition. Then $v_{N\backslash\{i\}}$ is a null game and hence $\Phi_j(v_{|S_t}) = 0$ for all $t \in \{1, ..., |N| - 1\}$ and $j \in S_t$. Also, since v is monotonic, $\Phi_j(v) \geq 0$ for all $j \in N$ which implies that the Shapley path scheme of $\mathbb{S}$ is population monotonic. □

Theorem 3.1 reveals that, in the class of simple games, the existence of veto players is a must for the existence of population monotonic Shapley path schemes and vice versa. We can interpret this result as follows. When a winning coalition is formed through a sequence of binding bilateral agreements, we know that the restriction of the TU-game to this coalition has veto players, that is, in this winning coalition, there is a subgroup of agents whose unanimous agreement/involvement is necessary to pass a decision. We also know that the formation of the grand coalition starting from this winning coalition via binding bilateral agreements requires the remaining players to be null players. But, this in turn implies that the veto players of the winning coalition are in fact the veto players of the whole game, i.e., the game is balanced.

The interpretation of Theorem 3.1 also brings out a useful hint for the characterization of the paths of balanced simple games which has population monotonic Shapley path schemes. If the veto players of the first winning coalition along a path which has a population monotonic Shapley path scheme are in fact the veto players of the whole game and if all the subsequent players have to be null players, then a path can have a population monotonic Shapley path scheme only when the first winning coalition along this path includes all the minimum winning coalitions of the game.

We will show in Theorem 3.2 that the hint stated above, i.e., the requirement that the first winning coalition along a path has to include all the minimum winning coalitions of the game is both necessary and sufficient for a path to have a population monotonic Shapley path scheme.

Theorem 3.2 *Let $v \in \mathcal{S}^N$ be a balanced simple game. A path $\mathbb{S}$ has a population monotonic Shapley path scheme if and only if the first winning coalition along $\mathbb{S}$ contains every minimal winning coalition of v.*

Proof. Let S^M denote the union of minimal winning coalitions of v. Because v is balanced we have that $veto(v) \neq \emptyset$.

We will first show the *only if* part. Let $\mathbb{S}$ be a path of coalitions with $S_m = \{i_1, ..., i_m\}$ for every $m \in \{1, ..., |N|\}$. Assume that $\mathbb{S}$ has a population monotonic Shapley path scheme and the first winning coalition along the path $\mathbb{S}$ is S_k ($k \in \{1, ..., |N|\}$). If $k = |N|$, obviously $S_N = N$ contains every minimal winning coalition. So assume that $k \in \{1, ..., |N| - 1\}$ and suppose $S_k \not\supset S^M$. Then there exists $m \in \{k + 1, ..., |N|\}$ such that S_m is the first path coalition that contains S^M. Now, on the one hand $(R1)$ implies that $\Phi_{i_m}(v_{|S_m}) = 0$. On the other hand, i_m is a member of a minimal winning coalition S since S_m is the first path coalition that contains S^M. Then, $S \backslash \{i_m\}$ is a losing coalition, i.e., $v(S) = 1$ and $v(S \backslash \{i_m\}) = 0$ implying that $\Phi_{i_m}(v_{|S_m}) > 0$, a contradiction. Thus, $S_k \supset S^M$.

We will now prove the *if* part. Let $\mathbb{S}$ be a path of coalitions with $S_m = \{i_1, ..., i_m\}$ for every $m \in \{1, ..., |N|\}$. Assume that the first winning coalition along the path $\mathbb{S}$ is S_k

($k \in \{1, ..., |N|\}$) and $S^M \subset S_k$. Now, $\Phi_j(v_{|S_t}) = 0$ for all $t \in \{1, ..., k-1\}$ and $j \in S_t$ since S_{k-1} is a losing coalition. Also, $\Phi_i(v_{|S_k}) \geq 0$ for all $i \in S_k$ since v is monotonic. Because each player i_m ($m \in \{k+1, ..., |N|\}$) is a null player since $S^M \subset S_k$, we know that $\Phi_{i_m}(v_{|S_p}) = 0$ for all $m \in \{k+1, ..., |N|\}$ and for all $p \in \{m, ..., |N|\}$. We will show that $\Phi_i(v_{|S_k}) = \Phi_i(v_{|S_m})$ for all $i \in S_k$ and for all $m \in \{k+1, ..., |N|\}$ to conclude that the Shapley path scheme of the path $\mathbb{S}$ is population monotonic.

Pick $i \in S_k$ and $t \in \{k, ..., |N|-1\}$. We will show that $\Phi_i(v_{|S_t}) = \Phi_i(v_{|S_{t+1}})$.

Let $\mathcal{A}$ denote the collection $\{S \subset (S_{t+1} \backslash \{i\}) | v(S) = 0, v(S \cup \{i\}) = 1\}$, let $\mathcal{B}$ denote the set $\{S \in \mathcal{A} | i_{t+1} \notin S\}$ and let $\mathcal{C}$ denote the set $\{S \in \mathcal{A} | i_{t+1} \in S\}$. Obviously $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$. Moreover, notice that $\mathcal{B} = \{S \subset (S_t \backslash \{i\}) | v(S) = 0, v(S \cup \{i\}) = 1\}$ and $\mathcal{C} = \{S \cup \{i_{t+1}\} | S \in \mathcal{B}\}$. Then,

$$\begin{aligned}
\Phi_i(v_{|S_{t+1}}) &= \sum_{S \in \mathcal{A}} \frac{|S|!(t-|S|)!}{(t+1)!} \\
&= \sum_{S \in \mathcal{B} \cup \mathcal{C}} \frac{|\, S \,|!(t-|S|)!}{(t+1)!} \\
&= \sum_{S \in \mathcal{B}} \left[\left(\frac{|S|!(t-|S|)!}{(t+1)!}\right) + \left(\frac{(|S|+1)!(t-|S|-1)!}{(t+1)!}\right) \right] \\
&= \sum_{S \in \mathcal{B}} \frac{|S|!(t-|S|-1)!}{t!} = \Phi_i(v_{|S_t})
\end{aligned}$$

where the first and the last equalities follow from (1). Hence, we can conclude that the Shapley path scheme of the path $\mathbb{S}$ is population monotonic. □

In the light of Theorem 3.2, we can answer one other important question in this context: For which simple games all Shapley path schemes are population monotonic?

Theorem 3.3 *Let $v \in \mathcal{S}^N$ be a simple game. All Shapley path schemes of v are population monotonic if and only if the set of veto players of v is a winning coalition.*

Proof. If $veto(v)$ is a winning coalition, it is the unique minimum winning coalition and clearly every path scheme's first winning coalition contains $S^M = veto(v)$. Hence, by Theorem 3.2 every Shapley path scheme is population monotonic. So, what remains to prove is the *only if* part.

Assume that all Shapley path schemes of v are population monotonic but $veto(v)$ is losing. There exists a minimum winning coalition $S = \{i_1, ..., i_m\}$ with $m \in \{1, ..., |N|-1\}$.

We know that $\Phi_i(v_{|S}) = \frac{1}{m}$ for every $i \in S$ since S is a minimal winning coalition. Pick a path of coalitions $\mathbb{S} = \{S_1, S_2, ..., S_{|N|}\}$ with $S_m = S$. The Shapley path scheme of $\mathbb{S}$ is population monotonic by assumption. Consequently, $\Phi_i(v) = \frac{1}{m}$ for every $i \in S$. Observe that there exists $i^* \in S$ such that $i^* \notin veto(v)$ since S is a minimal winning coalition and $veto(v)$ is losing. Then, there exists another minimal winning coalition $T \subsetneq N$ such that $i^* \notin T$. Pick a path of coalitions $\mathbb{S}' = \{S_1', S_2', ..., S_{|N|}'\}$ with $S_{|T|}' = T$. Now, the Shapley path scheme of $\mathbb{S}'$ is also population monotonic by assumption. Then, $(R1)$ implies that $\Phi_{i^*}(v) = 0$ since $i^* \notin T$, a contradiction with $\Phi_{i^*}(v) = \frac{1}{m}$ as derived earlier. □

4 Extensions of the Results to Probabilistic Values

Probabilistic values, introduced and characterized by Weber (1988), are generalizations of the Shapley value for finite TU-games. These values keep one essential feature of the Shapley value, they assign each player an average of his marginal contributions. They, however, fail to satisfy either the efficiency or anonymity property. In fact, the Shapley value is the unique probabilistic value satisfying both anonymity and efficiency. Probabilistic values can be classified into two groups: Quasi-values which are efficient probabilistic values and Semi-values, the probabilistic values which satisfy anonymity (see Weber (1988)). We refer to Monderer and Samet (2002) for a detailed discussion of probabilistic values.

Probabilistic values are formally defined as follows. Given N and $i \in N$, let P_N^i denote the set of probability distributions on $2^{N \setminus \{i\}}$, the family of coalitions not containing i. A value F (defined on $\mathcal{G}^N$) is called a *probabilistic value* (Weber, 1988) if for every $v \in \mathcal{G}^N$ and $i \in N$

$$F_i(v) = \sum_{T \subset N \setminus \{i\}} p^i(T) \left(v(T \cup \{i\}) - v(T) \right), \tag{2}$$

for some $p^i \in P_N^i$ for all $i \in N$. Here $p^i \in P_N^i$ can be interpreted as the player's subjective evaluation of the probability of joining different coalitions. For example, the probabilistic value which is defined by $p^i(T) = \frac{1}{|N|} \binom{|N|-1}{|T|}^{-1}$ for all $i \in N$ is the Shapley value.

In the following two subsections we will discuss the extensions of the results obtained for the Shapley value on quasi-values and on semi-values, respectively.

4.1 Population Monotonic Path Schemes of Quasi-values

Let $\mathcal{P}(\Pi(N))$ denote the set of probability distributions on the set of permutations of the player set N. Given $S \subset N$ and $i \in S$, we will denote by $\Pi^{S,i}(N)$ the set

$$\{\tau \in \Pi(N) | \tau(j) < \tau(i) \text{ if and only if } j \in S\}.$$

The following characterization of quasi-values is provided by Weber (1988).

Theorem 4.1 *(Weber (1988)) Let F be a probabilistic value as given in (2) defined by $p = \{p^i\}_{i \in N}$ with $p^i \in P_N^i$ for every $i \in N$. Then F is a quasi-value if and only if there exists $b \in \mathcal{P}(\Pi(N))$ such that*

$$p^i(S) = \sum_{\tau \in \Pi^{S,i}(N)} b(\tau) \tag{3}$$

for every $i \in N$ and $S \in 2^{N \setminus \{i\}}$.

Observe that probabilistic values are originally defined for a fixed player set. However, our analysis requires the values to be defined on every subset of the player set under consideration. Because, for every simple game, we want to be able to compare the payoffs assigned by a value to the players at every subgame of the game.We now extend quasi-values in such a way that the players' subjective evaluation of the probability of joining different coalitions will be consistent in the sense defined below. For this aim we will define the *restrictions* of a probabilistic value to subgames.

Let $F : \mathcal{G}^N \to \mathbb{R}^N$ be a probabilistic value defined by $\{p_N^i\}_{i \in N}$ where $p_N^i \in P_N^i$ for every $i \in N$. For each $S \subset N$, the *restriction* of F to $\mathcal{G}^S$ is denoted by F_S and for each player $i \in S$, his restricted evaluations $p_S^i \in P_S^i$ are constructed by using the following consistency condition.

$$p_S^i(T) = \sum_{T' \subset N \setminus S} p_N^i(T \cup T'), \tag{4}$$

for all $T \subset S \setminus \{i\}$.

We first illustrate the notion of restrictions of a quasi-value in the following example.

Example 4.1 Let F be a probabilistic value on $N = \{1, 2, 3\}$. Assume that F is defined by the following subjective evaluations of players.

$$p_N^1(\{2,3\}) = \tfrac{5}{16},\ p_N^1(\{2\}) = \tfrac{1}{16},\ p_N^1(\{3\}) = \tfrac{4}{16} \text{ and } p_N^1(\emptyset) = \tfrac{6}{16}.$$
$$p_N^2(\{1,3\}) = \tfrac{8}{16},\ p_N^2(\{1\}) = \tfrac{2}{16},\ p_N^2(\{3\}) = \tfrac{4}{16} \text{ and } p_N^2(\emptyset) = \tfrac{2}{16}.$$
$$p_N^3(\{1,2\}) = \tfrac{3}{16},\ p_N^3(\{1\}) = \tfrac{4}{16},\ p_N^3(\{2\}) = \tfrac{1}{16} \text{ and } p_N^3(\emptyset) = \tfrac{8}{16}.$$

F satisfies (3) by taking the following probability distribution on the set of permutations on the player set:

$$b(123) = \tfrac{2}{16}, b(132) = \tfrac{4}{16}, b(213) = \tfrac{1}{16}, b(231) = \tfrac{1}{16}, b(312) = \tfrac{4}{16}, \text{ and } b(321) = \tfrac{4}{16}.$$

Hence F is a quasi-value.

Now consider $S = \{1, 2\}$. According to (4), the restriction F_S is defined by:

$$\begin{array}{l} p_S^1(\{2\}) = \tfrac{3}{8} = p_N^1(\{2\}) + p_N^1(\{2,3\}) \text{ and } p_S^1(\emptyset) = \tfrac{5}{8} = p_N^1(\emptyset) + p_N^1(\{3\}). \\ p_S^2(\{1\}) = \tfrac{5}{8} = p_N^2(\{1\}) + p_N^2(\{1,3\}) \text{ and } p_S^2(\emptyset) = \tfrac{3}{8} = p_N^2(\emptyset) + p_N^2(\{3\}). \end{array}$$

Notice that F_S can be described via (3) by taking:

$$b(12) = \tfrac{5}{8} \text{ and } b(21) = \tfrac{3}{8}.$$

So also F_S is a quasi-value on $\mathcal{G}^S$. $\diamond$

In the previous example, we have shown that the specific restriction under consideration is again a quasi-value. Indeed, every restriction of a quasi-value is a quasi-value for the corresponding subgame as shown in the following proposition.

Proposition 4.1 *Let F be a quasi-value defined by $\{p_N^i\}_{i \in N}$ where $p_N^i \in P_N^i$ for every $i \in N$. Then, F_S is a quasi-value for every $S \subset N$, $S \neq \emptyset$.*

Proof. By Theorem 4.1 there exists $b \in \mathcal{P}(\Pi(N))$ such that $p_N^i(T) = \sum_{\tau \in \Pi^{T,i}(N)} b(\tau)$. Take $S \subset N, S \neq \emptyset$. Given $\tau \in \Pi(N)$, $\tau_{|S}$ denotes the restriction of τ to S, i.e., $\tau_{|S} = \pi$ for some $\pi \in \Pi(S)$ with $\pi(i) < \pi(j)$ if and only if $\tau(i) < \tau(j)$, for all $i, j \in S$. We can induce a probability distibution c on $\Pi(S)$ from b as follows.

$$c(\pi) = \sum_{\tau \in \Pi(N): \tau_{|S} = \pi} b(\tau), \text{ for all } \pi \in \Pi(S). \tag{5}$$

Let F_S be defined by $\{p_S^i\}_{i \in S}$ as determined by (4). Pick $i \in S$ and $T \subset S \backslash \{i\}$. Obviously,

$$\bigcup_{T' \subset N \backslash S} \Pi^{(T \cup T'),i}(N) = \bigcup_{\pi \in \Pi^{T,i}(S)} \{\tau \in \Pi(N) | \tau_{|S} = \pi\} \tag{6}$$

Notice that

$$\Pi^{(T \cup T'),i}(N) \cap \Pi^{(T \cup T''),i}(N) = \emptyset \text{ for every } T', T'' \subset N \backslash S \text{ with } T' \neq T''$$

and

$$\{\tau \in \Pi(N) | \tau_{|S} = \pi\} \cap \{\tau \in \Pi(N) | \tau_{|S} = \pi'\} = \emptyset \text{ for every } \pi, \pi' \in \Pi^{T,i}(S) \text{ with } \pi \neq \pi'.$$

Then,

$$\begin{aligned} p_S^i(T) &= \sum_{T' \subset N \setminus S} p_N^i(T \cup T') \\ &= \sum_{T' \subset N \setminus S} \sum_{\tau \in \Pi^{(T \cup T'),i}(N)} b(\tau) \\ &= \sum_{\pi \in \Pi^{T,i}(S)} \sum_{\tau \in \Pi(N): \tau_{|S} = \pi} b(\tau) \\ &= \sum_{\pi \in \Pi^{T,i}(S)} c(\pi) \end{aligned}$$

where the first equality follows from (4) and the last but one equality follows from (6) and the remarks below it. Then, Theorem 4.1 implies that F_S is a quasi-value on $\mathcal{G}^S$. □

Having defined the restrictions of a quasi-value, we can now illustrate the path schemes associated with these values in the following example.

Example 4.2 Consider the quasi-value F defined in Example 4.1 and let $v \in \mathcal{S}^N$ with $N = \{1, 2, 3\}$ be defined by $MWC(v) = \{\{1, 2\}, \{2, 3\}\}$. From Table 2 it can easily be observed that this balanced game has two population monotonic F-path schemes related to the paths {{1},{1,3},N} and {{3},{1,3},N}. ◇

Coalition	Player 1	Player 2	Player 3
{1}	0	-	-
{2}	-	0	-
{3}	-	-	0
{1,2}	$\frac{3}{8}$	$\frac{5}{8}$	-
{1,3}	0	-	0
{2,3}	-	$\frac{6}{8}$	$\frac{2}{8}$
N	$\frac{1}{16}$	$\frac{14}{16}$	$\frac{1}{16}$

Table 2: The restrictions of F for v and its subgames in Example 4.2

The following theorem states that the results for population monotonic Shapley path schemes in fact can be extended to quasi-values which are defined by strictly positive subjective evaluations of joining different coalitions for each player.

Theorem 4.2 *Let $F : \mathcal{G}^N \to \mathbb{R}^N$ be a quasi-value defined by $\{p^i_N\}_{i \in N}$ with $p^i_N > 0$ for all $i \in N$. Then*

(1) A simple game $v \in \mathcal{S}^N$ has a population monotonic F-path scheme if and only if v is balanced.

(2) Let v be balanced. Then a path $\mathbb{S}$ of v has a population monotonic F-path scheme if and only if the first winning coalition along $\mathbb{S}$ contains every minimal winning coalition of v.

(3) All F-path schemes of v are population monotonic if and only if the set of veto players of v is a winning coalition.

The proof of Theorem 4.1 is similar to the proofs of Theorems 3.1, 3.2 and 3.3, respectively and is therefore omitted.

It is important at this point to observe that if for a quasi-value F, $p^i_N(S) = 0$ for some $S \subset N$, and $i \in N \backslash S$, then an unbalanced simple game may have population monotonic F-path schemes. This is illustrated in Example 4.3.

Example 4.3 Let $N = \{1, 2, 3\}$. Let F be the quasi-value determined by

$$p^1_N(S) = \frac{1}{4} \text{ for all } S \subset N\backslash\{1\}; p^2_N(S) = \frac{1}{4} \text{ for all } S \subset N\backslash\{2\} \text{ and}$$
$$p^3_N(\{1, 2\}) = p^3_N(\emptyset) = \frac{1}{2}, p^3_N(\{1\}) = p^3_N(\{2\}) = 0.$$

Consider $v \in \mathcal{S}^N$ defined by $MWC(v) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. Clearly $veto(v) = \emptyset$. Then v has population monotonic F-path schemes on the paths $\{\{1\}, \{1, 2\}, N\}$ and $\{\{2\}, \{1, 2\}, N\}$ as can be seen in Table 3. ◇

Coalition	Player 1	Player 2	Player 3
{1}	0	-	-
{2}	-	0	-
{3}	-	-	0
{1,2}	$\frac{1}{2}$	$\frac{1}{2}$	-
{1,3}	$\frac{1}{2}$	-	$\frac{1}{2}$
{2,3}	-	$\frac{1}{2}$	$\frac{1}{2}$
N	$\frac{1}{2}$	$\frac{1}{2}$	0

Table 3: The restrictions of F for v and its subgames in Example 4.3

4.2 Population Monotonic Path Schemes of Semi-values

In this section, we will focus on one particular, well-known semi-value, the Banzhaf value (Banzhaf, 1965). Given $v \in \mathcal{G}^N$, the Banzhaf value β assigns to player $i \in N$

$$\beta_i(v) = \sum_{S \subset N \setminus \{i\}} \frac{1}{2^{|N|-1}} \left(v(S \cup \{i\}) - v(S)\right).$$

It can easily be observed that the Banzhaf value is defined for every finite player set, and in particular also for all subgames of a specific game. In fact, every semi-value is defined for every finite player set, and hence for all subgames of a specific game. Moreover, the restriction of a semi-value obtained by using the consistency condition (4) boils down to the definition of the same semi-value for the corresponding subgame. This can be readily verified from the characterization of semi-values on TU-games with finite support provided by Dubey et al. (1981).

For population monotonic Banzhaf path schemes the situation essentially differs from the population monotonic Shapley path schemes. This is illustrated in examples 4.4 and 4.5.

Example 4.4 Let $N = \{1, 2, 3\}$ and $v \in \mathcal{S}^N$ be defined by $MWC(v) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. The Banzhaf value of v and its subgames are provided in the table below.

Coalition	Player 1	Player 2	Player 3
{1}	0	-	-
{2}	-	0	-
{3}	-	-	0
{1,2}	$\frac{1}{2}$	$\frac{1}{2}$	-
{1,3}	$\frac{1}{2}$	-	$\frac{1}{2}$
{2,3}	-	$\frac{1}{2}$	$\frac{1}{2}$
N	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Table 4: The Banzhaf value of v and its subgames in Example 4.4

Notice that v is not balanced since $veto(v) = \emptyset$ but that every Banzhaf path scheme of v is population monotonic. $\diamond$

Example 4.5 Let $N = \{1, 2, 3, 4\}$ and consider the simple game $v \in \mathcal{S}^N$ defined by $MWC(v) = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}\}$. Clearly, v is balanced. The Banzhaf value of

v and its subgames are provided in Table 5. The Banzhaf values of the subgames corresponding to losing coalitions are omitted. Then, every Banzhaf path scheme is population

Coalition	Player 1	Player 2	Player 3	Player 4
{1,2,3}	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	-
{1,2,4}	$\frac{1}{4}$	$\frac{1}{4}$	-	$\frac{1}{4}$
{1,3,4}	$\frac{1}{4}$	$\frac{1}{4}$	-	$\frac{1}{4}$
N	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

Table 5: The Banzhaf value of v and its subgames in Example 4.5

monotonic although the set of veto players of v is a losing coalition. Secondly, there are path schemes of v, like the one related to path $\{\{1\},\{1,2\},\{1,2,3\},\{1,2,3,4\}\}$, which are population monotonic but the first winning coalition along these paths does not contain the union of minimal winning coalitions v. $\diamond$

The results for population monotonic Shapley path schemes can be extended only partly to the Banzhaf value. This is reflected in Theorem 4.3.

Theorem 4.3 *Let $v \in \mathcal{S}^N$ be balanced.*

(1) If the first winning coalition along a path $\mathbb{S}$ contains every minimal winning coalition of v, then the path scheme of $\mathbb{S}$ has a population monotonic Banzhaf-path scheme.

(2) If the set of veto players of v is a winning coalition, then all Banzhaf-path schemes of v are population monotonic.

The proof of Theorem 4.3 is similar to the corresponding parts of the proofs of Theorems 3.2 and 3.3, respectively and is therefore omitted.

By making use of the characterization of semi-values provided by Dubey et al. (1981), one can show that Theorem 4.3 can be extended to every semi-value.

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