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Abstract

We study the optimal consumption and portfolio choice problem over an individual's life-cycle taking into account annuity risk at retirement. Optimally, the investor allocates wealth at retirement to nominal, inflation-linked, and variable annuities and conditions this choice on the state of the economy. We also consider the case in which there are, either for behavioral or institutional reasons, limitations in the types of annuities that are available at retirement. Subsequently, we determine how the investor optimally anticipates annuitization before retirement. We find that i) using information on term structure variables and risk premia significantly improves the optimal annuity choice, ii) restricting the annuity menu to nominal or inflation-linked annuities is costly for both conservative and more aggressive investors, and iii) adjustments in the optimal investment strategy before retirement induced by the annuity demand due to inflation risk and time-varying risk premia are economically significant. This holds as well for sub-optimal annuity choices. The adjustment to hedge real interest rate risk is negligible. We estimate that the welfare costs of not taking these three factors into account at retirement are 9% for an individual with an average risk aversion ($\gamma = 5$). Not hedging annuity risk before retirement causes an additional welfare costs between 1% and 13%, depending on the annuitization strategy implemented at retirement.

Keywords: optimal life-cycle portfolio choice, annuity risk
JEL classification: D91, G0, G11, G23

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1 Introduction

In a seminal paper on annuity choice, Yaari (1965) shows that, in the absence of a bequest motive, it is optimal to transfer retirement wealth fully into annuities at retirement. Davidoff, Brown, and Diamond (2005) extend this result and show that full annuitization is optimal if every asset is available in annuitized form. Annuities allow investors to shift wealth from states of the world in which no utility is derived to states in which the investor is still alive. Even if annuity markets are incomplete, it is often optimal to annuitize a sizeable part of retirement benefits, see Davidoff et al. (2005). A reasonably complete annuity menu comprises nominal, inflation-linked, and variable annuities linked to a broad equity index. Using these life-contingent instruments, investors are able to manage exposures to interest rate, inflation, and equity risk, including the corresponding risk premia, at retirement. Annuitization of retirement wealth exposes the investor to annuity risk: the utility derived from the annuity payoffs may disappoint if financial market conditions turn out to be unfavorable at retirement.[1] This paper addresses the welfare gains of optimal annuity choice at retirement. Moreover, we explore how investors optimally anticipate annuity risk, for both optimal and sub-optimal annuity choices, before retirement.

Annuity risk is important because insurance companies generally do not repurchase annuity products. Irreversibility is a direct consequence of the adverse selection problem in annuity markets as annuitants generally possess better information concerning their health status, in particular when they are in bad health. It is therefore hard, if not impossible, to dynamically rebalance the annuity portfolio in response to changes in financial markets after retirement. Due to the illiquidity of annuity products after retirement, the investor can essentially manage annuity risk along two lines.[2] First, the optimal annuity portfolio at retirement can incorporate financial market conditions at this date. Second, by trading equity and bonds in the period before retirement, the investor can construct hedging portfolios which pay off in unfavorable states of the economy at retirement. Obviously, the optimal investment strategy in the period before retirement depends on the annuitization strategy followed. Even if, either for behavioral or institutional reasons, investors restrict attention to only part of the annuity menu mentioned above, see for instance Diamond (1997) and Brown, Mitchell, and Poterba (2001), hedging risks before retirement is welfare improving.

Optimal annuity choice has been addressed before in Charupat and Milevsky (2002), Brown et al. (2001), and Blake, Cairns, and Dowd (2003). Charupat and Milevsky (2002) assume interest rates and risk premia to be constant and abstract from inflation risk. They show that the optimal allocation to fixed (i.e., nominal or inflation-linked) and variable annuities coincides with the

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[1]Soares and Warshawsky (2002) illustrate annuity risk in value terms by determining the historical initial payouts of both nominal and inflation-linked annuities.

[2]Browne, Milevsky, and Salisbury (2003) determine a liquidity premium required by investors to compensate for the illiquidity of annuities.

optimal allocation to stocks and bonds in the period before retirement. Brown et al. (2001) are mainly interested in the welfare effects of having access to inflation-linked and variable annuities. They restrict attention to full annuitization using a single product or wealth is split evenly between inflation-linked and either nominal or variable annuities. Brown et al. (2001) find that both variable annuities and inflation-linked annuities can be welfare enhancing, depending on the risk preferences of the annuitant. Blake et al. (2003) consider various distribution programs of retirement wealth in a model similar to the one in Charupat and Milevsky (2002). They consider portfolios containing equity and fixed annuities and show that the ability to invest in equities during retirement can improve welfare significantly. However, both Brown et al. (2001) and Blake et al. (2003) assume risk premia to be constant and do not explore the possibility to tailor the annuity portfolio to the state of the economy at retirement. We relax both assumptions below and find significant additional gains in welfare terms.

Before retirement, the investor can use equity and bond markets to engage in (dynamic) trading strategies to hedge annuity risk. The recent long-term asset allocation literature does not (explicitly) account for the state dependence of the value function at retirement as a result of annuity risk. Notable exceptions are Boulier, Huang, and Taillard (2001), Deelstra, Grasselli, and Koehl (2003), and Cairns, Blake, and Dowd (2006) in which the investor respectively hedges a minimal guarantee or the interest rate risk induced by an annuity-like product at retirement. These papers, however, restrict attention to a single annuity product at retirement and abstract from risks relevant for long-term investors like inflation and changes in risk premia. In fact, recent developments in the dynamic asset allocation literature emphasize the importance for long-term asset allocation of time variation in interest rates, see Brennan and Xia (2000), and Wachter (2003), inflation rates, see Campbell and Viceira (2001) and Brennan and Xia (2002), and risk premia, see Brandt (1999), Campbell and Viceira (1999), Wachter (2002), Campbell, Chan, and Viceira (2003), Sangvinatsos and Wachter (2005), and Brennan and Xia (2005). It is likely that the same risk factors play a role in the optimal (conditional) demand for annuities at retirement and the induced hedging strategies in the period before retirement.[3]

In this paper, we study the optimal portfolio, consumption, and annuity choice over the investor's life-cycle. Our financial market model allows for time-varying interest rates, inflation rates, and risk premia. In order to keep the problem tractable, we exogenously fix the date at which the individual converts wealth into annuities. Milevsky and Young (2003), Neuberger (2003), and Blake et al. (2003) relax this assumption by allowing the investor to select the optimal moment to invest wealth in annuities. At retirement, we determine the optimal allocation to all three annuity products, conditionally upon the state of the economy, thus extending the class of annuitization strategies considered so far in the literature. In addition, we estimate the welfare loss of ignoring

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[3]Bodie and Pesando (1983) show that annuity products inherit the risk-return characteristics from the asset underlying the annuity.

conditioning information or, as often observed in real-world annuity markets, restricting attention to only part of the annuity menu. We solve subsequently for the optimal investment and consumption strategy in the period before retirement. During this stage of the investor's life-cycle, the investor receives a stream of labor income of which a fixed fraction is saved for retirement purposes. The savings are allocated dynamically to stocks, nominal and inflation-linked bonds, and a nominal cash account. We derive optimal investment strategies using equity and bonds to hedge annuity risk induced by both optimal and sub-optimal annuitization strategies. Again, it is important to explore the optimal responses to sub-optimal annuitization strategies as well. Finally, we calculate the welfare costs of ignoring annuity risk all together in the investment strategy in the period before retirement.

This paper provides three main contributions to the extant literature. First of all, we show that conditioning the annuity choice on financial market conditions improves welfare significantly. The welfare costs of ignoring information on the term structure and risk premia at retirement range from 7%-9% of total wealth, depending on the investor's risk preferences. The optimal conditional annuity strategy turns out to be a complex function of the state variables, which may limit its practical use. In fact, we show that 75%-95% of the gains due to incorporating conditioning information can be obtained by following a simple linear portfolio rule. Secondly, restricting the annuity menu to nominal or inflation-linked annuities alone can induce significant welfare costs. Restricting access to nominal annuities is mainly costly for conservative investors, while only having access to inflation-linked annuities is most harmful for aggressive investors. For an individual with an average risk aversion ($\gamma = 5$), the costs of investing retirement wealth fully in nominal annuities is estimated to be 28% and likewise 14% for inflation-linked annuities. This corroborates the results of Brown et al. (2001) and Blake et al. (2003) who confine attention to a few possible retirement strategies. Thirdly, we determine the optimal hedging strategy in the period before retirement for four different annuitization strategies. The optimal conditional annuitization strategy invests in all three annuities and the weights are state dependent. The optimal unconditional strategy may also use all annuity products, but its allocation is independent of the state of the economy at retirement. The third and fourth annuitization strategies invest all wealth accumulated in nominal or inflation-linked annuities, respectively. We find that the welfare costs of not hedging annuity risk before retirement equal 9% for the first two annuitization strategies and 1% for nominal annuitization for an individual with an average risk aversion ($\gamma = 5$). However, the welfare costs of not hedging the annuitization strategy which invests all wealth in inflation-linked annuities is negligible. This leads to the conclusion that hedging annuity risk induced by time variation in inflation and risk premia is welfare improving, while the annuity risk caused by real interest rates is only of minor importance. The limited impact of real interest rate risk is a consequence of the relatively strong mean-reversion implied by our estimates. We find a half-life for the real interest rate of only 8 months. Obviously, our results depend on the investor's preferences and the assumptions we have made concerning the

financial market. However, our paper is the first to provide an integral solution to the investment problems caused by annuity risk in a market which is characterized by time-varying interest rates, inflation, and risk premia.

Our results have important implications for risk management and product design of defined benefit (DB) and defined contribution (DC) pension plans. In case of DB pension plans, participants are generally entitled to a number of nominal or inflation-linked annuities. The number of annuities obtained is linked to the participant's average or final wage. This liability induces significant annuity risk at the fund level, predominantly caused by time variation in inflation rates. In addition, restricting the annuity menu to either nominal or inflation-linked annuities is costly in welfare terms, as mentioned before. This pleads for more flexible payout options in DB pension plans. Concerning DC schemes, we show that it is possible to design simple products which largely implement the optimal conditional annuitization strategy. Likewise, given the importance of hedging annuity risk in the period before retirement, DC pension plans may design products which hedge the most important sources of annuity risk identified, namely inflation risk and changes in risk premia, instead of simply aiming at wealth at retirement which is then subject to interest rate and inflation risk at conversion.

We simplify the problem along various dimensions for ease of exposition, even though these aspects may be relevant for either when to annuitize wealth or which fraction of wealth to be annuitized. First, we assume that annuities are priced fairly, which is (from the annuitant's perspective) an attractive representation of observed annuity markets. Friedman and Warshawsky (1990) and Mitchell, Poterba, Warshawsky, and Brown (1999) show that annuity products may be expensive in fair value terms due to the adverse selection problem in the annuity market. However, Mitchell et al. (1999) provide evidence that the deviation from the fair value of the annuity decreased substantially during the last decade. Second, the investor annuitizes wealth at a single point in time. Milevsky and Young (2003) and Neuberger (2003) show that, once annuitization is irreversible, it may be optimal to transfer retirement wealth gradually into annuities. Third, we abstract from bequest motives, health shocks, and the amount of wealth pre-annuitized, see Brown (2001), Brown et al. (2001), and Lopes (2005), which may affect the fraction of wealth annuitized. Finally, we abstract from joint life annuities, see for instance Brown and Poterba (2000) and Brown (2001), and consider only individual immediate annuities. We leave these extensions for future research.

This paper proceeds as follows. In Section 2 we provide our model of the financial market, the investor's preferences, and discuss the annuity market we consider. Next, we determine in Section 3 the optimal conditional and unconditional annuitization strategy at retirement. In addition, we determine the welfare costs of sub-optimal strategies. In Section 4, we solve for the optimal policies before retirement with annuity risk induced by both optimal and sub-optimal annuitization

strategies. We determine furthermore the welfare costs of not accounting for annuity risk in the period before retirement. Finally, Section 5 concludes. Five appendices contain technical details and proofs. All tables and figures are presented in Appendix F.

2 Financial market, annuity market, and preferences

2.1 Financial market

Our financial market accommodates time variation in real interest rates, inflation rates, and risk premia. The financial market model we consider is closely related to the models of Brennan and Xia (2002) and, in discrete time, Campbell and Viceira (2001). These papers propose two factor models of the term structure, where the factors are identified with the real interest rate (r) and expected inflation (π). Both models assume that bond risk premia are constant. We accommodate time variation in both equity and bond risk premia. The investor's asset menu comprises stocks, nominal and inflation-linked bonds, and a nominal cash account.

We assume that the real rate is driven by a single factor, X_1,

$$r_t = \delta_r + X_{1t}, \delta_r > 0, \tag{1}$$

and expected inflation is affine in a second, possibly correlated, factor, X_2,

$$\pi_t = \delta_\pi + X_{2t}, \delta_\pi > 0. \tag{2}$$

It is well-known that both the real interest rate and expected inflation are persistent processes. Therefore, we model both factors as Ornstein-Uhlenbeck processes, with $i = 1, 2$,

$$\mathrm{d}X_{it} = -\kappa_i X_{it}\mathrm{d}t + \sigma_i'\mathrm{d}Z_t, \kappa_i > 0, \tag{3}$$

in which Z denotes a five-dimensional vector of independent Brownian motions and $\sigma_i \in \mathbb{R}^5$. Realized inflation is subsequently modeled as

$$\frac{\mathrm{d}\Pi_t}{\Pi_t} = \pi_t\mathrm{d}t + \sigma_\Pi'\mathrm{d}Z_t, \tag{4}$$

in which Π_t denotes the value of a (commodity) price index at time t and $\sigma_\Pi \in \mathbb{R}^5$.

The value of the equity index at time t is denoted by S_t, with dynamics

$$\frac{\mathrm{d}S_t}{S_t} = \mu_t\mathrm{d}t + \sigma_S'\mathrm{d}Z_t, \tag{5}$$

where $\mu_t = R_t + \mu_0 + \mu_1'Y_t$, in which R_t denotes the instantaneous nominal short rate, which

is derived in (12) below, and Y a vector of forecasting variables. Risk premia are allowed to depend on the term structure variables, (X_1, X_2), and the dividend yield, D. The use of the dividend yield is not fully uncontroversial, see Ang and Bekaert (2006), Goyal and Welch (2003), Campbell and Yogo (2005), and Campbell and Thompson (2005), but Ang and Bekaert (2006) show that its predictive power is enhanced in a joint model with the short rate. We take therefore $Y = (X_1, X_2, D)'$. In order to accommodate first order autocorrelation in the dividend yield, we model the dividend yield using an Ornstein-Uhlenbeck process

$$\mathrm{d}D_t = \kappa_D \left(\mu_D - D_t \right) \mathrm{d}t + \sigma_D' \mathrm{d}Z_t, \tag{6}$$

with $\sigma_D \in \mathbb{R}^5$. Without further restrictions, the volatility vectors of the different processes are statistically not identified. Therefore, we impose the volatility matrix $(\sigma_1', \sigma_2', \sigma_\Pi', \sigma_S', \sigma_D')$ to be lower triangular.

To derive the prices of both nominal and inflation-linked bonds, we assume that the prices of real interest rate and inflation risk are affine in the state variables. Formally, in the nominal state price density, ϕ, with corresponding dynamics

$$\frac{\mathrm{d}\phi_t}{\phi_t} = -R_t \mathrm{d}t - \Lambda_t' \mathrm{d}Z_t, \tag{7}$$

we assume the prices of risk, Λ_t, to be affine in the term structure variables and the dividend yield,

$$\Lambda_t = \Lambda_0 + \Lambda_1 Y_t. \tag{8}$$

Bond risk premia are allowed to be time varying, but we impose restrictions on Λ_1 such that the risk premium on inflation-linked bonds is driven only by the real rate. Similarly, the risk premium on nominal bonds depends on both the real rate and expected inflation, in line with Koijen, Nijman, and Werker (2006). We assume, in addition, that the dividend yield does not drive the term structure of interest rates, which requires the price of unexpected inflation, $\sigma_\Pi' \Lambda_t$, to be independent of the dividend yield. More formally, these restriction imply that we parameterize Λ_1 as

$$\Lambda_1 = \begin{pmatrix} \Lambda_{1(1,1)} & 0 & 0 \\ 0 & \Lambda_{1(2,2)} & 0 \\ 0 & 0 & 0 \\ \Lambda_{1(4,1)} & \Lambda_{1(4,2)} & \Lambda_{1(4,3)} \\ 0 & 0 & \Lambda_{1(5,3)} \end{pmatrix}, \tag{9}$$

in which the parameters in the last two rows are identified via $\sigma_S' \Lambda_1 = \mu_1'$ and $\sigma_\Pi' \Lambda_{1(:,3)} = 0$.

Given the nominal state price density in (7), we find for the dynamics of the real state price density, ϕ^R,

$$\frac{\mathrm{d}\phi_t^R}{\phi_t^R} = -(R_t - \pi_t + \sigma_\Pi'\Lambda_t)\mathrm{d}t - (\Lambda_t' - \sigma_\Pi')\mathrm{d}Z_t \qquad (10)$$

$$= -r_t\mathrm{d}t - (\Lambda_t' - \sigma_\Pi')\mathrm{d}Z_t, \qquad (11)$$

which implies for the instantaneous nominal short rate

$$R_t = \delta_R + \left(\iota_2' - \sigma_\Pi'\tilde{\Lambda}_1\right)X_t, \qquad (12)$$

where $\delta_R = \delta_r + \delta_\pi - \sigma_\Pi'\Lambda_0$ and $\tilde{\Lambda}_1$ denotes the first two columns of Λ_1.[4] The conditions specified in Duffie and Kan (1996) are satisfied, implying that both nominal and real bond prices are exponentially affine in the state variables. Hence, we find for the prices of a nominal bond at time t, which matures at time $t + \tau$,

$$P(X_t, t, t + \tau) = \exp(A_\tau + B_\tau'X_t), \qquad (13)$$

and, similarly, for an inflation-linked bond

$$P^R(X_t, t, t + \tau) = \exp(A_\tau^R + B_\tau^{R'}X_t), \qquad (14)$$

where A_τ, B_τ, A_τ^R, B_τ^R, and the corresponding derivations, are provided in Appendix A.

The financial market model is estimated using monthly data on bond yields, inflation, and stock returns over the period January 1952 up to May 2002. The government yield data up to February 1993 is the McCulloch and Kwon data and we extend the sample using data provided by Rob Bliss.[5] We use 3-month, 6-month, 1-year, 2-year, 5-year, and 10-year nominal yields in estimation. Inflation data is obtained via CRSP and is based on the CPI-U price index. Finally, we use returns on the CRSP value-weighted NYSE/Amex/Nasdaq index for stock returns. We construct the dividend yield on the basis of the index with and without dividends, along the lines of Campbell et al. (2003). The model is estimated using standard Gaussian Kalman filtering techniques as detailed in Appendix B. The estimation results are portrayed in Table 1.

In line with Brennan and Xia (2002) and Campbell and Viceira (2001), we find that expected inflation is far more persistent than the real interest rate. The innovations to the real rate and expected inflation are negatively correlated. For the unconditional prices of real interest rate risk and expected inflation risk, Λ_0, we find that the former is rewarded a much higher price of risk

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[4]Recall that we have assumed that the price of unexpected inflation risk does not depend on the dividend yield, i.e. $\sigma_\Pi'\Lambda_{1(:,3)} = 0$.

[5]We are grateful to Rob Bliss for providing the data.

than the latter, in line with Campbell and Viceira (2001) and Brennan and Xia (2002). This implies that real interest rate risk is a highly priced risk factor in comparison to the expected inflation factor. The unconditional equity risk premium is estimated to be 4.3%. The unconditional nominal bond risk premium equals 1.8% for a 10-year nominal bond. Likewise, the risk premium on 10-year inflation-linked bonds equals 1.2%. Concerning the dynamics of bond risk premia, we find that the real bond risk premium increases with the real rate and the nominal bond risk premium increases with both the real rate and expected inflation. However, due to the negative correlation between the real rate and expected inflation, the exposure of nominal bonds to the real rate will be smaller than for inflation-linked bonds, with the same maturity. This implies that the inflation risk premium is decreasing in the real rate and increasing in expected inflation, in line with Buraschi and Jiltsov (2005) and Koijen et al. (2006). Next, we find that the equity risk premium decreases with the real rate ($\mu_{1(1)} < 0$) and expected inflation ($\mu_{1(2)} < 0$) and is increasing in the dividend yield ($\mu_{1(3)} > 0$). Finally, the estimates of the dividend yield process reveal the well-documented persistence of financial ratios and the high (negative) correlation of its innovations with equity returns.

2.2 Annuities market

Currently, annuities markets provide, broadly speaking, three types of individual immediate annuity products, see also Brown et al. (2001).[6] Nominal annuities ensure a constant nominal periodic payment during the remainder of the annuitant's life. Inflation-linked annuities, on the other hand, provide payments which are constant in real terms. Consequently, inflation-linked annuities can protect the annuitant against inflation risk. The third annuity product we consider is a so-called variable annuity. The payments provided by variable annuities are linked to a broad equity index.[7] In this way, the annuitant is able to benefit from the (possibly) attractive investment opportunities offered by equity markets during the retirement phase. Using nominal, inflation-linked, and variable annuities, the annuitant is able to manage exposures to the various risk factors like interest rate risk, inflation risk, and equity risk. The remainder of this section discusses the pricing of annuities as well as the income streams provided in more detail.

We price the annuity products in the before-mentioned financial market. We assume throughout that annuities are priced fairly and the proper survival probabilities are taken into account.[8] We

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[6]In this paper, we confine attention to immediate annuities, which implies that the payments start the period after the annuity has been purchased. Alternatively, investors can purchase deferred annuities during the accumulation phase. These products may be particularly interesting from a tax perspective. Poterba (1997) and Blake (1999) provide a detailed overview of the different annuity products offered.

[7]More generally, variable annuities are backed by a portfolio of assets. For instance, Bodie and Pesando (1983) consider variable annuities which are backed by a diversified portfolio of both stocks and bonds. In this paper, we restrict attention to variable annuities which are linked to a broad equity index.

[8]Despite the evidence that annuities are expensive once compared to their value due to the adverse selection problems in annuity markets, see for instance Friedman and Warshawsky (1990) and Mitchell et al. (1999), Mitchell

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assume in addition that longevity risk is idiosyncratic. Denote the probability that the annuitant, who is currently T years old, survives at least another s years by $_sp_T$. We normalize the nominal and real annuity payments of respectively nominal and inflation-linked annuities to one. Formally, the nominal rate of income provided by the nominal annuity at time $T+s$ for an annuity purchased at time T, $I^N(T+s,T)$, is given by $I^N(T+s,T)=1$. For inflation-linked annuities, the nominal rate of income, $I^R(T+s,T)$, is given by $I^R(T+s,T) = \Pi_{T+s}\Pi_T^{-1}$. Consequently, the price of a nominal annuity starting at time t for an annuitant of age T is given by

$$A^N(X_t,T) \quad = \quad \mathbb{E}_t\left(\int_0^\infty {}_sp_T I^N(T+s,T)\frac{\phi_{t+s}}{\phi_t}\mathrm{d}s\right) = \int_0^\infty {}_sp_T P(X_t,t,t+s)\,\mathrm{d}s. \qquad (15)$$

The price of an inflation-linked annuity starting at time t for an annuitant of age T equals

$$A^R(X_t,T) \quad = \quad \mathbb{E}_t\left(\int_0^\infty {}_sp_T I^R(T+s,T)\frac{\phi_{t+s}}{\phi_t}\mathrm{d}s\right) = \int_0^\infty {}_sp_T P^R(X_t,t,t+s)\,\mathrm{d}s. \qquad (16)$$

The pricing and payout structure of variable annuities is somewhat more involved, see also Bodie and Pesando (1983) and Brown et al. (2001). A variable annuity is parameterized by a so-called assumed interest rate (AIR), h. The AIR is an actuarial construct to determine the number of contracts obtained per dollar invested. Formally, for every dollar invested in a variable annuity, the annuitant receives $A^V(h,T)^{-1}$ contracts, with

$$A^V(h,T) = \int_0^\infty {}_sp_T e^{-hs}\mathrm{d}s. \qquad (17)$$

The rate of income provided at time $T+s$ for a variable annuity purchased at time T is given by $I^V(h,T+s,T)$, with

$$I^V(h,T+s,T) = \frac{1}{A^V(h,T)}\frac{S_{T+s}}{S_T}e^{-hs}. \qquad (18)$$

Bodie and Pesando (1983) select the AIR equal to the expected return on the portfolio underlying the variable annuity. Brown et al. (2001) remark that commonly observed values of the AIR are around 3-4%. The choice of the AIR is not unimportant as it affects the distributional properties of the income stream offered by variable annuities. For instance, if the AIR is chosen relatively high, the number of contracts obtained will be high as well. This implies in turn that the initial payments will be high, but the income stream is expected to decline rapidly. Likewise, for low values of the AIR, the initial payouts are low, but increase in expectation. Appendix C provides a rigorous discussion of the role of the AIR in a model with constant investment opportunities. Throughout the paper, we consider variable annuities with an AIR of $h=4\%$. Finally, in order to

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et al. (1999) also provide evidence that the deviation from the fair value decreased during the last decade.

calibrate the survival probabilities, we use mortality rates observed in 1999 provided by the human mortality database.[9]

Figure 1 portrays the mean and volatility of the real (monthly) annuity income provided by nominal, inflation-linked, and variable annuities if the investor converts $100,000 at retirement. The horizontal axis portrays the investor's age and the vertical axis the corresponding annuity income. The vector of state variables is set equal to its unconditional expectation. Inflation-linked annuities provide a riskless payoff stream, but the level is considerably lower than the mean payoff of variable annuities for all ages and that of nominal annuities up to age 75. The real payoffs provided by nominal and variable annuities are risky and the volatility of the payoffs increases with the investor's age.

Figure 2 displays the average, real (monthly) annuity income provided by nominal, inflation-linked, and variable annuities for different initial values of the real rate, expected inflation, and the dividend yield if the investor converts $100,000 at retirement. Different values of the real rate (Panel A and Panel B) have a small effect on the level of the payoffs and the main patterns are unaffected. This is markedly different for expected inflation. When initial expected inflation is low (Panel C), the initial payoff of nominal annuities is low and the expected payoff stream is more stable in real terms. When initial expected inflation is high (Panel D), the initial payoffs of nominal annuities are high, but decline rapidly in expectation. The opposite occurs for variable annuities, since the equity risk premium is negatively related to the level of expected inflation. The level of expected inflation is therefore likely to impact the investor's annuity choice and utility derived from inflation-sensitive annuity products. Finally, Panel E and Panel F portray the impact of different initial values of the dividend yield. Different values of the dividend yield have a substantial impact on the expected annuity payoffs of variable annuities, which is a consequence its high persistence.

2.3 Investor's preferences and labor income

The investor is assumed to participate in the labor market during the period $[t_0, T]$ and the retirement date T is specified exogenously. The nominal rate of income is denoted by $L_t^{\$}$ and its real counterpart by $L_t = L_t^{\$} \Pi_t^{-1}$. Before retirement, the investor allocates wealth dynamically to stocks, two long-term nominal bonds, and a long-term inflation-linked bond.[10] The optimal proportion of wealth allocated to these assets at time t is denoted by x_t. The remainder, $1 - x_t' \iota$, is invested in a nominal cash account. In addition, the investor optimally decides upon the amount to consume at time t, C_t. At age T the investor retires and annuitizes all wealth accumulated. The fractions allocated to the nominal, inflation-linked, and variable annuity at time T are denoted by α_T^N, α_T^R, and α_T^V, respectively.

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[9]We refer to http://www.mortality.org/ for further information.

[10]Any additional bond is redundant as the term structure of interest rates is driven by two factors, i.e. (X_1, X_2). In order to complete the market, one inflation-linked bond is required to hedge unexpected inflation.

The investor derives utility from real consumption during the life-cycle, in line with Brennan and Xia (2002) and Sangvinatsos and Wachter (2005). The preferences are represented by a time-separable CRRA utility index, i.e., the value function of the problem is

$$J\left(W_{t_0}, Y_{t_0}, L_{t_0}, t_0\right) = \max_{(C_t)_{t \in [t_0,\infty)}, (x_t)_{t \in [t_0,T]}, \alpha_T^N, \alpha_T^R, \alpha_T^V} \mathbb{E}_{t_0}\left(\int_{t_0}^{\infty} {}_{t-t_0}p_{t_0} \frac{e^{-\beta t}}{1-\gamma}\left(\frac{C_t}{\Pi_t}\right)^{1-\gamma} \mathrm{d}t\right), \quad (19)$$

where β denotes the subjective discount factor. We assume throughout that ${}_{T-t}p_t = 1$, for all $t \in [t_0, T]$, i.e., the investor survives up to retirement with probability one. The optimization in the period before retirement is subject to a dynamic budget constraint. Let W_t denote wealth accumulated and $L_t^\$$ the nominal rate of labor income at time t. The budget constraint is

$$\mathrm{d}W_t = W_t\left(x_t'\Sigma\Lambda_t + R_t\right)\mathrm{d}t + \left(L_t^\$ - C_t\right)\mathrm{d}t + W_t x_t'\Sigma\mathrm{d}Z_t, \quad (20)$$

and Σ the volatility matrix of the traded assets. During the retirement phase, the investor receives annuity income. Part of this annuity income can be saved in order to smooth consumption. We assume that the wealth accumulated is invested in a nominal cash account. This leads to the budget constraint during

$$\mathrm{d}W_t = W_t R_t \mathrm{d}t + (Y_t - C_t)\,\mathrm{d}t, \quad (21)$$

with Y_t indicating nominal annuity income at time t. Further, we assume that the investor annuitizes fully at retirement, i.e., for $t \geq T$,

$$\frac{Y_t}{W_{T-}} = \frac{\alpha_T^N}{A^N\left(X_{1T}, X_{2T}, T\right)} + \frac{\alpha_T^R}{A^R\left(X_{1T}, X_{2T}, T\right)}\frac{\Pi_t}{\Pi_T} + \alpha_T^V I^V(h, t, T), \text{ with } \alpha_T^N + \alpha_T^R + \alpha_T^V = 1. \ (22)$$

The budget constraint in (21) is subject to the initial condition $W_T = 0$, since the investor converts all wealth into annuities. Note that W_{T-} in (22) refers to retirement wealth just *prior to* conversion. Summarizing, the investor annuitizes fully at retirement and can smooth annuity income optimally during retirement using a nominal cash account, following for instance Brown et al. (2001).

The optimization is subject to the (institutional) constraint that annuities cannot be shorted, i.e.,

$$\alpha_T^N, \ \alpha_T^R, \ \alpha_T^V \geq 0. \quad (23)$$

We assume that the investor cannot capitalize future annuity income to increase today's consumption. Therefore, we impose that the investor is liquidity constrained, which formally

implies, for $t > T$,

$$W_t \geq 0. \tag{24}$$

The dynamics of real labor income, $L_t = L_t^\$ \Pi_t^{-1}$, are given by, with $t \in [t_0, T]$ indicating the investor's age

$$\frac{\mathrm{d}L_t}{L_t} = g_t \mathrm{d}t, \tag{25}$$

where g_t is calibrated on the basis of Cocco et al. (2005) and Munk and Sørensen (2005) to capture the hump-shaped pattern in labor income over the life-cycle. The (deterministic) growth rate of labor income is given by, with $t \in [t_0, T]$

$$g_t = 0.1682 - 0.00646t + 0.00006t^2, \tag{26}$$

which corresponds to an individual with high school education in the estimates of Cocco et al. (2005) and Munk and Sørensen (2005). The income rate at age t_0 is normalized to $L_{t_0} = 1$ and initial wealth $W_{t_0} = 0$. We assume that the investor does not receive any additional form of income during retirement, i.e. $L_t = 0$ for $t \geq T$. As we are mainly concerned with the impact of the annuitization decision at retirement on the optimal investment strategies before retirement, we abstract from idiosyncratic labor income risk. We refer to Viceira (2001), Cocco et al. (2005), Munk and Sørensen (2005), and Koijen et al. (2006) for an analysis of the impact of idiosyncratic labor income on optimal portfolio choice.[11] Note that the labor income stream is equivalent to a particular portfolio of inflation-linked bonds in our model.

The investor's problem can be decomposed conveniently as

$$J\left(W_{t_0}, Y_{t_0}, L_{t_0}, t_0\right) = \max_{(C_t, x_t)_{t \in [t_0, T]}} \mathbb{E}_{t_0}\left(\int_{t_0}^T \frac{e^{-\beta t}}{1-\gamma}\left(\frac{C_t}{\Pi_t}\right)^{1-\gamma} \mathrm{d}t\right) + e^{-\beta T}\mathbb{E}_{t_0}\left(J\left(W_{T-}^R, Y_T, 0, T\right)\right), \tag{27}$$

which disentangles the problem before and after retirement. We define $W_{T-}^R = W_{T-}\Pi_{T-}^{-1}$ to denote real wealth just before retirement. For future reference, we formulate after retirement the problem as, for $\gamma > 1$,

$$
\begin{aligned}
J\left(W_{T-}^R, Y_T, 0, T\right) &= \max_{(C_t)_{t \in (T, \infty)}, \alpha_T^N, \alpha_T^R, \alpha_T^V} \mathbb{E}_T\left(\int_T^\infty {}_{t-T}p_T \frac{e^{-\beta(t-T)}}{1-\gamma}\left(\frac{C_t}{\Pi_t}\right)^{1-\gamma} \mathrm{d}t\right) \\
&= \frac{1}{1-\gamma}\left(W_{T-}^R\right)^{1-\gamma} \min_{(C_t^R)_{t \in (T, \infty)}, \alpha_T^N, \alpha_T^R, \alpha_T^V} \mathbb{E}_T\left(\int_T^\infty {}_{t-T}p_T e^{-\beta(t-T)}\left(C_t^R\right)^{1-\gamma} \mathrm{d}t\right),
\end{aligned}
\tag{28}
$$

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[11]When labor income risk is idiosyncratic, the model cannot be solved in closed-form.

with $C_t^R = C_t W_{T-}^{-1} \Pi_{T-} \Pi_t^{-1}$, i.e. C_t^R denotes the real fraction of wealth consumed at time t. In terms of preference parameters, we consider three values for the coefficient of relative risk aversion, namely $\gamma = 2, 5$, and 10. The time preference parameter is set, in accordance with Cocco et al. (2005), to $\beta = 0.04$.

3   Optimal retirement choice

3.1   Optimal annuity choice

In this section, we first determine the optimal annuity choice at retirement. Optimally, the investor's annuity menu contains nominal, inflation-linked, and variable annuities with an assumed interest rate (AIR) of $h = 4\%$. This menu allows the investor to construct the proper exposures to all important risk factors in the economy. The investor determines the optimal allocation conditional on the current state of the economy. Next, we assess the welfare cost of sub-optimal annuitization strategies, like ignoring conditioning information or restricting the annuity menu to nominal or inflation-linked annuities alone. These latter annuitization strategies are particularly interesting for at least two reasons. First of all, individual investors tend to restrict attention, either for behavioral or institutional reasons, to nominal annuities, see Diamond (1997) and Brown et al. (2001). Secondly, Defined Benefit (DB) pension plans generally offer their participants a nominal or an inflation-linked annuity. Therefore, it is important to explore welfare costs induced by restricting retirement choice, as well as its implications for the optimal investment strategies before retirement. This latter question is the subject of Section 4.

We now briefly outline the optimization problem faced by the investor during retirement as described in Section 2.3. The investor has to decide upon the annuity choice at retirement and, given the annuity strategy selected at retirement, the consumption strategy during the retirement phase. The investor optimally takes into account the economic conditions, the current income level, and wealth level, which implies that the consumption-savings problem during retirement is characterized by six state variables (the real rate, expected inflation, the dividend yield, the income level provided by nominal and variable annuities, and the current wealth level[12]). This problem is computationally very hard to solve using standard dynamic programming tools. We therefore opt for a simulation-based approach which is essentially an extension of the methods developed in Brandt, Goyal, Santa-Clara, and Stroud (2005). First, we simulate 10,000 trajectories of state variables and annuity income. Subsequently, we proceed as in case of numerical dynamic programming, but we approximate the conditional expectations we encounter by expansions in a set of basis functions in the state variables. This approach is virtually insensitive to the number of state variables and therefore perfectly suited to solve the problem at hand. By proceeding

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[12]Note that the homogeneity property of the power utility index allows us to reduce the number of state variables to five.

backwards, we obtain an estimate of the value function at retirement for a certain retirement state and annuity portfolio. This allows us to optimize over the annuity portfolio at retirement. For the sake of robustness, we optimize over a grid with step sizes of 5%. Our numerical procedure then results in a set of optimal annuity choices for all of the 10,000 initial states of the economy at retirement. A detailed discussion of the approach is provided in Appendix D.

It turns out that the optimal annuitization strategy is a non-linear function of the state variables, say $\alpha_T^i(Y)$, with $i \in \{N, R, V\}$.[13] In order to summarize the results, we consider a first order approximation of the optimal annuitization strategy. We will show in the next section that this approximation is close to optimal. More specifically, we run (cross-sectional) regressions for the optimal weights in each of the annuity products on the state variables (real rate, expected inflation, and the dividend yield). We standardize the state variables to enhance the interpretation of the coefficients. Thus, we have

$$\alpha_T^i(Y) \simeq \alpha_0^i + \sum_{j=1}^{3} \alpha_j^i \tilde{Y}_j, \tag{29}$$

with $i \in \{N, R, V\}$ and $\tilde{Y} = (Y - \mathbb{E}(Y))/\sigma(Y)$. The constant term (α_0^i) can then be interpreted as the unconditional allocation to a particular annuity product. The slope coefficients (α_j^i) present the percentage change in the allocation for a one standard deviation increase in the j-th state variable. As all retirement wealth is annuitized at retirement, we have by construction that the sum (over the three annuity products) of the constant terms equals one, i.e., $\sum_{i=1}^{3} \alpha_0^i = 1$, and the sum of the slope coefficients for a particular state variable is zero, i.e., for $j \in \{1, 2, 3\}$, $\sum_{i=1}^{3} \alpha_j^i = 0$. These regressions therefore facilitate a clear interpretation of the resulting optimal annuity choice and its dependence on the state of the economy.

Table 2 presents the optimal annuity strategy at retirement. Recall that the constants indicate the unconditional allocation to a particular annuity product. For instance, an investor with a coefficient of relative risk aversion equal to $\gamma = 5$ allocates 50% to inflation-linked annuities, 42% to variable annuities, and 8% to nominal annuities. In all cases, the unconditional allocation to nominal annuities is marginal. Since expected inflation is a persistent and relatively low priced factor, risk averse investors are not willing to bear the inflation risk. Similarly, more aggressive investors consider variable annuities, which are linked to an equity index, as more attractive from a risk-return perspective. This explains the minor role of nominal annuities in the optimal retirement choice. Table 2 shows that the current state of the economy has significant consequences for the optimal annuity choice. In particular changes in expected inflation and dividend yield, which are both highly persistent, can alter the optimal allocation substantially. For an investor with a risk aversion of $\gamma = 5$, the optimal allocation to variable annuities reduces by 11% in absolute

---

[13]Recall that 'N' refers to nominal, 'R' to inflation-linked or real, and 'V' to variable annuities.

terms if expected inflation increases with one (unconditional) standard deviation. In response, the allocation to nominal and variable annuities increases. Recall that nominal bond risk premia are increasing in expected inflation, whereas the equity risk premium relates negatively to expected inflation. Likewise, a one (unconditional) standard deviation increase in the dividend yield leads to 28% increase in the allocation to variable annuities at the cost of the allocation to nominal and especially inflation-linked annuities.

Finally, we determine the optimal annuity choice for individuals who do not incorporate conditioning information in their portfolio choice. More specifically, the optimal annuity choice is determined using the unconditional value function as opposed to the conditional value function. The results are depicted in Table 3. The optimal unconditional annuitization strategy is comparable to the optimal conditional annuitization strategy, when the state variable equal their unconditional allocation.

3.2 Welfare costs of sub-optimal annuitization strategies

We calculate welfare costs induced by adopting one of the four sub-optimal annuitization strategies described above. First, we consider the unconditional annuity choice, where wealth is allocated to all three annuity products, but independently of the state of the economy. Alternatively, the annuity choice may be restricted to either nominal or inflation-linked annuities alone. Finally, we investigate whether it is possible to approximate the optimal conditional annuity choice with a simple linear portfolio rule.

The welfare costs of the sub-optimal annuitization strategies are presented in Table 4. The optimal conditional annuity choice serves as the benchmark strategy. It turns out that taking into account information about the term structure and risk premia at retirement is significantly welfare improving. The welfare costs range from 7% up to almost 9% of certainty equivalent consumption, depending on the investor's risk preferences. In other words, investors are willing to give up 7-9% of their retirement wealth in order to optimally incorporate the economic conditions at retirement. If the retirement choice is restricted to inflation-linked, even conservative investors are faced with welfare costs close to 10%. Restricting attention to only nominal annuities induces large welfare costs, especially for conservative investors. In that case, the decrease in certainty equivalent consumption ranges from 22% for aggressive investors to 55% for conservative investors. These results imply that both equity exposure during retirement and the possibility to insure inflation risk are valuable for individuals. While these results are in line with Brown et al. (2001) and Blake et al. (2003), recall that these studies assume risk premia to be constant and do not explore the possibility to tailor the annuity portfolio to the state of the economy at retirement.

The optimal conditional annuity choice may be a complex function of the state variables. Finally, we discuss whether a first order approximation to the optimal annuity weights as in (29) may be

used to design a simple conditional allocation rule. More precisely, we consider

$$\alpha_T^{\text{Linear},i}(Y) = \frac{\max(\alpha_T^i(Y), 0)}{\sum_{i=1}^3 \max(\alpha_T^i(Y), 0)}, \tag{30}$$

with α_T^i given in (29). The results are presented in the final row of Table 4. It turns out that this simple rule reduces the welfare loss relative to the unconditional annuity choice by 75% to 95%, depending on the risk attitude of the investor. This illustrates that it is possible to design simple rules, which are very close to the optimal conditional retirement strategy.

4 Optimal policies before retirement

4.1 The optimal investment and consumption strategy

The main conclusion of Section 3 is that the optimal annuity strategy depends on the economic conditions at retirement and that it is costly is welfare terms to ignore this. However, even if the optimal annuity choice does not depend on the state of the economy, for instance if the investor restricts attention to nominal or inflation-linked annuities, the value function at retirement does. As a result, an investor optimally anticipates this dependence before retirement and uses financial markets to hedge exposures to interest rates, inflation rates, and risk premia. In this section, we study the induced optimal investment and consumption strategy in the period before retirement. The existing literature either abstracts from the investor's desire to annuitize wealth, see among others Viceira (2001), Cocco et al. (2005), and Koijen et al. (2006), or assumes that the investor restricts attention to a single annuity product, like in Cairns et al. (2006).

The optimal investment and consumption policies are the solution to (27), subject to the dynamic budget constraint given in (20). We assume that the investor can allocate wealth to stocks, two nominal bonds, and an inflation-linked bond. The maturities of the two nominal bonds are assumed to be three and ten years and the maturity of the inflation-linked bonds is set to ten years. The remainder is allocated to a nominal cash account. It is well-known, see for instance Wachter (2002) and Liu (2006), that this optimal consumption problem cannot be solved explicitly in incomplete markets. Therefore, we assume that the investor consumes a fixed fraction of labor income, θ, and saves the remainder, $1-\theta$, before retirement. We determine the optimal investment strategy and savings rate jointly. For the remainder of this section, it will turn out to be useful to define the real present value of future savings until retirement, for $t \in [t_0, T]$,

$$H(L_t, Y_t, \theta, t, T) = \int_t^T (1-\theta) L_s P^R(X_t, t, s) \mathrm{d}s. \tag{31}$$

Appendix E derives the optimal solution to the investment problem before retirement. The

optimal investment strategy at time t is given by

$$
\begin{aligned}
x_t^\star &= \left(\frac{W_t^R + H_t}{W_t^R}\right) \frac{1}{\gamma} \left(\Sigma\Sigma'\right)^{-1} \Sigma\Lambda_t + \\
&\quad \left(\frac{W_t^R + H_t}{W_t^R}\right) \left(1 - \frac{1}{\gamma}\right) \left(\Sigma\Sigma'\right)^{-1} \Sigma \left(\Sigma_Y' \left(\xi_1 + \xi_2 Y_t\right) + \sigma_\Pi\right) + \\
&\quad \left(\frac{W_t^R + H_t}{W_t^R}\right) \frac{1}{\gamma} \left(\Sigma\Sigma'\right)^{-1} \Sigma\Sigma_Y' \left(\Gamma_1(\tau) + \frac{1}{2} \left(\Gamma_2(\tau) + \Gamma_2(\tau)'\right) Y_t\right) - \\
&\quad \left(\Sigma\Sigma'\right)^{-1} \Sigma\Sigma_Y' \frac{H_{Yt}}{W_t^R} - \left(\frac{H_t}{W_t^R}\right) \left(\Sigma\Sigma'\right)^{-1} \Sigma\sigma_\Pi,
\end{aligned} \tag{32}
$$

where Σ and Σ_Y are the volatility matrices of respectively the traded assets and the state variables, $H_t = H(L_t, Y_t, \theta, t, T)$, ξ_1, ξ_2, $\Gamma_1(\tau)$ and $\Gamma_2(\tau)$ are given in Appendix E, $\tau = T - t$, and H_Y denotes the partial derivative of H with respect to Y. The optimal portfolio in (32) contains four components which allow for an intuitive interpretation. The first component is the standard myopic demand, which optimally exploits the risk-return trade-off provided by the assets. The second component constitutes the minimum variance portfolio in an investment problem in which $K(Y_T)$ serves as the numéraire. The third component hedges changes in the state variables which affect future investment opportunity sets. This component depends on the investment horizon. The first three component are pre-multiplied by the ratio of current real financial wealth (W_t^R) plus the real present value of future savings (H_t) to current real financial wealth. This tilt of the optimal portfolio is a consequence of the fact that the investor has to implement the optimal strategy via financial wealth rather than total wealth, see also Bodie et al. (1992) and Munk and Sørensen (2005). The relative importance of the first three components is determined by the investor's risk attitude. The fourth and final component (the last two terms in (32)) corrects the optimal investment strategy for the exposures to the risk factors of the savings stream. Finally, the optimal savings rate is determined in Appendix E.

It is important to emphasize the impact of the investor's desire to annuitize wealth at retirement. The minimum variance portfolio (the second component in (32)) is affected by the investor's desire to annuitize wealth at retirement. We show in Appendix E that both $\Gamma_1(T-t)$ and $\Gamma_2(T-t)$ depend on the value function at retirement, (28). This implies that the second and third component of the optimal investment strategy anticipate the investor's retirement strategy.

The optimal investment strategy in (32) shows that the investor's optimal portfolio is composed by three general funds and an investor-specific fund, namely the fourth component. The weights of the different components is investor-specific and depends on the income path as well as on the risk attitude of the investor.

In Section 4.2 we examine the adjustments in the optimal investment strategy before retirement as a result of annuity risk for both optimal and sub-optimal annuitization strategies. Subsequently,

we determine in Section 4.3 the utility costs of sub-optimal investment strategies which abstract from the investor's preference to annuitize wealth at retirement.

4.2 Optimal investment and consumption with annuity risk

We now assess empirically the properties of the optimal portfolio strategy before retirement. We focus in particular on the additional hedging demands caused by annuity risk. We first of all determine the optimal investment strategy if the individual ignores annuity risk. Subsequently, we provide the adjustments to the optimal portfolio if the individual allocates retirement wealth to either the optimal conditional annuitization strategy, the optimal unconditional annuitization strategy, inflation-linked annuities, or nominal annuities.

Figure 3 portrays the optimal allocation to 3-year and 10-year nominal bonds, 10-year inflation-linked bonds, and stocks in absence of annuity risk. The remainder is invested in the nominal cash account. The optimal investment strategy at time t as given in (32) depends on the state variables (Y_t), the investor's horizon $(T - t)$, the income to wealth ratio (L_t/W_t^R), and the investor's risk attitude (γ). In all graphs, the state variables are set to their unconditional expectation, the investment horizon ranges from zero to thirty years, and the income to wealth ratio equals 0.01, 0.25, 0.50, or 1. An income to wealth ratio of 0.5 implies that the investor accumulated an amount equal to two annual incomes. In the first case $(L_t/W_t^R = 0.01)$, the impact of human capital on the optimal portfolio is negligible. This allows us to separate the horizon effects generated by depletion of human capital and hedging demands as a consequence of time variation in investment opportunities. The investor's coefficient of relative risk aversion is set to $\gamma = 5$.[14] The optimal savings rate used is determined by assuming that the investor implements the optimal conditional annuitization strategy and starts saving as of age 35.

We find that the investor holds a long position in 3-year nominal bonds, which is financed by a short position in cash and 10-year nominal bonds, in line with Campbell and Viceira (2001), Brennan and Xia (2002), and Sangvinatsos and Wachter (2005). The role of inflation-linked bonds is limited to hedging unexpected inflation risk. Both nominal bonds and stocks exhibit strong horizon effects due to time variation in investment opportunities (i.e. $L_t/W_t = 0.01$), see also Campbell and Viceira (1999), Wachter (2002), and Sangvinatsos and Wachter (2005). The horizon effects are further amplified by the savings stream to which the investor is entitled. Note further that the optimal portfolio entails rather large long and short positions, in particular for nominal bonds. This is a consequence of the high correlation between the returns on these assets. Since all that matters are the induced exposures to the risk factors, the positions are in fact in economic terms less extreme than suggested by the portfolio weights. Rather than physically taking extreme

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[14]To conserve space, we restrict attention to an investor with coefficient of relative risk aversion equal to $\gamma = 5$. Results for any other configuration of the parameters are available upon request.

bond positions, these exposures can also be created through the use of swaps or other interest rate and inflation-linked derivatives.

Next, we illustrate the impact of the annuity choice on the investor's investment strategy before retirement for the same configuration of parameters as before. In determining the hedging strategy before retirement of a particular annuitization strategy, we set the savings rate in accordance with the particular annuity choice made at retirement. The income-to-wealth ratio equals 0.5.[15] Figure 4 portrays the hedging strategies for, respectively, the optimal conditional annuitization strategy (Panel A), the optimal unconditional annuitization strategy (Panel B), and either only inflation-linked (Panel C) or only nominal annuities (Panel D). The hedging strategy is defined as the difference between the optimal strategy which does account for annuity risk and the strategy which ignores it. Hence, by adding the portfolios in Figure 3 to the hedging portfolio, we obtain the total optimal portfolio in the presence of annuity risk. First, Panel A of Figure 4 presents the results for the optimal conditional annuitization strategy. The hedging strategy holds long positions in 3-year nominal bonds and stocks, while 10-year nominal bonds and cash are shorted. The allocation to inflation-linked bonds is hardly affected. The long-short position in nominal bonds pays off in states of the economy where either expected inflation is high or real interest rates are low. In addition, when the investor anticipates the preference to annuitize wealth at retirement, the allocation to equities increases. An important determinant of the investment opportunity set at retirement is the level of the dividend yield. Low levels of the dividend yield correspond to low expected returns on variable annuities. Given the negative correlation between innovations in equity returns and dividend yields, a long equity position in the period before retirement can hedge adverse changes of the dividend yield. Interestingly, the additional hedging demands are already substantial during early stages of the investor's life-cycle. Panel B of Figure 4 portrays the optimal hedging strategy for an individual implementing the optimal unconditional hedging strategy at retirement. The optimal hedging strategy of the optimal conditional and unconditional annuitization strategy are close. Next, Panel C displays the optimal hedging strategy when the investor allocates all wealth at retirement to inflation-linked annuities. The hedging portfolio is strikingly different than when the full annuity menu is at the individual's disposal. First, neither inflation-linked nor stocks are present in the hedging strategy. When the investor allocates all capital at retirement to inflation-linked annuities, all that matters is the exposure to the real interest rate, which is managed by positions in the 3-year and 10-year nominal bond. Second, the allocation to both nominal bonds changes. Note that the hedging strategy has only significant weights as of, say, age 60. This is a consequence of the relatively low persistence of real interest rates in comparison with expected inflation and the dividend yield implied by our estimates. This suggests that hedging inflation-linked annuities, i.e. hedging real interest rate risk, is far less important than hedging the optimal (un)conditional

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[15]The results hardly change for a reasonable range of income-to-wealth ratios. Additional results are available upon request.

annuitization strategy which (possibly) allocates wealth to all three annuity products. Finally, Panel D portrays the optimal hedging strategy if the investor allocates all wealth at retirement to nominal annuities. At early stages of the investor's life-cycle, it turns out to be optimal to hold long 10-year nominal bonds, and short 3-year nominal bonds to hedge time-varying bond risk premia. Close to retirement, hedging real rate risk is more important and the hedging strategy is long in 3-year nominal bonds and shorts 10-year nominal bonds, as is the case for all other annuitization strategies.

4.3 Welfare costs of not hedging annuity risk

We now calculate the welfare costs of not hedging annuity risk. As before, the welfare metric employed is the amount of retirement wealth an investor is willing to give up in order to implement the optimal investment strategy before retirement. We refer to Appendix E for further details. For both investment strategies, the savings rate is determined on the basis of the investment strategy which accounts for annuity risk.

The welfare costs are presented in Table 5. First, we find that hedging the optimal conditional annuitization strategy, the unconditional annuitization strategy, and nominal annuities before retirement is valuable, in particular for conservative investors. Individuals with an average risk aversion ($\gamma = 5$) are willing to give up 9% of retirement to hedge annuity risk induced by the optimal conditional or unconditional strategy. The welfare costs of not hedging annuity risk induced by nominal annuities amounts to 1% of retirement. Surprisingly, we find that the costs of not hedging inflation-linked annuities are small. This was suggested already by the small additional hedging demands generated by this strategy in Figure 4. The bottom line is that hedging annuity risk before retirement is welfare improving due to persistent expected inflation and time variation in risk premia. The welfare costs of not hedging the annuity risk caused by inflation-linked annuities is marginal. The fact that hedging real interest rate risk is of minor importance is caused by the relatively low persistence of the real interest rate implied by our estimates.

The welfare costs of not hedging annuity risk before retirement have been calculated for the situation in which the state variables equal their unconditional expectation. However, the costs of not hedging annuity risk are obviously dependent on the prevailing term structure and risk premia. To highlight the impact of changing state variables, we consider the case where all equal their unconditional expectation up to age 55, and subsequently one of the state variables is perturbed by a one (unconditional) standard deviation, positive shock. We take into account the correlation with the other state variables by perturbing one of the state variables. After the shock, the state variables equal their expected value, conditional on the shock at age 55. The results are portrayed in Figure 5 for all four annuitization strategies.

Panel A portrays the results for the optimal conditional annuitization strategy. A positive

shock at age 55 to expected inflation, leading to a lower expected returns on equity and a higher expected return on long-term nominal bonds, decreases the welfare costs of not hedging annuity risk. For a positive shock to the dividend yield, it becomes more costly for the individual to abstract from hedging before retirement. This is in line with, for instance, Brandt et al. (2005) who illustrate that the economic value of hedging demands are higher in good economic environments. This immediately implies that in this stage of the investor's life cycle, high expected inflation corresponds to a deteriorated investment opportunity set. Panel B presents the results for the optimal unconditional hedging strategy. Note that the welfare costs hardly change in response to a shock in the state variables. This implies that whenever the individual is not able to tailor the annuitization strategy to the economic conditions at retirement, the welfare costs are almost independent of the state of the economy before retirement. This is also the case for the annuity strategies which convert all capital at retirement into inflation-linked annuities (Panel C) or nominal annuities (Panel D).

5   Conclusions

Investors facing longevity risk prefer to annuitize wealth at retirement if annuity markets are sufficiently complete. However, this exposes investors to, what is known as, annuity risk. The utility derived from annuitized wealth may disappoint if market conditions turn out to be unfavorable at retirement. We show that investors may mitigate the welfare loss due to annuity risk by conditioning the annuity portfolio on the state of the economy and by hedging certain risks before retirement.

First, we show that it is optimal for individuals to incorporate information on the term structure and risk premia at retirement in the annuity choice. The welfare costs of ignoring this information range from 7% to 9%, depending on the investor's risk preferences. However, the optimal conditional annuitization strategy may depend in a complex way on the underlying state variables. We show that it is possible to design a simple linear rule which allocates wealth to nominal, inflation-linked, and variable annuities contingent on the state of the economy. In fact, 75%-95% of the gains due to incorporating conditioning information can be obtained by following this simple rule. In addition, restricting the annuity menu to only nominal or only inflation-linked annuities increases welfare costs even further. This implies that equity exposure during retirement, as well as the ability to insure inflation risk, are welfare enhancing. These conclusions may have serious implications for both DC and DB pension plans. Concerning DC pension plans, investors tend to restrict attention to nominal annuities, if they annuitize at all. On the other hand, DB pension plans usually offer participants either nominal or inflation-linked annuities. Therefore, the annuity portfolio is not diversified and participants cannot take advantage of the (possibly) investment opportunities offered by equity markets.

Whether or not investors adopt optimal annuitization strategies at retirement, investors face

annuity risk before retirement that can be hedged by trading in equity and bond markets. We consider the case where the investor's menu contains stocks, nominal and inflation-linked bonds, and a cash account. We determine the optimal hedging strategies for annuity risk induced by the optimal conditional annuitization strategy, the optimal unconditional annuity strategy, and investing all wealth in inflation-linked or nominal annuities. The optimal hedging strategy entails significant positions in the various securities already in early stages of the investor's life cycle, unless the investor allocates all wealth to inflation-linked annuities. This result is confirmed by a welfare analysis in which we determine the welfare costs induced by not hedging annuity risk. These welfare costs range from 1% to over 10%, depending on the risk attitude of the investor, unless the investor allocates all wealth to inflation-linked annuities at retirement. In other words, hedging inflation risk and time variation in risk premia before retirement can be significantly welfare enhancing. Hedging annuity risk induced by time variation in real interest rates turns out to be only of minor importance.

Future research can extend this paper in various directions. First, the investor considered annuitizes all wealth at age 65. Neuberger (2003) and Milevsky and Young (2003) have shown that it may be optimal to gradually transfer wealth accumulated to annuities. This relates directly to the literature in which investors endogenously select their retirement age, see for instance Farhi and Panageas (2005) and the references therein. Incorporating this additional flexibility may provide a more complete answer to the dynamic life-cycle investment and consumption problem. Second, we have restricted our analysis to immediate individual annuities, ignore bequests, and uncertainty concerning the investor's health during retirement. The annuity menu may be extended by joint annuities for married couples, deferred annuities, or annuities which embed particular derivative structures like escalating annuities. Finally, we have assumed that annuities are priced fairly, while realistic annuity markets usually include charge substantial load factors. Lopes (2005) shows that load factors and, in addition, minimum size restrictions, may have a substantial impact on the annuitization decision.

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A   Pricing of nominal and inflation-linked bonds

We derive the nominal prices of both nominal and inflation-linked bonds in the financial market described in Section 2, following the results on affine term structure models in, for instance, Duffie and Kan (1996), Sangvinatsos and Wachter (2005), and Koijen et al. (2006).

To that extent, we assume that both nominal and inflation-linked bond prices are smooth functions of time and the term structure factors X, which satisfy

$$\mathrm{d}X_t = -K_X \mathrm{d}t + \Sigma_X \mathrm{d}Z_t. \tag{A.1}$$

Denote the price of a nominal bond at time t that matures at time T by $P(X_t, t, T)$. Since nominal bonds are traded assets, we must have that $\phi_t P(X_t, t, T)$ is a martingale, where ϕ is given in (7). This implies[16]

$$-P_X K_X X + P_t + \frac{1}{2} tr\left(\Sigma'_X P_{XX} \Sigma_X\right) - RP - P'_X \Sigma_X \Lambda = 0, \tag{A.2}$$

where the subscripts of P denote partial derivatives with respect to the different arguments. Using the results of Duffie and Kan (1996), we obtain nominal bond prices that are exponentially affine in the state variables, like in (13). Substituting this expression in (A.2) and matching the coefficients on the constant and the state variables X, we obtain the following set of ordinary differential equations

$$\dot{A}(\tau) = -B(\tau)' \Sigma_X \Lambda_0 + \frac{1}{2} B(\tau)' \Sigma_X \Sigma'_X B(\tau) - \delta_R, \tag{A.3}$$

$$\dot{B}(\tau) = -\left(K'_X + \tilde{\Lambda}'_1 \Sigma'_X\right) B(\tau) - \left(\iota_2 - \sigma'_\Pi \tilde{\Lambda}_1\right), \tag{A.4}$$

where ι_2 denotes a two dimensional vector of ones and $\tilde{\Lambda}_1$ the first two columns of Λ_1. The boundary conditions to the differential equations are given by $A(0) = 0$, $B(0) = 0$.

The price of inflation-linked bonds can be derived along the same lines. The nominal price of a real bond is denoted by the product $\Pi_t P^R(X, t, T)$. The martingale property of $\phi_t \Pi_t P^R(X_t, t, T)$ implies

$$-P^R_X K_X X + P^R_t + \frac{1}{2} tr\left(\Sigma'_X P^R_{XX} \Sigma_X\right) - (R - \pi + \sigma'_\Pi \Lambda)P^R + P^{R\prime}_X \Sigma_X \left(\sigma_\Pi - \Lambda\right) = 0, \tag{A.5}$$

Since prices of inflation-linked bonds are affine in the state variables,[17] (A.5) reduces to

$$-B^R(\tau)' K_X X - \dot{A}^R(\tau) - \dot{B}^R(\tau)' X + \frac{1}{2} B^R(\tau)' \Sigma_X \Sigma'_X B^R(\tau) - r + B^R(\tau)' \Sigma_X \left(\sigma_\Pi - \Lambda\right) = 0. \tag{A.6}$$

---

[16]For notational convenience, we omit the time subscripts.

[17]This is a consequence of the fact that instantaneous expected inflation is affine in the state variables.

We again match the coefficients on the constant and the state variables X, which leads to the following set of ordinary differential equations

$$\dot{A}^R(\tau) = -B^R(\tau)'\Sigma_X(\Lambda_0 - \sigma_\Pi) + \frac{1}{2}B^R(\tau)'\Sigma_X\Sigma_X'B^R(\tau) - \delta_r, \tag{A.7}$$

$$\dot{B}^R(\tau) = -\left(K_X' + \tilde{\Lambda}_1'\Sigma_X'\right)B^R(\tau) - e_1, \tag{A.8}$$

where e_i denotes the i-th unit vector. The boundary conditions to the differential equations are given by $A^R(0) = 0$, $B^R(0) = 0$.

B Details estimation procedure

Our estimation procedure in closely related to Sangvinatsos and Wachter (2005) and Koijen et al. (2006). The main difference is that we allow all yields to be measured with error, following Brennan and Xia (2002) and Campbell and Viceira (2001), rather than assuming that some yields are measured without error. We assume that the measurement errors are independent, both sequentially and cross-sectionally. The continuous time equations underlying the financial market in Section 2 can be written as

$$
\mathrm{d}\begin{bmatrix} X_t \\ \log \Pi_t \\ \log S_t \\ D_t \end{bmatrix} = \left(\begin{bmatrix} 0_{2\times 1} \\ \delta_\pi - \frac{1}{2}\sigma_\Pi'\sigma_\Pi \\ \delta_R + \mu_0 - \frac{1}{2}\sigma_S'\sigma_S \\ \mu_D\kappa_D \end{bmatrix} + \begin{bmatrix} -K_X & 0_{2\times 2} & 0 \\ e_2' & 0_{1\times 2} & 0 \\ \left(\iota_2' - \sigma_\Pi'\tilde{\Lambda}_1 + \mu_{1(1:2)}'\right) & 0_{1\times 2} & \mu_{1(3)} \\ 0_{1\times 2} & 0_{1\times 2} & -\kappa_D \end{bmatrix} \begin{bmatrix} X_t \\ \log \Pi_t \\ \log S_t \\ D_t \end{bmatrix} \right) \mathrm{d}t
$$
$$
+ \begin{bmatrix} \Sigma_X \\ \sigma_\Pi' \\ \sigma_S' \\ \sigma_D' \end{bmatrix} \mathrm{d}Z_t
$$
$$
= (\Theta_0 + \Theta_1 K_t)\mathrm{d}t + \Sigma_K \mathrm{d}Z_t. \tag{B.1}
$$

with $K_t = (X_t', \log \Pi_t, \log S_t, D_t)'$ and $K_X \in \mathbb{R}^{2\times 2}$ is a diagonal matrix with elements κ_1 and κ_2. As K_t follows a standard multivariate multivariate Ornstein-Uhlenbeck process, we may write the exact h-period discretization (see for instance Sangvinatsos and Wachter (2005))

$$K_{t+h} = \mu^{(h)} + \Gamma^{(h)}K_t + \varepsilon_{t+h}, \tag{B.2}$$

where $\varepsilon_{t+h} \overset{i.i.d.}{\sim} N\left(0_{5\times 1}, \Sigma^{(h)}\right)$ for appropriate $\mu^{(h)}$, $\Gamma^{(h)}$, and $\Sigma^{(h)}$ which we derive below. To derive the discrete time parameters, we consider the eigenvalue decomposition $\Theta_1 = UDU^{-1}$. The

parameters in the VAR(1) - model relate to the structural parameters via

$$\Gamma^{(h)} = \exp\left(\Theta_1 h\right) = U \exp\left(Dh\right) U^{-1}, \tag{B.3}$$

$$\mu^{(h)} = \left[\int_t^{t+h} \exp\left(\Theta_1 \left[t + h - s\right]\right) ds\right] \Theta_0$$

$$= U F U^{-1} \Theta_0,$$

where F is a diagonal matrix with elements $F_{ii} = h\alpha\left(D_{ii}h\right)$, with

$$\alpha(x) = \frac{\exp(x) - 1}{x},$$

and $\alpha(0) = 1$. Finally. the derivation of $\Sigma^{(h)}$ is slightly more involved. We have

$$\Sigma^{(h)} = \int_t^{t+h} \exp\left(\Theta_1 \left[t + h - s\right]\right) \Sigma_K \Sigma_K' \exp\left(\Theta_1 \left[t + h - s\right]\right)' ds \tag{B.4}$$

$$= U V U',$$

where V is a matrix with elements

$$V_{ij} = \left[\int_t^{t+h} \exp\left(D\left[t + h - s\right]\right) U^{-1} \Sigma_K \Sigma_K' \left(U^{-1}\right)' \exp\left(D\left[t + h - s\right]\right) ds\right]_{ij} \tag{B.5}$$

$$= \left[U^{-1} \Sigma_K \Sigma_K' \left(U^{-1}\right)'\right]_{ij} \int_t^{t+h} \exp\left(\left[D_{ii} + D_{jj}\right]\left[t + h - s\right]\right) ds$$

$$= \left[U^{-1} \Sigma_K \Sigma_K' \left(U^{-1}\right)'\right]_{ij} h\alpha\left(\left[D_{ii} + D_{jj}\right] h\right).$$

Using data on six yields, stock returns, and inflation, we estimate the model using the Kalman filter. The transition equation is given by (B.2). We assume that all yields are measured with measurement error, in line with Brennan and Xia (2002) and Campbell and Viceira (2001). The likelihood can subsequently be constructed using the error-prediction decomposition, see for instance Harvey (1989).

C   Digression on the AIR

In this section, we succinctly summarize the role of the AIR in a simple model. Reducing (5), we find

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t, \tag{C.1}$$

with $\sigma = \| \sigma_S \|$ and Z a univariate Brownian motion. For the sake of exposition, we consider in this appendix that the remaining life-time of an individual of age T is exponentially distributed with parameter λ, implying for the survival probabilities

$$_sp_T = e^{-\lambda s}. \tag{C.2}$$

Therefore, we immediately have, with $s \geq 0$,

$$A^V(h, T) = \frac{1}{\lambda + h}, \tag{C.3}$$

$$I^V(h, T+s, T) = (\lambda + h) \exp\left(\left(\mu - \frac{1}{2}\sigma^2 - h\right) s + \sigma(Z_{T+s} - Z_T)\right), \tag{C.4}$$

see also Charupat and Milevsky (2002). The latter expression reveals that the choice of the AIR affects both the expectation as well as the dispersion of the payments provided by the variable annuity. The expectation and α-quantiles, Q_α, of the payout are given by

$$\mathbb{E}\left(I^V(h, T+s, T)\right) = (\lambda + h) \exp\left((\mu - h) s\right), \tag{C.5}$$

$$Q_\alpha = (\lambda + h) e^{(\mu - \frac{1}{2}\sigma^2 - h)s} e^{\sqrt{s}\sigma\Phi^{-1}(\alpha)}, \tag{C.6}$$

with Φ denoting the CDF of the standard normal distribution. Obviously, $\mu = h$ is the knife-edge case for which the expected income stream is constant. For $\mu > h$, the initial expected payout is low, but expected to increase during the retirement phase. The early payout is less risky than for $\mu = h$, but increases as the investor ages. On the contrary, for $\mu < h$, the initial payout is high, but expected to decrease. The quantiles for the initial payouts will be more wide-spread, but will widen less rapidly as the investor ages. In sum, a low AIR corresponds to low and not too risky initial payouts, but future payouts are expected to be higher and more risky. A high AIR corresponds to high and relatively risky initial payouts, but future payouts are likely to be lower, albeit less risky.

D   Optimal policies after retirement: numerical details

We discretize the consumption-savings problem after retirement (28) at an annual frequency. We thus consider

$$J(1, Y_T, 0, T) = \max_{(C_t)_{t \in (T, \infty)}} \mathbb{E}_T\left(\sum_{t=T+1}^{T_{\max}} {}_{t-T}p_T e^{-\beta(t-T)} \frac{\left(C_t^R\right)^{1-\gamma}}{1-\gamma}\right), \tag{D.1}$$

where we normalize real retirement wealth before annuitization to unity and $T_{\max} = 100$. The maximization is subject to the discretized real budget constraint

$$W_{t+1}^R = (W_t^R - C_t^R)R_{t+1} + Y_{t+1}^R, \ t = T, T+1, \ldots, \text{ and } W_T^R = 0, \tag{D.2}$$

with $Y_t^R = Y_t\Pi_T/\Pi_t$ indicating real annuity income, Y_t nominal annuity income, and $R_{t+1} = \Pi_t/(P(X_t,t,t+1)\Pi_{t+1})$ the real return on a nominal cash account. Further, we impose the liquidity constraint

$$C_t^R \le W_t^R, \tag{D.3}$$

which implies that the investor cannot capitalize future annuity income to increase today's consumption. Concerning the return on investment, we assume that the investor has access to a cash account, which is riskless in nominal terms, but risky in real terms.

This problem leads to the following Bellman equation, for $t \ge T$

$$J(W_t^R, Y_t, 0, t) = \max_{C_t^R} \left\{ \frac{\left(C_t^R\right)^{1-\gamma}}{1-\gamma} + \left(\frac{t+1p_T}{tp_T}\right)\mathbb{E}_t\left(e^{-\beta}J(W_{t+1}^R, Y_{t+1}, 0, t+1)\right) \right\}, \tag{D.4}$$

and

$$J(W_{T_{\max}}, Y_{T_{\max}}, 0, T_{\max}) = \frac{\left(C_{T_{\max}}^R\right)^{1-\gamma}}{1-\gamma}. \tag{D.5}$$

Applying the envelope theorem results in the optimal consumption

$$\left(C_t^{R\star}\right)^{-\gamma} = \left(\frac{t+1p_T}{tp_T}\right)\mathbb{E}_t\left(e^{-\beta}\left(C_{t+1}^{R\star}\right)^{-\gamma}R_{t+1}\right), \tag{D.6}$$

i.e.

$$C_t^{R\star} = \left\{\left(\frac{t+1p_T}{tp_T}\right)\mathbb{E}_t\left(e^{-\beta}\left(C_{t+1}^{R\star}\right)^{-\gamma}R_{t+1}\right)\right\}^{-1/\gamma}. \tag{D.7}$$

This result shows that we can determine the optimal consumption policy in a "myopic" fashion. The common procedure is to specify a grid over wealth and subsequently apply numerical dynamic programming to solve for the optimal policy. This results in a strategy $C_{t+1}^{R\star}(W_{t+1}^R)$ which can be used to determine the consumption policy in period t for a given initial wealth level W_t^R. However, solving for the optimal consumption policy then still entails solving for the root of (D.7) as W_{t+1}^R depends on C_t^R. Carroll (2006) provides an alternative approach to circumvent this problem. Carroll (2006) suggests to consider a grid in $a_t = W_t^R - C_t^R$ rather than W_t^R. The budget constraint

(D.2) now reads

$$W_{t+1}^R = R_{t+1}a_t + Y_{t+1}^R, \tag{D.8}$$

which no longer depends on C_t^R. As a consequence, once we can approximate the conditional expectation in (D.7), we can solve immediately for the optimal consumption policy

$$C_t^{R\star}(a_t) = \left\{ \left(\frac{t+1p_T}{tp_T} \right) \mathbb{E}_t \left(e^{-\beta} \left(C_{t+1}^{R\star} \right)^{-\gamma} R_{t+1} \right) \mid a_t \right\}^{-1/\gamma}. \tag{D.9}$$

Second, once we determine the optimal consumption policy on all m grid points, indicated by $a_1, \ldots, a_m$, we determine an endogenous wealth grid via

$$W_t^R = C_t^{R\star}(a_t) + a_t. \tag{D.10}$$

To approximate the conditional expectations we encounter, we consider expansions of the conditional expectations in a set of basis function of the state variables. This approach has been introduced by Brandt et al. (2005) for optimal portfolio choice. To review, we first simulate N trajectories and indicate the trajectories by $\omega_1, \ldots, \omega_N$. Second, select a grid of after consumption wealth a, indicated by $a_1, \ldots, a_m$.[18] At time T, the optimal policy is trivial and $C_{T_{\max}}^R = W_{T_{\max}}^R$, $\forall \omega_i$, implying $a_{T_{\max}} = 0$. At times $t = T+1, \ldots, T_{\max}-1$, we estimate the conditional expectation for each a_j using cross-sectional regressions. For wealth levels W_{t+1} in between the (endogenous) grid points, we employ linear interpolation. This leads to the optimal policy for each given a_j. Given all a_j, we determine the endogenous wealth grid W_{jt}^R, $j = 1, \ldots, m$. Along these lines, we proceed backwards. Since the conditional expectation in (D.9) should remain strictly positive, we approximate the conditional expectation not linearly in basis functions, but rather an exponentially affine combination of basis functions. We consider exponentially affine expansions in the real rate, expected inflation, the dividend yield, and log annuity income. The first three state variables are de-meaned and normalized by their (unconditional standard deviation). Higher order expansions hardly change the results at the reported precision.

This numerical procedure results in N trajectories of realized utility. These trajectories are in turn used to estimate the coefficients of the exponentially affine approximation of the value function at retirement. Further details on the numerical procedure are available upon request. Using this approximation, we are able to determine welfare costs of sub-optimal annuitization strategies and the optimal investment and consumption strategies in the period before retirement.

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[18]This grid possibly depends on time as well.

More specifically, we consider an approximation of the form[19]

$$[(1-\gamma)J(1,Y_T,0,T)]^{\frac{1}{\gamma-1}} \simeq \exp\left(\xi_0 + \xi_1'Y_T + \frac{1}{2}Y_T'\xi_2 Y_T\right) = K(Y_T). \qquad (D.11)$$

Using this approximation, the value function at retirement can be written conveniently as

$$J(W_T^R, Y_T, 0, T) = \frac{1}{1-\gamma}\left(\frac{W_T^R}{K(Y_T)}\right)^{1-\gamma}. \qquad (D.12)$$

The utility loss we report (ϕ) is defined as the decrease in certainty equivalent consumption

$$\phi(Y_T) = \left\{\frac{J^{\text{Opt}}(1,Y_T,0,T)}{J^{\text{Sub}}(1,Y_T,0,T)}\right\}^{1/(\gamma-1)} - 1. \qquad (D.13)$$

Alternatively, $\phi(Y_T)$ can be interpreted as the fraction of retirement wealth an investor is willing to give up in order to be able to implement the optimal conditional annuitization strategy. Since this measure depends on the state of the economy at retirement, we report its unconditional expectation.[20]

E Optimal policies before retirement: analytical solution

In this appendix we derive the optimal investment strategy in the period before retirement using dynamic programming. We initially solve the problem without labor income. The approximate objective function is given by

$$\mathbb{E}_0\left\{\frac{1}{1-\gamma}\left(\frac{W_T^R}{K(Y_T)}\right)^{1-\gamma}\right\}, \qquad (E.1)$$

with $W_T^R = W_T \Pi_T^{-1}$, subject to the dynamic budget constraint of real wealth

$$\frac{dW_t^R}{W_t^R} = \left(\delta_r + e_1'Y_t + \sigma_\Pi'(\sigma_\Pi - \Lambda_t) + x_t'\Sigma(\Lambda_t - \sigma_\Pi)\right)dt + \left(x_t'\Sigma - \sigma_\Pi'\right)dZ_t \qquad (E.2)$$

$$= \mu_{W^R}dt + \sigma_{W^R}'dZ_t, \qquad (E.3)$$

---

[19]This approximation can be shown to be highly accurate. However, the accuracy deteriorates for high risk aversion levels and annuity portfolios which are concentrated in nominal annuities. Results on the accuracy of the value function are available upon request.

[20]We use a lemma in Campbell et al. (2003) to determine $\mathbb{E}(\phi(Y_T))$.

with e_i denoting the i-th unit vector and $Y = (X_1, X_2, D)'$. The diffusion of the state vector is given by

$$dY_t = (\zeta_0 + \zeta_1 Y_t)\, dt + \Sigma_Y dZ_t, \tag{E.4}$$

with

$$\zeta_0 = \begin{pmatrix} 0 \\ 0 \\ \kappa_D \mu_D \end{pmatrix}, \quad \zeta_1 = \begin{pmatrix} -\kappa_1 & 0 & 0 \\ 0 & -\kappa_2 & 0 \\ 0 & 0 & -\kappa_D \end{pmatrix}, \quad \Sigma_Y = \begin{pmatrix} \sigma'_1 \\ \sigma'_2 \\ \sigma'_D \end{pmatrix}. \tag{E.5}$$

Using the definition of $K(Y)$ in (D.11), the dynamics of $K_t = K(Y_t)$ is given by

$$\begin{aligned} \frac{dK_t}{K_t} &= \left((\xi_1 + \xi_2 Y_t)'(\zeta_0 + \zeta_1 Y_t) + \frac{1}{2} tr\left(\Sigma'_Y \xi_2 \Sigma_Y\right) + \frac{1}{2}(\xi_1 + \xi_2 Y_t)' \Sigma_Y \Sigma'_Y (\xi_1 + \xi_2 Y_t) \right) dt \\ &\quad + (\xi_1 + \xi_2 Y_t)' \Sigma_Y dZ_t \end{aligned} \tag{E.6}$$

$$= \mu_K dt + \sigma'_K dZ_t. \tag{E.7}$$

We first of all derive the dynamics of wealth relative to K_t, which we denote by W^K. Next, we derive the optimal portfolio and the induced value function before retirement. Importantly, we show in (E.18) that the optimal portfolio policy is affine in the state variables, Y, and we introduce the notation

$$x_t = \alpha_0 + \alpha_1 Y_t. \tag{E.8}$$

The dynamics of scaled real wealth, W^K, is given by

$$\frac{dW_t^K}{W_t^K} = \left(\mu_{W^K}^C + \mu_{W^K}^Y Y_t + Y_t' \mu_{W^K}^{YY} Y_t\right) dt + \left(\sigma_{W^K}^C + \sigma_{W^K}^Y Y_t\right)' dZ_t. \tag{E.9}$$

with drift coefficients

$$\begin{aligned} \mu_{W^K}^C &= \delta_r + \left(\Sigma'\alpha_0 - \sigma_\Pi\right)'(\Lambda_0 - \sigma_\Pi) - \xi_1'\zeta_0 - \frac{1}{2}tr\left(\Sigma_Y \xi_2 \Sigma'_Y\right) + \frac{1}{2}\xi_1' \Sigma_Y \Sigma'_Y \xi_1 - \\ &\quad \xi_1' \Sigma_Y \left(\Sigma'\alpha_0 - \sigma_\Pi\right), \end{aligned} \tag{E.10}$$

$$\begin{aligned} \mu_{W^K}^Y &= e_1' - \sigma'_\Pi \Lambda_1 + \alpha_0' \Sigma \Lambda_1 + (\Lambda_0 - \sigma_\Pi)' \Sigma'\alpha_1 - \xi_1'\zeta_1 - \zeta_0'\xi_2 + \xi_1' \Sigma_Y \Sigma'_Y \xi_2 - \\ &\quad \xi_1' \Sigma_Y \Sigma'\alpha_1 - \left(\alpha_0'\Sigma - \sigma'_\Pi\right)\Sigma'_Y \xi_2, \end{aligned} \tag{E.11}$$

$$\mu_{W^K}^{YY} = \alpha_1' \Sigma \Lambda_1 - \xi_2'\zeta_1 + \frac{1}{2}\xi_2' \Sigma_Y \Sigma'_Y \xi_2 - \xi_2 \Sigma_Y \Sigma'\alpha_1, \tag{E.12}$$

and diffusion coefficients

$$\sigma^C_{WK} = \Sigma'\alpha_0 - \sigma_\Pi - \Sigma'_Y\xi_1, \tag{E.13}$$

$$\sigma^Y_{WK} = \Sigma'\alpha_1 - \Sigma'_Y\xi_2. \tag{E.14}$$

The value function is conjectured to be of the form

$$J(W^R_t, Y_t, t) = \frac{1}{1-\gamma}\left(\frac{W^R_t}{K(Y_t)}\right)^{1-\gamma}\exp\left(\Gamma_0(\tau) + \Gamma_1(\tau)'Y_t + \frac{1}{2}Y_t'\Gamma_2(\tau)Y_t\right), \tag{E.15}$$

with $\tau = T - t$ the remaining time up to retirement. The optimal investment policy is subsequently derived via the Hamilton-Jacobi-Bellman (HJB) equation

$$\sup_x \left(\begin{array}{c} J_{W^K}W^K\left(\mu^C_{WK} + \mu^Y_{WK}Y_t + Y_t'\mu^{YY}_{WK}Y_t\right) + J'_{W^KY}\Sigma_Y\left(\sigma^C_{WK} + \sigma^Y_{WK}Y_t\right) + \\ \frac{1}{2}J_{W^KW^K}W^{K2}\left(\sigma^C_{WK} + \sigma^Y_{WK}Y_t\right)'\left(\sigma^C_{WK} + \sigma^Y_{WK}Y_t\right) + J'_Y\left(\zeta_0 + \zeta_1 Y\right) + \\ \frac{1}{2}tr\left(\Sigma'_Y J_{YY}\Sigma_Y\right) + J_t \end{array} \right) = 0, \tag{E.16}$$

subject to the boundary condition

$$J(W^R_T, Y_T, T) = \frac{1}{1-\gamma}\left(\frac{W^R_T}{K(Y_T)}\right)^{1-\gamma}. \tag{E.17}$$

Using the first order conditions of (E.16) and the value function as given in (E.15), the optimal investment strategy is given by

$$\begin{aligned} x^\star_t &= \frac{1}{\gamma}\left(\Sigma\Sigma'\right)^{-1}\Sigma\Lambda_t + \left(1 - \frac{1}{\gamma}\right)\left(\Sigma\Sigma'\right)^{-1}\Sigma\left(\sigma_\Pi + \Sigma'_Y\left(\xi_1 + \xi_2 Y_t\right)\right) + \\ &\quad \frac{1}{\gamma}\left(\Sigma\Sigma'\right)^{-1}\Sigma\Sigma'_Y\left(\Gamma_1(\tau) + \frac{1}{2}\left(\Gamma_2(\tau) + \Gamma_2(\tau)'\right)Y_t\right), \end{aligned} \tag{E.18}$$

so that the coefficients in (E.8) are given by

$$\begin{aligned} \alpha_0 &= \frac{1}{\gamma}\left(\Sigma\Sigma'\right)^{-1}\Sigma\Lambda_0 + \left(1 - \frac{1}{\gamma}\right)\left(\Sigma\Sigma'\right)^{-1}\Sigma\left(\sigma_\Pi + \Sigma'_Y\xi_1\right) + \\ &\quad \frac{1}{\gamma}\left(\Sigma\Sigma'\right)^{-1}\Sigma\Sigma'_Y\Gamma_1(\tau), \end{aligned} \tag{E.19}$$

$$\begin{aligned} \alpha_1 &= \frac{1}{\gamma}\left(\Sigma\Sigma'\right)^{-1}\Sigma\Lambda_1 + \left(1 - \frac{1}{\gamma}\right)\left(\Sigma\Sigma'\right)^{-1}\Sigma\Sigma'_Y\xi_2 + \\ &\quad \frac{1}{2}\frac{1}{\gamma}\left(\Sigma\Sigma'\right)^{-1}\Sigma\Sigma'_Y\left(\Gamma_2(\tau) + \Gamma_2(\tau)'\right). \end{aligned} \tag{E.20}$$

Substitution of the optimal policy into the HJB-equation (E.16) implies for the coefficients of

the value function

$$
\begin{aligned}
\dot{\Gamma}_0(\tau) &= (1-\gamma)\mu_{W^K}^C - \frac{1}{2}\gamma(1-\gamma)\sigma_{W^K}^{C\prime}\sigma_{W^K}^C + \Gamma_1(\tau)'\zeta_0 + \frac{1}{2}\Gamma_1(\tau)'\Sigma_Y\Sigma_Y'\Gamma_1(\tau) + \\
&\quad \frac{1}{4}tr\left(\Sigma_Y'\left(\Gamma_2(\tau)+\Gamma_2(\tau)'\right)\Sigma_Y\right) + (1-\gamma)\Gamma_1(\tau)'\Sigma_Y\sigma_{W^K}^C, \quad\quad\quad (E.21)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\Gamma}_1(\tau)' &= (1-\gamma)\mu_{W^K}^Y - \gamma(1-\gamma)\sigma_{W^K}^{C\prime}\sigma_{W^K}^Y + \Gamma_1(\tau)'\zeta_1 + \frac{1}{2}\zeta_0'\left(\Gamma_2(\tau)+\Gamma_2(\tau)'\right) + \\
&\quad \frac{1}{2}\Gamma_1(\tau)'\Sigma_Y\Sigma_Y'\left(\Gamma_2(\tau)+\Gamma_2(\tau)'\right) + (1-\gamma)\Gamma_1(\tau)'\Sigma_Y\sigma_{W^K}^Y + \\
&\quad \frac{1}{2}(1-\gamma)\sigma_{W^K}^{C\prime}\Sigma_Y'\left(\Gamma_2(\tau)+\Gamma_2(\tau)'\right) \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (E.22)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\Gamma}_2(\tau) &= 2(1-\gamma)\mu_{W^K}^{YY} - \gamma(1-\gamma)\sigma_{W^K}^{Y\prime}\sigma_{W^K}^Y + \left(\Gamma_2(\tau)+\Gamma_2(\tau)'\right)\zeta_1 + \\
&\quad \frac{1}{4}\left(\Gamma_2(\tau)+\Gamma_2(\tau)'\right)\Sigma_Y\Sigma_Y'\left(\Gamma_2(\tau)+\Gamma_2(\tau)'\right) + \\
&\quad (1-\gamma)\left(\Gamma_2(\tau)+\Gamma_2(\tau)'\right)\Sigma_Y\sigma_{W^K}^Y, \quad\quad\quad\quad\quad\quad\quad\quad (E.23)
\end{aligned}
$$

subject to the boundary conditions $\Gamma_0(0) = 0$, $\Gamma_1(0) = 0$, $\Gamma_2(0) = 0$. Note that the value function corresponding to sub-optimal strategies satisfies the same ODEs, although the expressions for α_0 and α_1 should be modified.

In order to solve the investment problem with labor income, Bodie et al. (1992) and Munk and Sørensen (2005) have shown that it is possible to recast this problem to one without labor income and initial real wealth $W_t^R + H_t$. On the one hand, the optimal exposures to the risk factors of total real wealth at time t are given by $(W_t^R + H_t)(x_t^{\star\prime}\Sigma - \sigma_\Pi')$. On the other hand, for any investment strategy, $\bar{x}_t$, we can derive another expression for the diffusion coefficient of real total wealth using (20) together with the dynamic of realized inflation in (4) to determine the dynamics of real financial wealth and (31) for the dynamics of the real present value of the savings stream, namely $W_t^R\left(\bar{x}_t'\Sigma - \sigma_\Pi\right) + H_{Yt}'\Sigma_Y$. By equating both diffusion coefficients and solving for the optimal portfolio $\bar{x}_t$, we immediately obtain the expression provided in (32).[21]

Next, we solve for the optimal fixed consumption rate, θ. The value function at time $t = t_0$ induced by the optimal investment strategy is given by

$$
\begin{aligned}
J(W_{t_0}, Y_{t_0}, L_{t_0}, t_0) &= \max_\theta \int_{t_0}^T e^{-\beta s}\frac{(\theta L_s)^{1-\gamma}}{1-\gamma}ds + \frac{e^{-\beta T}}{1-\gamma}\left(\frac{W_{t_0}^R + H(L_{t_0}, Y_{t_0}, \theta, t_0, T)}{K(Y_{t_0})}\right)^{1-\gamma} \times \\
&\quad \exp\left(\Gamma_0(\tau) + \Gamma_1(\tau)'Y_{t_0} + \frac{1}{2}Y_{t_0}'\Gamma_2(\tau)Y_{t_0}\right). \quad\quad\quad (E.24)
\end{aligned}
$$

where $\tau = T - t_0$. Using the definition of H_t in (31), it follows immediately that the optimal

---

[21]In the main text, the optimal portfolio in the presence of labor income is indicated by $x_t^\star$, with slight abuse of notation.

consumption rate is given by

$$\theta^\star = \frac{\left(\frac{\rho_1}{\rho_2}\right)^{\frac{1}{\gamma}}}{1 + \left(\frac{\rho_1}{\rho_2}\right)^{\frac{1}{\gamma}}}, \tag{E.25}$$

with, assuming $W_{t_0}^R = 0$,

$$\rho_1 = \int_{t_0}^{T} e^{-\beta s} (L_s)^{1-\gamma} \, ds, \tag{E.26}$$

$$\rho_2 = e^{-\beta T} \left(\frac{H(L_{t_0}, Y_{t_0}, 0, t_0, T)}{K(Y_{t_0})} \right)^{1-\gamma} \exp\left(\Gamma_0(\tau) + \Gamma_1(\tau)' Y_{t_0} + \frac{1}{2} Y_{t_0}' \Gamma_2(\tau) Y_{t_0} \right), \tag{E.27}$$

and $\tau = T - t_0$. We will assume throughout that initial wealth, i.e. $W_{t_0}^R$, equals zero.

Finally, using the expressions for the value function in (E.24), we can determine the fraction of retirement wealth an investor is willing to give up to hedge annuity risk is given by, with $\tau = T - t_0$,

$$\phi(Y_{t_0}) = \tag{E.28}$$
$$\exp\left(\Gamma_0^{\text{Opt}}(\tau) - \Gamma_0^{\text{Sub}}(\tau) + \left(\Gamma_1^{\text{Opt}}(\tau) - \Gamma_1^{\text{Sub}}(\tau) \right)' Y_{t_0} + \frac{1}{2} Y_{t_0}' \left(\Gamma_2^{\text{Opt}}(\tau) - \Gamma_2^{\text{Sub}}(\tau) \right) Y_{t_0} \right)^{1/(\gamma-1)} - 1,$$

where the coefficients with superscripts 'Opt' originate from the value function corresponding to the optimal strategy which anticipates the investor's desire to annuitize wealth at retirement. Likewise, the superscripts 'Sub' correspond to the value function generated by a strategy which perceives ξ_1 and ξ_2 to be equal to zero, see (32). The welfare costs are calculated assuming that the initial state vector, Y_{t_0}, equals its unconditional expectation.

F   Tables and figures

Parameter	Estimate	Parameter	Estimate

Means nominal short rate and expected inflation: $\mathbb{E}\left(R_t\right) = \delta_R$, $\mathbb{E}\left(\pi_t\right) = \delta_\pi$

Parameter	Estimate	Parameter	Estimate
δ_R	4.69%	δ_π	3.02%

Process real interest rate and expected inflation: $\mathrm{d}X_{it} = -\kappa_i X_{it}\mathrm{d}t + \sigma_i'\mathrm{d}Z_t$

Parameter	Estimate	Parameter	Estimate
κ_1	1.07	σ_1	1.98%
κ_2	0.08	σ_{12}	-0.13%
		σ_2	1.05%

Realized inflation process: $\mathrm{d}\Pi_t/\Pi_t = \pi_t\mathrm{d}t + \sigma_\Pi'\mathrm{d}Z_t$

Parameter	Estimate	Parameter	Estimate
$\sigma_{\Pi(1)}$	0.10%	$\sigma_{\Pi(3)}$	1.08%
$\sigma_{\Pi(2)}$	0.18%		

Stock return process: $\mathrm{d}S_t/S_t = (R_t + \mu_0 + \mu_1'Y_t)\mathrm{d}t + \sigma_S'\mathrm{d}Z_t$

Parameter	Estimate	Parameter	Estimate
μ_0	0.44	$\sigma_{S(1)}$	-1.50%
$\mu_{1(1)}$	-0.56	$\sigma_{S(2)}$	-2.49%
$\mu_{1(2)}$	-0.80	$\sigma_{S(3)}$	-1.53%
$\mu_{1(3)}$	0.11	$\sigma_{S(4)}$	14.48%

Prices of real rate and inflation risk: $\Lambda_t = \Lambda_0 + \Lambda_1 Y_t$

Parameter	Estimate	Parameter	Estimate
$\Lambda_{0(1)}$	-0.35	$\Lambda_{1(1,1)}$	-26.01
$\Lambda_{0(2)}$	-0.13	$\Lambda_{1(2,2)}$	-7.07

Dividend yield process: $\mathrm{d}D_t = \kappa_D(\mu_D - D_t)\mathrm{d}t + \sigma_D'\mathrm{d}Z_t$

Parameter	Estimate	Parameter	Estimate
κ_D	0.05	$\sigma_{D(3)}$	1.48%
μ_D	-3.50	$\sigma_{D(4)}$	-14.40%
$\sigma_{D(1)}$	1.73%	$\sigma_{D(5)}$	3.73%
$\sigma_{D(2)}$	2.44%		

Table 1: Estimation results for the financial market model
Parameter estimates of the financial market model. The model is estimated using monthly data on six bond yields, inflation, and stock returns over the period from January 1952 up to May 2002. Details on the estimation procedure are provided in Appendix B.

$\gamma = 2$	Constant	Real rate	Exp. inflation	Div. yield
Nominal annuity	12%	-1%	12%	-10%
Inflation-linked annuity	23%	2%	2%	-26%
Variable annuity	65%	-2%	-14%	36%
$\gamma = 5$	Constant	Real rate	Exp. inflation	Div. yield
Nominal annuity	8%	0%	8%	-1%
Inflation-linked annuity	50%	2%	3%	-27%
Variable annuity	42%	-2%	-11%	28%
$\gamma = 10$	Constant	Real rate	Exp. inflation	Div. yield
Nominal annuity	6%	0%	5%	2%
Inflation-linked annuity	70%	1%	2%	-20%
Variable annuity	25%	-1%	-7%	18%

Table 2: Optimal retirement choice
Optimal annuity choice *conditional* on the economic conditions at retirement. The optimal allocation to nominal, inflation-linked, and variable annuities with an AIR of $h = 4\%$ is presented. We present the coefficients of a regression of the optimal allocation to the three annuity products on the three state variables. The time preference parameter equals $\beta = 0.04$ and the coefficient of relative risk aversion equals $\gamma = 2$, 5, or 10. The main text provides further details.

	$\gamma = 2$	$\gamma = 5$	$\gamma = 10$
Nominal annuity	0%	0%	0%
Inflation-linked annuity	30%	60%	75%
Variable annuity	70%	40%	25%

Table 3: Optimal unconditional retirement choice
Optimal *unconditional* annuity choice at retirement. The optimal allocation to nominal, inflation-linked, and variable annuities with an AIR of $h = 4\%$ is presented. The time preference parameter equals $\beta = 0.04$ and the coefficient of relative risk aversion equals $\gamma = 2$, 5, or 10. The main text provides further details.

	$\gamma = 2$	$\gamma = 5$	$\gamma = 10$
Optimal unconditional	-8.91%	-8.64%	-6.88%
Inflation-linked	-18.85%	-13.89%	-9.91%
Nominal	-22.38%	-27.51%	-54.84%
Linear rule	-2.10%	-0.47%	-0.29%

Table 4: Welfare costs of sub-optimal annuitization strategies

Welfare costs of *sub-optimal* annuitization strategies relative to the optimal conditional annuitization strategy. Welfare costs are determined as the decrease in certainty equivalent consumption during retirement. As this welfare metric depends on the state of the economy, we report unconditional expectations. The sub-optimal annuitization strategies are either not using conditioning information ('Optimal unconditional'), investing all wealth in either inflation-linked ('Inflation-linked') or nominal ('Nominal') annuities, or the linear rule based on the first order approximation of the optimal conditional annuitization strategy. The time preference parameter equals $\beta = 0.04$ and the coefficient of relative risk aversion equals $\gamma = 2$, 5, or 10. The main text provides further details.

	$\gamma = 2$	$\gamma = 5$	$\gamma = 10$
Optimal conditional	-2.03%	-9.20%	-11.98%
Optimal unconditional	-2.43%	-9.12%	-13.35%
Inflation-linked	-0.02%	-0.13%	-0.35%
Nominal	-0.10%	-0.59%	-1.79%

Table 5: Welfare costs of not hedging annuity risk before retirement

Welfare costs of not hedging annuity risk before retirement. Welfare costs are determined as the amount of wealth an investor is willing to give up in order to follow the optimal investment strategy. The reported numbers are based on the condition that the initial vector of state variables, Y_{t_0}, equals its unconditional expectation. Apart from the optimal annuitization strategy ('Optimal conditional'), the sub-optimal annuitization strategies either ignore information on the term structure and risk premia ('Optimal unconditional') or invest all wealth in either inflation-linked ('Inflation-linked') or nominal ('Nominal') annuities. The state variables are set to their unconditional expectation. The time preference parameter equals $\beta = 0.04$ and the coefficient of relative risk aversion ranges from $\gamma = 2$, 5 or 10.
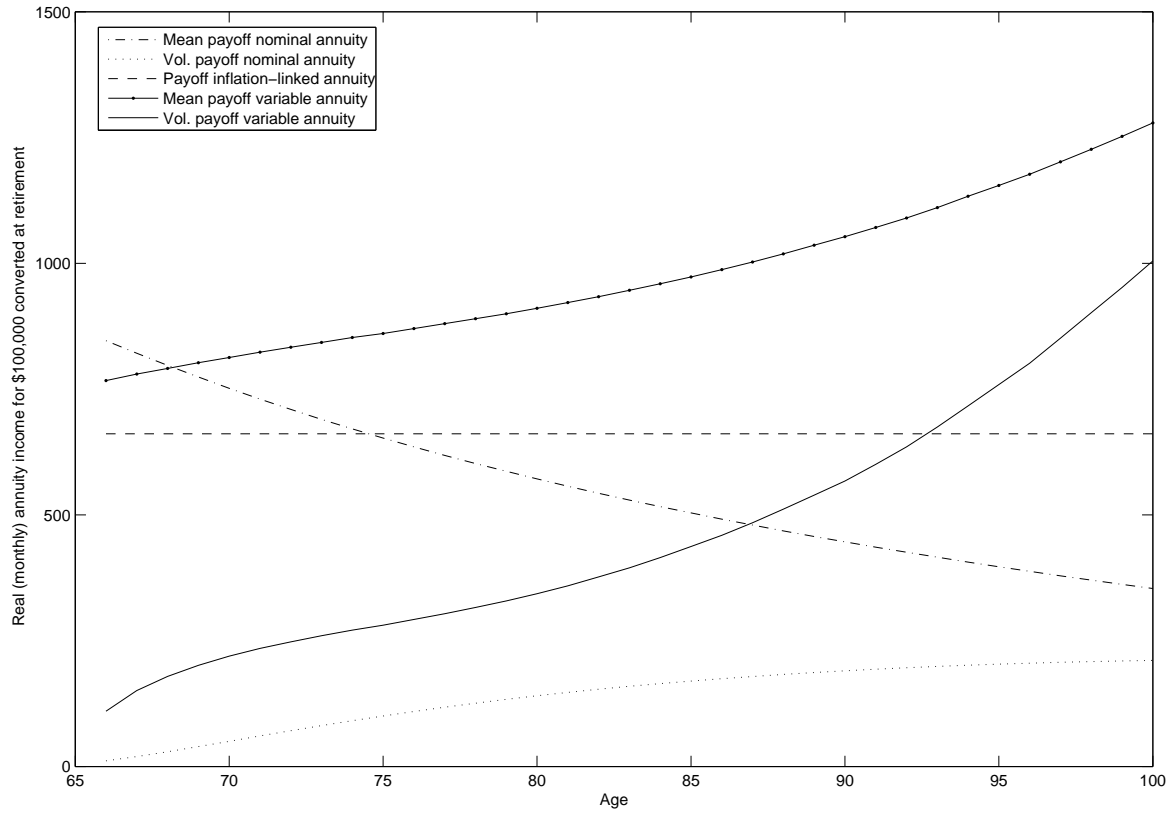
Figure 1: Mean and volatility of real (monthly) annuity benefits provided by various annuities
Mean and volatility of real (monthly) annuity benefits provided by nominal, inflation-linked, and variable annuities ($h = 4\%$) for \$100,000 converted at retirement. The real income stream provided by an inflation-linked annuity is by definition constant and therefore only its level is reported. The horizontal axis depicts the investor's age and the vertical axis indicates the level of the real annuity payout. The state variables equal their unconditional expectation.
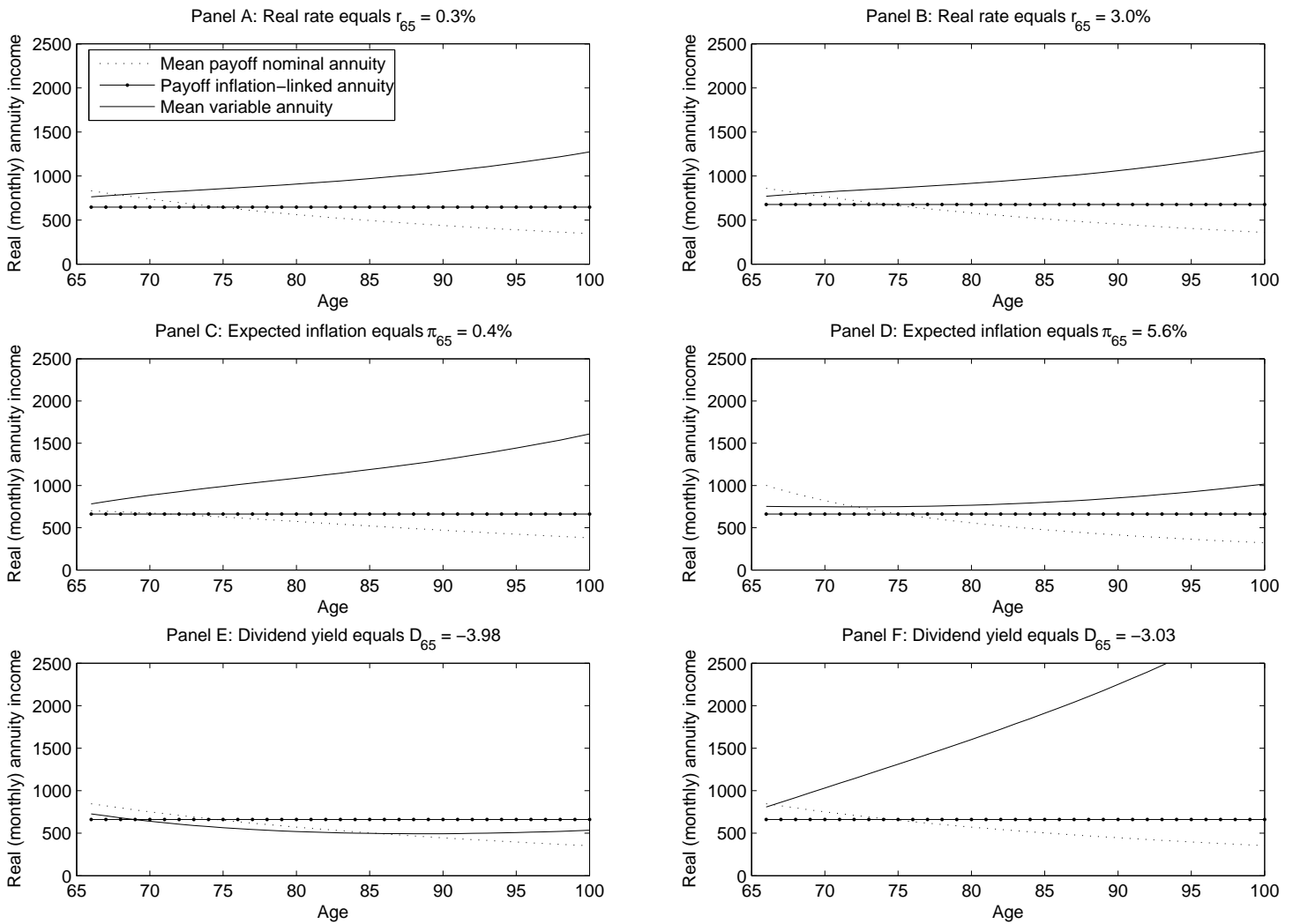
Figure 2: Mean real (monthly) annuity benefits provided by nominal, inflation-linked, and variable annuities for different economic conditions

Mean real (monthly) annuity benefits provided by nominal, inflation-linked, and variable annuities for various economic conditions at retirement if the investor converts $100,000 into annuities. The top panels portray the characteristics of the income stream when the real rate is either one (unconditional) standard deviations below (Panel A) or above (Panel B) its unconditional expectation. The middle panels display the results when expected inflation is either one (unconditional) standard deviations below (Panel C) or above (Panel D) its unconditional expectation. Likewise, the bottom panels present the characteristics of the income stream for an initial level of the dividend yield which is one (unconditional) standard deviations below (Panel E) or above (Panel F) its unconditional expectation. The horizontal axis depicts the investor's age and the vertical axis indicates the level of the real annuity payout.
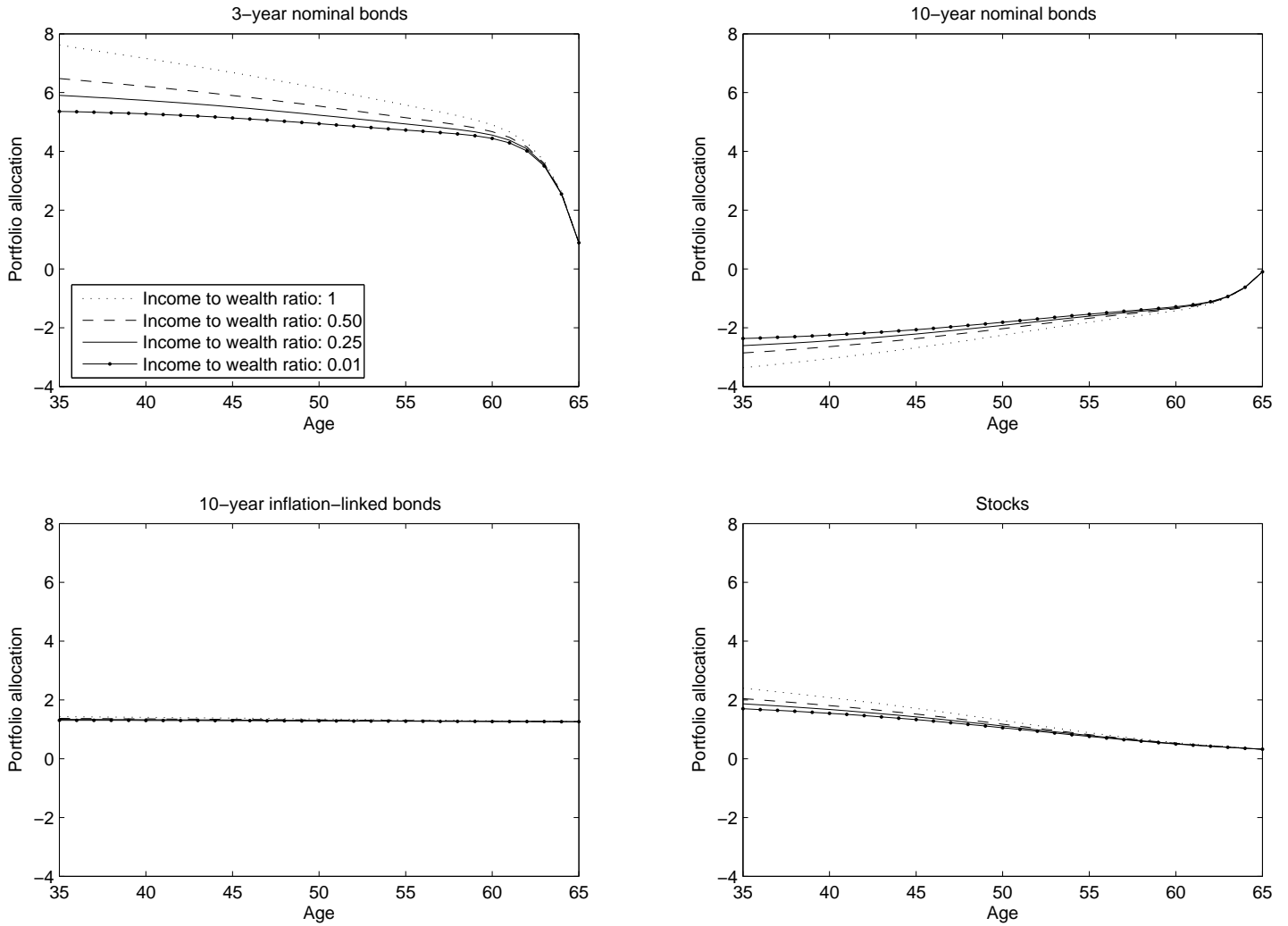
Figure 3: Optimal portfolio choice before retirement without annuity risk
Optimal portfolio choice before retirement without annuity risk using 3-year and 10-year nominal bonds, 10-year inflation-linked bonds, and stocks. The remainder is invested in a nominal cash account. The different lines in each of the graphs correspond to income to wealth ratios (L_t/W_t^R) of 1, 0.5, 0.25, and 0.01. The investor's coefficient of relative risk aversion equals $\gamma = 5$. In all graphs, the horizontal axis depicts the investor's age and the vertical axis indicates the optimal allocation to a particular asset. The main text provides further details.
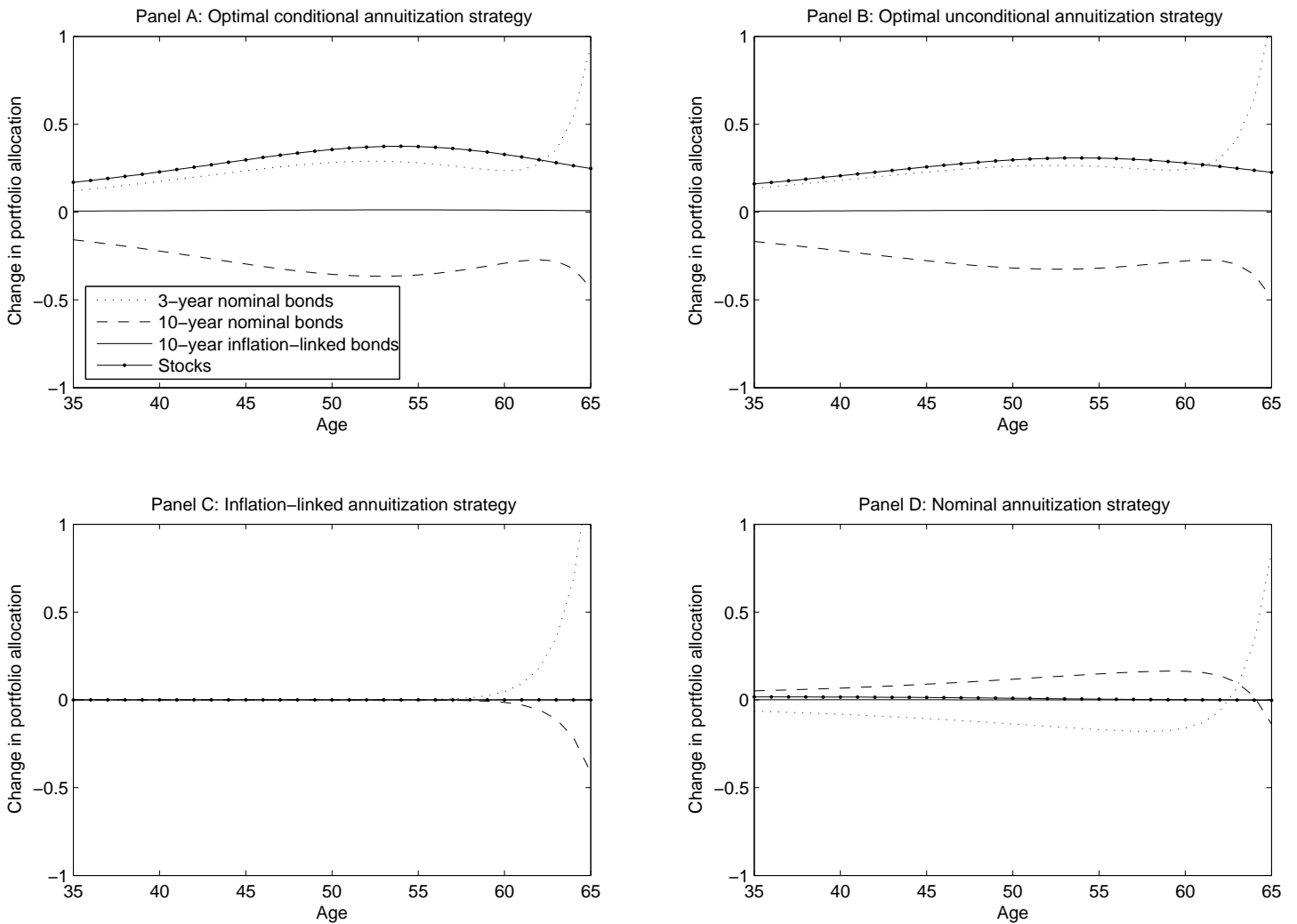
Figure 4: Optimal hedging strategy before retirement corresponding to four annuitization strategies
Optimal hedging strategy before retirement to hedge the annuity risk caused by the optimal conditional annuitization strategy (Panel A), the optimal unconditional annuitization strategy (Panel B), inflation-linked annuitization (Panel C), and finally nominal annuitization (Panel D). The optimal hedging strategy is defined the difference between the optimal investment strategy which does and does not account for annuity risk. The asset menu contains 3-year and 10-year nominal bonds, 10-year inflation-linked bonds, and stocks. The remainder is invested in a nominal cash account. The income-to-wealth ratio is set to 0.5 and the investor's coefficient of relative risk aversion equals $\gamma = 5$. In all graphs, the horizontal axis depicts the investor's age and the vertical axis indicates the optimal allocation to a particular asset.
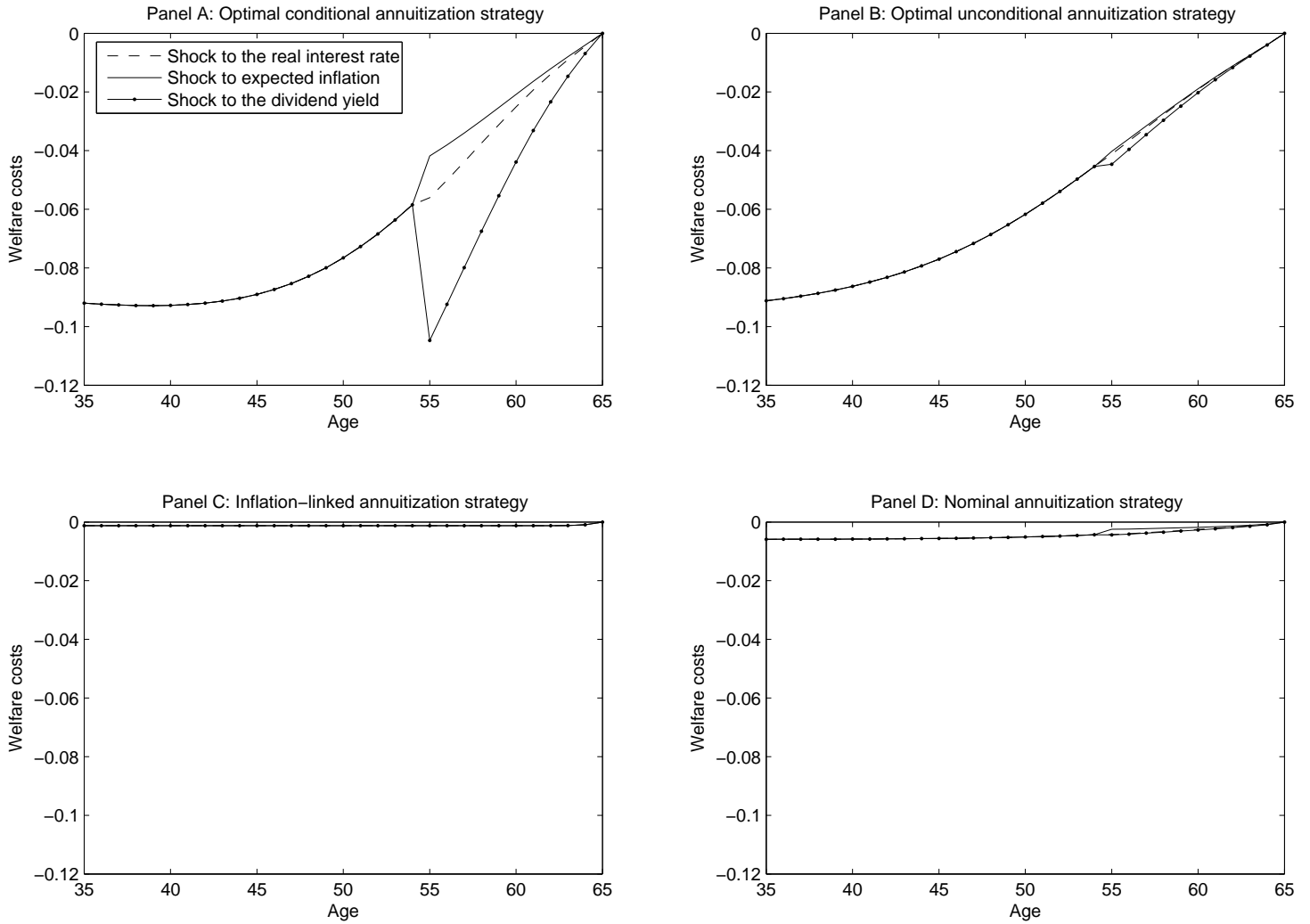
Figure 5: Welfare costs of not hedging annuity risk before retirement for different state variables
Welfare costs of not hedging annuity risk before retirement. Welfare costs are determined as the amount of wealth an investor is willing to give up in order to follow the optimal investment strategy. The reported numbers are based on the condition that the initial vector of state variables, Y_{t_0}, equals its unconditional expectation. Apart from the optimal annuitization strategy ('Optimal conditional'), the sub-optimal annuitization strategies either ignore information on the term structure and risk premia ('Optimal unconditional') or invest all wealth in either inflation-linked ('Inflation-linked') or nominal ('Nominal') annuities. The state variables are set to their unconditional expectation up to age 55, perturbed by a one (unconditional) standard deviation, positive shock at age 55, and their expected value, conditional upon the shock, afterwards. The time preference parameter equals $\beta = 0.04$ and the coefficient of relative risk aversion ranges from $\gamma = 2$, 5 or 10. The main text provides further details.

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