CentER

Discussion Paper

No. 2003–11

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February 2003

ISSN 0924-7815

TILBURG UNIVERSITY

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Abstract

In this paper we study the relation between convexity of TU games and marginal vectors. We show that if specific marginal vectors are core elements, then the game is convex. We characterize sets of marginal vectors satisfying this property, and we derive the formula for the minimum number of marginal vectors in such sets.

Keywords: convexity, marginal vector, TU game

1 Introduction

This paper studies the relation between convexity of TU games and marginal vectors. Shapley (1971) and Ichiishi (1981) showed that a game is convex if and only if all marginal vectors are core elements. In Rafels, Ybern (1995) it is shown that if all even marginal vectors are core elements, then all odd marginal vectors are core elements as well, and vice versa. Hence, if all even or all odd marginal vectors are core elements, then the game is convex. In Van Velzen, Hamers, Norde (2002) other sets of marginal vectors are constructed such that the requirement that these marginal vectors are core elements is a sufficient condition for convexity of a game. This construction is based on a neighbour argument, i.e. it is shown that if two specific neighbours of a marginal vector are core elements, then this marginal vector is a core element as well. In this way they characterize convexity using a fraction of the total number of marginal vectors. Moreover, they show that this fraction converges to zero.

In this paper we use combinatorial arguments to obtain sets of marginal vectors that characterize convexity. We characterize the sets of marginal vectors satisfying this property. Furthermore we present the formula for the minimum cardinality of sets of marginal vectors that characterize convexity.

In cooperative game theory, convexity is studied because for convex games many solution concepts possess nice properties. For instance, it is established that the core is nonempty, that the Shapley value is the barycentre of the core and that the set of marginal vectors coincides with the set of extreme points of the core (Shapley (1971)). Furthermore it is shown that the bargaining set and the core coincide as well as the kernel and the nucleolus (Maschler, Peleg, Shapley (1972)) and that the τ-value can easily be calculated (Tijs (1981)). There are several classes of cooperative games that are included in the class of convex games. For instance, the class of convex games contains bankruptcy games (Aumann, Maschler (1985), Curiel, Maschler, Tijs (1987)), sequencing games (Curiel, Pederzoli, Tijs (1989)), airport games (Littlechild, Owen (1973)) and standard fixed tree games (Granot, Maschler, Owen, Zhu (1996)). Convexity is also characterized for chinese postman games (Granot, Hamers (2000)) and travelling salesman games (Granot, Granot, Zhu (2000)).

2 Preliminaries

In this section we recall some notions from cooperative game theory and introduce some notation.

A *cooperative TU game* is a pair (N, v) where $N = \{1, \ldots, n\}$ is a finite (player-)set and the characteristic function $v : 2^N \to \mathbb{R}$ assigns to each subset $S \subset N$, called a coalition, a real number $v(S)$, called the worth of coalition S, where $v(\emptyset) = 0$. The *core* of a game (N, v) is the set of payoff vectors for which no coalition has an incentive to leave the grand coalition N, i.e.

$$C(v) = \{x \in \mathbb{R}^N | \sum_{j \in S} x_j \geq v(S) \text{ for every } S \subset N, \sum_{j \in N} x_j = v(N)\}.$$

Note that the core of a game can be empty. A game (N, v) is called *convex* if the marginal contribution of any player to any coalition is less than his marginal contribution to a larger coalition, i.e. if it holds that

$$v(S \cup \{i\}) - v(S) \leq v(S \cup \{i, j\}) - v(S \cup \{j\}) \quad \text{for all } i, j \in N, i \neq j, \text{ and } S \subset N \backslash \{i, j\}. \tag{1}$$

Before we introduce marginal vectors, we first introduce orders. An *order* σ of N is a bijection $\sigma : \{1, \ldots, n\} \to N$. This order is denoted by $\sigma(1) \cdots \sigma(n)$, where $\sigma(i) = j$ means that with respect to σ, player j is in the i-th position. An order is called *even* if it can be turned into the identity order e by an even number of neighbourswitches, where e is such that $e(i) = i$ for every $i \in \{1, \ldots, n\}$. An order which is not even is called *odd*. Note that the set of all orders S_n contains as many even orders as odd orders.

For notational convenience we introduce a special set of orders. Let $i, j \in N$ with $i \neq j$, and let $S \subset N \backslash \{i, j\}$. Then $M(S, \{i, j\})$ contains those orders which begin with the players in S, followed by the players in $\{i, j\}$ and end with the players in $N \backslash (S \cup \{i, j\})$, i.e.

$$\begin{aligned} M(S, \{i, j\}) \quad &= \{\sigma \in S_n : \quad \sigma(k) \in S \text{ for } 1 \leq k \leq |S|, \sigma(k) \in \{i, j\} \text{ for } |S| + 1 \leq k \leq |S| + 2 \\ &\qquad \text{and } \sigma(k) \in N \backslash (S \cup \{i, j\}) \text{ for } |S| + 3 \leq k \leq n\}. \end{aligned}$$

Note that we allow for $S = \emptyset$ and $S = N \backslash \{i, j\}$.

Example 1 Let $N = \{1, 2, 3, 4, 5\}$, $S = \{3\}$ and $\{i, j\} = \{1, 5\}$. Now $M(\{3\}, \{1, 5\}) = \{31524, 31542, 35124, 35142\}$.

For each $k \in \{0, \ldots, n-2\}$ let $G_n(k)$ consist of those $M(S, \{i, j\})$ with S containing k players, i.e.

$$G_n(k) = \{M(S, \{i, j\}) : i, j \in N \text{ with } i \neq j, S \subset N \backslash \{i, j\} \text{ and } |S| = k\}.$$

Let $k \in \{0, \ldots, n-2\}$. Obviously for each $\sigma \in S_n$ it holds that there is precisely one $M(S, \{i, j\}) \in G_n(k)$ such that $\sigma \in M(S, \{i, j\})$, i.e. $G_n(k)$ is a partitioning of S_n. Furthermore we have that $|G_n(k)| = \binom{n}{k}\binom{n-k}{2}$.

Let (N, v) be a game. For $\sigma \in S_n$, the *marginal vector* $m^\sigma(v)$ is defined by

$$m_i^\sigma(v) = v([i, \sigma]) - v((i, \sigma)) \text{ for all } i \in N,$$

where $[i, \sigma] = \{j \in N : \sigma^{-1}(j) \leq \sigma^{-1}(i)\}$ is the set of predecessors of i with respect to σ including i, and $(i, \sigma) = \{j \in N : \sigma^{-1}(j) < \sigma^{-1}(i)\}$ is the set of predecessors of i with respect to σ excluding i. A marginal vector is called even (odd) if the corresponding order is even (odd).

3 Characterizing convexity of games with marginal vectors

In this section we present our main results. First we recall theorems which deal with the relation between convexity and marginal vectors. Secondly, we characterize sets of orders for which the requirement that the marginal vectors corresponding to the orders in such a set are core elements is a sufficient condition for all marginal vectors to be core elements. Finally we derive the formula for the minimum cardinality of a set of orders satisfying this property.

The following well-known theorem deals with the relation between marginal vectors and convex games. It states that a game is convex if and only if all marginal vectors are core elements.

Theorem 1 ***(Shapley (1971), Ichiishi (1981))*** *Let (N, v) be a game. It holds that (N, v) is convex if and only if $m^\sigma(v) \in C(v)$ for all $\sigma \in S_n$.*

The following theorem also deals with the relation between marginal vectors and convexity of games. It states that the even marginal vectors, as well as the odd marginal vectors, characterize convexity. In other words, if all even marginal vectors or all odd marginal vectors are core elements, then the game is convex.

Theorem 2 ***(Rafels, Ybern (1995))*** *For every game (N, v) the following statements are equivalent:*

1. *(N, v) is convex,*
2. *$m^\sigma(v) \in C(v)$ for all even $\sigma \in S_n$,*
3. *$m^\sigma(v) \in C(v)$ for all odd $\sigma \in S_n$.*

Van Velzen et al.(2002) construct other sets of marginal vectors that characterize convexity. This construction is based on a neighbour argument. It is shown that if two specific neighbours of some marginal vector are core elements, then this marginal vector is a core element as well.

Sets of marginal vectors that characterize convexity are baptized *complete* sets. In other words, a set $A \subset S_n$ is called complete if for every game (N, v) the following statements are equivalent:

1. (N, v) is convex,
2. $m^\sigma(v) \in C(v)$ for all $\sigma \in A$.

Theorem 2 states that the set of even orders and the set of odd orders are complete sets. Let M_n be the minimum cardinality of a complete set, i.e.

$$M_n = \min_{A \subset S_n:\ A \text{ is complete}} |A|,$$

and let the fraction of M_n with respect to all orders be denoted by $g_n = \frac{M_n}{n!}$.

From Theorem 2 we obtain that $M_n \leq \frac{1}{2}n!$. Van Velzen et al.(2002) show that for $n \geq 5$ it holds that $M_n \leq \frac{1}{4}n!$ and that $g_n \to 0$ if $n \to \infty$. However, they only provide upper bounds for M_n and g_n. Some of these upper bounds are presented in Table 1.

n	3	4	5	6	7	8	9	10
$n!$	6	24	120	720	5040	40320	362880	3628800
M_n	≤ 3	≤ 12	≤ 30	≤ 180	≤ 1260	≤ 5040	≤ 45360	≤ 226800
g_n	$\leq \frac{1}{2}$	$\leq \frac{1}{2}$	$\leq \frac{1}{4}$	$\leq \frac{1}{4}$	$\leq \frac{1}{4}$	$\leq \frac{1}{8}$	$\leq \frac{1}{8}$	$\leq \frac{1}{16}$

Table 1: Upper bounds for M_n and g_n.

To obtain the main result of this paper we need the following characterization of complete sets.

Lemma 1 *Let $A \subset S_n$. Then A is complete if and only if*

$$A \cap M(S, \{i, j\}) \neq \emptyset \qquad \text{for all } i, j \in N, i \neq j, \text{ and } S \subset N \backslash \{i, j\}. \tag{2}$$

Proof: First we show the "if" part. Let A be such that (2) holds, and let (N, v) be a game. Assume that $m^\sigma(v) \in C(v)$ for each $\sigma \in A$. Let $i, j \in N$ with $i \neq j$, and let $S \subset N \backslash \{i, j\}$. Then there is a $\sigma \in A$ such that $\sigma \in M(S, \{i, j\})$. Without loss of generality assume that $\sigma(|S| + 1) = i$ and $\sigma(|S| + 2) = j$. It follows that

$$v(S \cup \{j\}) \leq \sum_{k \in S \cup \{j\}} m_k^\sigma(v) = v(S \cup \{i, j\}) - v(S \cup \{i\}) + v(S),$$

where the inequality holds because $m^\sigma(v) \in C(v)$. Therefore (1) holds and hence (N, v) is convex. Using Theorem 1 we obtain that $m^\sigma(v) \in C(v)$ for all $\sigma \in S_n$. We conclude that A is complete.

To show the "only if" part, suppose that (2) does not hold. We will show that A is not complete by constructing a nonconvex game for which all marginal vectors corresponding to orders in A are core elements.

Because (2) does not hold, there are $i, j \in N$ with $i \neq j$, and $S \subset N\backslash\{i,j\}$ such that $A \cap M(S, \{i,j\}) = \emptyset$. Let

$$v(T) = \begin{cases} 1 & \text{if} \quad T = S \cup \{i\}, S \cup \{j\} \\ \max(0, |T| - |S| - 1) & \text{otherwise.} \end{cases} \tag{3}$$

We will show that (N, v) is such that $m^\sigma(v) \in C(v)$ if and only if $\sigma \notin M(S, \{i,j\})$.

Let $\sigma \in M(S, \{i,j\})$. Without loss of generality assume that σ is such that $\sigma(|S|+1) = i$ and $\sigma(|S|+2) = j$. It follows that $\sum_{k \in S \cup \{j\}} m_k^\sigma(v) = v(S \cup \{i,j\}) - v(S \cup \{i\}) + v(S) = 1 - 1 + 0 < 1 = v(S \cup \{j\})$. Hence, $m^\sigma(v) \notin C(v)$. We now show that the other marginal vectors are core elements.

Let $\sigma \notin M(S, \{i,j\})$. For all $k \in N$ and $T \subset N\backslash\{k\}$ it holds that $v(T \cup \{k\}) - v(T) \in \{0,1\}$. This implies that

$$m_k^\sigma(v) \in \{0,1\} \text{ for each } k \in N. \tag{4}$$

It is sufficient to show that $\sum_{k \in T} m_k^\sigma(v) \geq v(T)$ for each $T \subset N$. We distinguish between two cases.

`Case 1:` $T \neq S \cup \{i\}, S \cup \{j\}$.

Suppose that $|T| \leq |S| + 1$. Then $v(T) = 0$ and therefore, using (4), we have that $\sum_{k \in T} m_k^\sigma(v) \geq v(T)$. So suppose that $|T| > |S| + 1$. Then, because of (4), it holds that $\sum_{k \in N\backslash T} m_k^\sigma(v) \leq |N\backslash T|$ and therefore

$$\sum_{k \in T} m_k^\sigma(v) = v(N) - \sum_{k \in N\backslash T} m_k^\sigma(v) \geq |N| - |S| - 1 - |N\backslash T| = |T| - |S| - 1 = v(T).$$

`Case 2:` $T = S \cup \{i\}$ or $T = S \cup \{j\}$.

Without loss of generality assume that $T = S \cup \{i\}$. It holds that $v(S \cup \{i\}) = 1$. To show that $\sum_{k \in S \cup \{i\}} m_k^\sigma(v) \geq 1$ it is sufficient, according to (4), to prove that there is a $k \in S \cup \{i\}$ such that $m_k^\sigma(v) = 1$. Let $\hat{h} \in S \cup \{i\}$ be such that all players in $S \cup \{i\}$ precede $\hat{h}$ with respect to σ, i.e. $\hat{h} \in S \cup \{i\}$ is such that $\sigma^{-1}(k) \leq \sigma^{-1}(\hat{h})$ for all $k \in S \cup \{i\}$. We distinguish between three subcases.

`Subcase 2a:` $\sigma^{-1}(\hat{h}) = |S| + 1$.

It holds that $[\hat{h}, \sigma] = S \cup \{i\}$. Therefore $m_{\hat{h}}^\sigma(v) = v(S \cup \{i\}) - v((S \cup \{i\})\backslash\{\hat{h}\}) = 1 - 0 = 1$.

`Subcase 2b:` $\sigma^{-1}(\hat{h}) = |S| + 2$.

First suppose that $\hat{h} = i$. If $\sigma^{-1}(j) > \sigma^{-1}(i)$, then it holds that $(i, \sigma) \neq S \cup \{j\}$, and $|[i, \sigma]| = |S| + 2$. Hence, $m_i^\sigma(v) = v([i, \sigma]) - v((i, \sigma)) = 1 - 0 = 1$.

If $\sigma^{-1}(j) < \sigma^{-1}(i)$ then, because $\sigma \notin M(S, \{i,j\})$, it follows that there is a $k \in S \cup \{i\}$ such that $\sigma(|S|+1) = k \neq j$. Hence $[k, \sigma] = S \cup \{j\}$. Therefore, $m_k^\sigma(v) = v([k, \sigma]) - v((k, \sigma)) = v(S \cup \{j\}) - v((k, \sigma)) = 1 - 0 = 1$.

So suppose that $\hat{h} \neq i$. From $|[\hat{h}, \sigma]| = |S| + 2$ it follows that $v([\hat{h}, \sigma]) = 1$. Because $\hat{h} \in S$ we have that $(\hat{h}, \sigma) \neq S \cup \{i\}, S \cup \{j\}$. Hence, $v((\hat{h}, \sigma)) = 0$. Therefore, $m_{\hat{h}}^\sigma(v) = v([\hat{h}, \sigma]) - v((\hat{h}, \sigma)) = 1 - 0 = 1$.

`Subcase 2c:` $\sigma^{-1}(\hat{h}) \geq |S| + 3$.

Then it holds that $|[\hat{h}, \sigma]| = \sigma^{-1}(\hat{h}) \geq |S| + 3$, and $|(\hat{h}, \sigma)| = \sigma^{-1}(\hat{h}) - 1$. Therefore, $m_{\hat{h}}^\sigma(v) = v([\hat{h}, \sigma]) - v((\hat{h}, \sigma)) = \sigma^{-1}(\hat{h}) - |S| - 1 - (\sigma^{-1}(\hat{h}) - 1 - |S| - 1) = 1$. □

The following example shows that for $n = 3$ the even orders form a complete set.

Example 2 Let $N = \{1,2,3\}$. Let $A \subset S_3$. From Lemma 1, it follows that A is complete if and only if $A \cap B \neq \emptyset$ for all $B \in \{M(\{1\}, \{2,3\}), M(\{2\}, \{1,3\}), M(\{3\}, \{1,2\}), M(\emptyset, \{1,2\}), M(\emptyset, \{1,3\}), M(\emptyset, \{2,3\})\}$.

For example the set of even orders $\{123, 312, 231\}$ satisfies this property. On the contrary, $\{123, 132, 213, 231\}$ does not satisfy this property because $\{123, 132, 213, 231\} \cap M(\{3\}, \{1, 2\}) = \emptyset$. Hence, $\{123, 132, 213, 231\}$ is not complete. This means that there is a nonconvex game for which the marginal vectors corresponding to the orders in $\{123, 132, 213, 231\}$ are core elements. Such a game is obtained by considering (3) with $S = \{3\}$, $i = 1$ and $j = 2$. This game is depicted in Table 2. It holds that $m^{123}(v) = m^{132}(v) = m^{213}(v) = m^{231}(v) = (0, 0, 1) \in C(v)$. However, $m^{312}(v) = (1, 0, 0) \notin C(v)$ and $m^{321}(v) = (0, 1, 0) \notin C(v)$.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	0	1	1	1

Table 2: A nonconvex game for which $m^{123}(v), m^{132}(v), m^{213}(v)$ and $m^{231}(v)$ are core elements.

From Lemma 1 it follows that to find a complete set, it is necessary and sufficient to find a set of orders that covers all elements of $G_n(k)$ for each $k \in \{0, \ldots, n-2\}$, i.e. a set $A \subset S_n$ such that $A \cap B \neq \emptyset$ for all $B \in \cup_{0 \leq p \leq n-2} G_n(p)$. Trivially, by choosing an order from each $B \in G_n(k)$ for each $k \in \{0, \ldots, n-2\}$ a complete set is obtained. In this way we can find a complete set containing at most $\sum_{0 \leq p \leq n-2} |G_n(p)|$ orders. However, there are complete sets containing less than $\sum_{0 \leq p \leq n-2} |G_n(p)|$ orders. The main result of this paper is the formula for the minimum cardinality of a complete set. To prove this result we distinguish between odd $n \in \mathbb{N}$ and even $n \in \mathbb{N}$.

First we focus on odd n. For the proof of the formula for odd n we need the concepts of right-hand side neighbours, left-hand side neighbours and perfect coverings. Let $n \geq 3$ be odd and let k be such that $n = 2k + 1$. Suppose the players are seated at a round table such that for all $j \in N$ it holds that the person on the right-hand side of player j is player $(j-1)$ *mod* n and the person on his left-hand side is player $(j+1)$ *mod* n. For each $j \in N$ let the *right-hand side neighbours* R_j of j be the first k players on the right-hand side of player j at the round table, i.e.

$$R_j = \{(j-1) \text{ } mod \text{ } n, \ldots, (j-k) \text{ } mod \text{ } n\}.$$

Similarly, let the *left-hand side neighbours* L_j of player j be the first k players on the left-hand side of player j, i.e.

$$L_j = \{(j+1) \text{ } mod \text{ } n, \ldots, (j+k) \text{ } mod \text{ } n\}.$$

The notion of left-hand side neighbours and right-hand side neighbours is illustrated by Example 3.

Example 3 If $n = 9$, $k = 4$ and $j = 3$, then it holds that $R_3 = \{1, 2, 8, 9\}$ and $L_3 = \{4, 5, 6, 7\}$. The sets R_3 and L_3 are illustrated in Figure 1.

Let $i, j \in N$ with $i \neq j$. The following properties of L_j and R_j can easily be verified.

(P1) $L_j \cap R_j = \emptyset$,

(P2) $L_j \cup R_j \cup \{j\} = N$,

(P3) $i \in L_j$ if and only if $j \in R_i$,

(P4) $i \in R_j$ if and only if $j \notin R_i$.

Now we introduce the concept of perfect coverings. Let $i, j \in N$ with $i \neq j$ and $T \subset N \backslash \{i, j\}$. Then $\sigma \in M(T, \{i, j\})$ is said to *perfectly cover* $M(T, \{i, j\})$ if $\sigma(|T| + 1) \in R_{\sigma(|T|+2)}$. Because of (P4) it holds that $M(T, \{i, j\})$ contains orders which perfectly cover this set. A set $A \subset S_n$ is called *perfect complete* if for each $M(S, \{l, m\})$ there is a $\sigma \in A$ that perfectly covers $M(S, \{l, m\})$. The concept of a perfect covering is illustrated in the following example.

Example 4 If $n = 9$, $T = \{1, 4, 5\}$, $i = 8$ and $j = 3$, then it holds that $8 \in R_3$ and $3 \notin R_8$. Hence, $\sigma \in M(\{1, 4, 5\}, \{3, 8\})$ is a perfect covering of this set if $\sigma(4) = 8$. In particular, 514832967 is a perfect covering of $M(\{1, 4, 5\}, \{3, 8\})$ and 514382967 is not a perfect covering of $M(\{1, 4, 5\}, \{3, 8\})$.

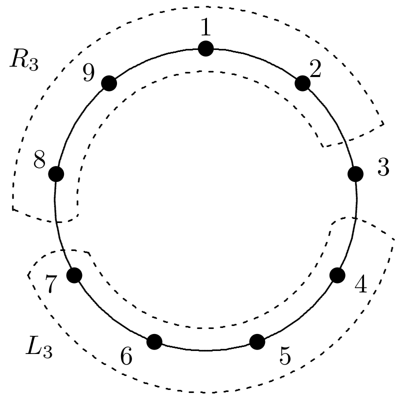

Figure 1: The left-hand side neighbours and right-hand side neighbours of player 3.

The following theorem gives the minimum cardinality of a complete set for odd n. The proof of this theorem is constructive in the sense that it contains a procedure to obtain a complete set of minimum cardinality.

Theorem 3 *Let $n \geq 3$ be odd. It holds that*

$$M_n = \frac{n!}{2(\frac{n-3}{2})!(\frac{n-1}{2})!}.$$

Proof: Let k be such that $n = 2k+1$. First we will show that $M_n \geq \frac{n!}{2(\frac{n-3}{2})!(\frac{n-1}{2})!}$. It holds that $G_n(k)$ forms a partition of S_n. This implies, using Lemma 1, that to cover all elements of $G_n(k)$ at least $|G_n(k)|$ orders are needed. It holds that $|G_n(k)| = \binom{n}{k}\binom{n-k}{2} = \frac{n!}{k!(n-k-2)!2!} = \frac{n!}{2(\frac{n-3}{2})!(\frac{n-1}{2})!}$. Therefore $M_n \geq \frac{n!}{2(\frac{n-3}{2})!(\frac{n-1}{2})!}$.

Now we will show that $M_n \leq \frac{n!}{2(\frac{n-3}{2})!(\frac{n-1}{2})!}$. We will do this by constructing a perfect complete set of size $|G_n(k)| = \frac{n!}{2(\frac{n-3}{2})!(\frac{n-1}{2})!}$. First we construct a set $A \subset S_n$ containing $|G_n(k)|$ orders that perfectly covers each element of $G_n(k)$. Using A we inductively show that we can construct a perfect complete set.

Each $M(S, \{l, m\}) \in G_n(k)$ contains perfect coverings. Therefore it is trivial to obtain a set A containing $|G_n(k)|$ orders that perfectly covers each element of $G_n(k)$. In particular, A can be obtained by choosing precisely one perfect covering from each element of $G_n(k)$.

Assume that A perfectly covers each element in $\cup_{m \leq p \leq k} G_n(p)$ for some $m \leq k$. Obviously $m = k$ satisfies this property. Suppose that $M(T, \{i, j\}) \in G_n(m-1)$ is not perfectly covered by A. We will replace one order $\sigma \in A$ by an order $\bar{\sigma} \in S_n \backslash A$ to obtain the set $\bar{A} = (A \backslash \{\sigma\}) \cup \{\bar{\sigma}\}$. Our selection of σ and $\bar{\sigma}$ is such that $\bar{A}$ perfectly covers one element of $\cup_{m-1 \leq p \leq k} G_n(p)$ more than A. In particular, $\bar{A}$ perfectly covers the same elements of $\cup_{m-1 \leq p \leq k} G_n(p)$ as A, except for $M(T, \{i, j\}) \in G_n(m-1)$ which is only perfectly covered by $\bar{A}$.

Without loss of generality assume that $i \in R_j$. This yields that if $\tau \in S_n$ perfectly covers $M(T, \{i, j\})$, then it holds that $\tau(|T|+1) = i$ and $\tau(|T|+2) = j$. Let B be the set of orders in A that begin with $T \cup \{i\}$ followed by j, i.e. $B = \{\tau \in A : \tau(p) \in T \cup \{i\} \text{ for all } p \leq |T|+1, \tau(|T|+2) = j\}$. We will replace an order $\sigma \in B$ with an order $\bar{\sigma} \in S_n \backslash A$.

Now first suppose that there is an order in B that is not a perfect covering of an element in $G_n(m-1)$, i.e. suppose there is a $\sigma \in B$ with $\sigma(|T|+1) \notin R_j$. Now interchange $\sigma(|T|+1)$ and i to obtain the order $\bar{\sigma}$. Note that $\bar{\sigma}$ and σ only differ in two positions, namely in position $\sigma^{-1}(i) \leq m$ and in position $|T|+1 = m$. This yields that $\bar{\sigma}$ perfectly covers the same elements of $\cup_{m \leq p \leq k} G_n(p)$ as σ. Furthermore, $\bar{\sigma}$ perfectly covers $M(T, \{i, j\})$. Because σ was not a perfect covering of an element of $G_n(m-1)$ it follows that $\bar{A} = (A \backslash \{\sigma\}) \cup \{\bar{\sigma}\}$ perfectly covers one element of $\cup_{m-1 \leq p \leq k} G_n(p)$ more than A.

Now suppose that all orders in B are perfect coverings of elements in $G_n(m-1)$, i.e. suppose that for all $\tau \in B$ it holds that $\tau(|T|+1) \in R_j$. We will show that there are $\pi, \rho \in B$ such that $\pi(|T|+1) = \rho(|T|+1) = h$ for some $h \in T$, i.e. that $M((T \cup \{i\}) \backslash \{h\}, \{h, j\}) \in G_n(m-1)$ is perfectly covered twice by orders in B. If we then take $\sigma = \pi$ and obtain $\bar{\sigma}$ by interchanging h and i, it follows that $\bar{A} = (A \backslash \{\sigma\}) \cup \{\bar{\sigma}\}$ still contains a perfect covering of $M((T\cup\{i\})\backslash\{h\},\{h,j\})$, namely ρ. Moreover, $\bar{A}$ perfectly covers $M(T,\{i,j\})$. Hence, $\bar{A}$ perfectly covers one element of $\cup_{m-1\le p\le k}G_n(p)$ more than A.

We will now show that there are orders $\pi,\rho\in B$ with $\pi(|T|+1)=\rho(|T|+1)$. Note that, by supposition, it holds that $\tau(|T|+1)\in R_j$ for all $\tau\in B$. Because we have assumed that $M(T,\{i,j\})$ was not perfectly covered by an order in A it also holds that $\tau(|T|+1)\neq i$ for all $\tau\in B$. Therefore we have that $\tau(|T|+1)\in T$ for all $\tau\in B$. This implies that $\tau(|T|+1)\in T\cap R_j$ for all $\tau\in B$. Hence, showing that there are orders $\pi,\rho\in B$ with $\pi(|T|+1)=\rho(|T|+1)$ boils down to showing that $|B|>|T\cap R_j|$.

First note that our assumption states that each element of $\cup_{m\le p\le k}G_n(p)$ is perfectly covered by A. This implies that $M(T\cup\{i\},\{j,l\})$ is perfectly covered for all $l\in N\backslash(T\cup\{i,j\})$. Therefore $M(T\cup\{i\},\{j,l\})$ is perfectly covered for all $l\in(N\backslash(T\cup\{i\}))\cap L_j$. Let $\tau\in A$ be a perfect covering of $M(T\cup\{i\},\{j,l\})$ for some $l\in(N\backslash(T\cup\{i\}))\cap L_j$. Because of (P3) it follows that $j\in R_l$. Therefore $\tau(p)\in T\cup\{i\}$ for all $p\le|T|+1$, $\tau(|T|+2)=j$ and $\tau(|T|+3)=l$. We conclude that $\tau\in B$. This implies that

$$|B|\ge|(N\backslash(T\cup\{i\}))\cap L_j|. \tag{5}$$

It holds that $|(N\backslash(T\cup\{i\}))\cap L_j|+|(T\cup\{i\})\cap L_j|=|L_j|=k$. Using (P2) it follows that $|(T\cup\{i\})\cap L_j|+|(T\cup\{i\})\cap R_j|=|T\cup\{i\}|$. From these two expressions we derive that

$$|(N\backslash(T\cup\{i\}))\cap L_j|=k-|(T\cup\{i\})\cap L_j|=k-|T\cup\{i\}|+|(T\cup\{i\})\cap R_j|\ge|(T\cup\{i\})\cap R_j|, \tag{6}$$

where the inequality holds because $k\ge m=|T|+1$. From (5) and (6) it follows that

$$|B|\ge|(T\cup\{i\})\cap R_j|>|T\cap R_j|, \tag{7}$$

where the strict inequality holds because $i\in T\cup\{i\}$ and $i\in R_j$.

So if we start with a set A containing $|G_n(k)|$ elements that perfectly covers each element of $\cup_{m\le p\le k}G_n(p)$, then we can find a set $\bar{A}$ that perfectly covers one element of $\cup_{m-1\le p\le k}G_n(p)$ more than A. This yields that we can construct a set of orders that perfectly covers all elements of $\cup_{0\le p\le k}G_n(p)$. Now let $m\ge k$ be such that A perfectly covers all elements of $\cup_{0\le p\le m}G_n(p)$. Obviously, $m=k$ satisfies this property. Suppose that some $M(T,\{i,j\})\in G_n(m+1)$ is not perfectly covered by A. It is now straightforward to show that there exists a set $\bar{A}$ that perfectly covers one element in $\cup_{0\le p\le m+1}G_n(p)$ more than A. It follows that there exists a set containing $|G_n(k)|$ orders that perfectly covers all elements of $\cup_{0\le p\le n-2}G_n(p)$. □

The following example illustrates the possibility in the proof of Theorem 3 that there is a $\sigma\in B$ with $\sigma(|T|+1)\notin R_j$.

Example 5 Let $n=5$ and $k=2$. According to the proof of Theorem 3, we first need to find a set $A\subset S_5$ that perfectly covers each element of $G_5(2)$. This can be done by taking one perfect cover from each $M(T,\{i,j\})\in G_5(2)$. For example, let

$$\begin{aligned} A = \{&12345,\ 14235,\ 23415,\ 52134,\ 35124\\ &12354,\ 14523,\ 23514,\ 25413,\ 35412\\ &12453,\ 14352,\ 23451,\ 25341,\ 35241\\ &13245,\ 15234,\ 24135,\ 34125,\ 45123\\ &13524,\ 15243,\ 24513,\ 34512,\ 45132\\ &13452,\ 15342,\ 24351,\ 34521,\ 45231\}.\end{aligned}$$

It is straightforward to check that A indeed perfectly covers all elements of $G_5(2)$. However, not all elements of $G_5(1)$ are perfectly covered. For instance, it holds that $M(\{5\},\{3,4\})\cap A=\emptyset$. Because A does not cover $M(\{5\},\{3,4\})$, it certainly does not perfectly cover this set. We will obtain a set $\bar{A}$ that perfectly covers $M(\{5\},\{3,4\})$.

Let $T=\{5\}$, $i=3$ and $j=4$. Note that $i\in R_j$. It holds that $B=\{\tau\in A:\tau(k)\in\{3,5\}$ for $k=1,2$ and $\tau(3)=4\}=\{35412\}$. It holds for $\sigma=35412\in B$ that $\sigma(|T|+1)=5\notin R_4$. According to the proof we need to interchange $\sigma(|T|+1)=5$ and $i=3$. This yields $\bar{\sigma}=53412$. Note that 53412 perfectly covers $M(\{5\},\{3,4\})$. Now let $\bar{A}=(A\backslash\{35412\})\cup\{53412\}$. It holds that $\bar{A}$ perfectly covers $M(\{5\},\{3,4\})$.

The following example illustrates the possibility that for all $\sigma \in B$ it holds that $\sigma(|T|+1) \in R_j$.

Example 6 Let $n = 5$, $k = 2$ and let A be the same set of orders as in Example 5. Although $52134 \in A$ covers $M(\{5\},\{1,2\}) \in G_5(1)$, it holds that $M(\{5\},\{1,2\})$ is not perfectly covered by A. Therefore let $T = \{5\}$, $i = 1$ and $j = 2$. Note that $i \in R_j$. It holds that $B = \{\tau \in A : \tau(k) \in \{1,5\}$ for $k = 1,2$ and $\tau(3) = 2\} = \{15243, 15234\}$. For all $\sigma \in B$ it holds that $\sigma(|T|+1) = 5 \in R_2$. So, $M(\{1\},\{2,5\})$ is perfectly covered twice by orders in B. Take $\sigma = 15243 \in B$. Now interchange $\sigma(|T|+1) = 5$ and $i = 1$ to obtain $\bar{\sigma} = 51243$ and let $\bar{A} = (A \backslash \{15243\}) \cup \{51243\}$. It holds that $\bar{A}$ still perfectly covers $M(\{1\},\{2,5\})$, and, moreover, $\bar{A}$ perfectly covers $M(\{5\},\{1,2\})$.

In the final part of this paper we deal with even $n \in \mathbb{N}$. Although the proof of the formula for even n is very similar to the proof for odd n, there is a subtle difference between these two proofs.

Let $n \geq 3$ be even and let k be such that $n = 2k+2$. For each $j \in N$ we define the right-hand side neighbours R_j by

$$R_j = \{(j-1) \; mod \; n, \ldots, (j-k) \; mod \; n, (j-k-1) \; mod \; n\}$$

and the left-hand side neighbours L_j by

$$L_j = \{(j+1) \; mod \; n, \ldots, (j+k) \; mod \; n, (j+k+1) \; mod \; n\}.$$

The intuition of L_j and R_j is similar as for odd n. For convenience, we define $o_j = (j+k+1) \; mod \; n$ for all $j \in N$. Intuitively, o_j is the player seated exactly opposite to player j. It is straightforward to show that $o_{(j+k+1) \; mod \; n} = j$ and that $o_j = (j-k-1) \; mod \; n$. The following properties can easily be verified.

(Q1) $L_j \cap R_j = \{o_j\}$,

(Q2) $L_j \cup R_j \cup \{j\} = N$,

(Q3) $i \in L_j$ if and only if $j \in R_i$,

(Q4) $i \in R_j$ or $j \in R_i$.

Player o_j is a member of L_j and R_j. This observation implies that (P1) does not hold anymore and that (P4) only holds in a weakened version. Let $i, j \in N$ with $i \neq j$ and $T \subset N \backslash \{i,j\}$. Then $\sigma \in M(T,\{i,j\})$ is said to *perfectly cover* $M(T,\{i,j\})$ if $\sigma(|T|+1) \in R_{\sigma(|T|+2)}$. Because of (Q4) it holds that $M(T,\{i,j\})$ contains orders which perfectly cover this set. Moreover, if $i = o_j$, then it holds that every order of $M(T,\{o_j,j\})$ perfectly covers this set. A set $A \subset S_n$ is called *perfect complete* if for each $M(S,\{l,m\})$ there is a $\sigma \in A$ that perfectly covers $M(S,\{l,m\})$. The following theorem gives the formula for the minimum cardinality of a complete set for even n.

Theorem 4 *Let $n \geq 3$ be even. It holds that*

$$M_n = \frac{n!}{2(\frac{n-2}{2})!(\frac{n-2}{2})!}.$$

Proof: Let k be such that $n = 2k+2$. First we will show that $M_n \geq \frac{n!}{2((\frac{n-2}{2})!)^2}$. It holds that $G_n(k)$ forms a partition of S_n. This implies, using Lemma 1, that to cover all elements of $G_n(k)$ at least $|G_n(k)| = \binom{n}{k}\binom{n-k}{2} = \frac{n!}{k!k!2!} = \frac{n!}{2!((\frac{n-2}{2})!)^2}$ orders are needed. It follows that $M_n \geq \frac{n!}{2((\frac{n-2}{2})!)^2}$.

It remains to show that $M_n \leq \frac{n!}{2((\frac{n-2}{2})!)^2}$. We do this similar as for odd n. It is straightforward to construct a set $A \subset S_n$ containing $|G_n(k)|$ orders which perfectly covers each element of $G_n(k)$. Assume that A perfectly covers each element of $\cup_{m \leq p \leq k} G_n(p)$ for some $m \leq k$. Suppose that $M(T,\{i,j\})$ is not perfectly covered by A. Assume without loss of generality that $i \in R_j$ and let $B = \{\tau \in A : \tau(p) \in T \cup \{i\}$ for all $p \leq |T|+1, \tau(|T|+2) = j\}$.

Suppose there is an order $\sigma \in B$ with $\sigma(|T|+1) \notin R_j$. Using the same technique as for odd n it is now straightforward to obtain a set $\bar{A}$ that perfectly covers one element of $\cup_{m-1 \leq p \leq k} G_n(p)$ more than A.

So suppose that for all $\tau \in B$ it holds that $\tau(|T|+1) \in R_j$, i.e. all orders in B are perfect coverings of some element of $G_n(m-1)$. Again, we will show that there are $\pi, \rho \in B$ with $\pi(|T|+1) = \rho(|T|+1) = h$ for some $h \in T$, i.e. that $M((T \cup \{i\})\backslash\{h\}, \{h, j\}) \in G_n(m-1)$ is perfectly covered twice by orders in B. This boils down to showing that $|B| > |T \cap R_j|$. We distinguish between two cases to show this inequality.

Case 1: $o_j \in N\backslash(T \cup \{i\})$.

We assumed that each element of $\cup_{m-1 \leq p \leq k} G_n(p)$ is perfectly covered by A. This implies that $M(T \cup \{i\}, \{j, l\})$ is perfectly covered for all $l \in N\backslash(T \cup \{i, j\})$. Hence, $M(T \cup \{i\}, \{j, l\})$ is perfectly covered for all $l \in (N\backslash(T \cup \{i, o_j\})) \cap L_j$. Let $l \in (N\backslash(T \cup \{i, o_j\})) \cap L_j$ and let $\tau \in A$ be a perfect covering of $M(T \cup \{i\}, \{j, l\})$.

Because $l \neq o_j$ it holds because of (Q1) that $l \notin R_j$. Because τ is a perfect covering it follows that $\tau(p) \in T \cup \{i\}$ for all $p \leq |T|+1$, $\tau(|T|+2) = j$ and $\tau(|T|+3) = l$. We conclude that $\tau \in B$.

It follows that $|B| \geq |(N\backslash(T \cup \{i, o_j\})) \cap L_j| = |(N\backslash(T \cup \{i\})) \cap L_j| - 1$, where the equality follows from $o_j \in (N\backslash(T \cup \{i\})) \cap L_j$. Trivially it holds that $|(N\backslash(T \cup \{i\})) \cap L_j| + |(T \cup \{i\}) \cap L_j| = k+1$. Because of $o_j \in N\backslash(T \cup \{i\})$, (Q1) and (Q2) it holds that $|(T \cup \{i\}) \cap L_j| + |(T \cup \{i\}) \cap R_j| = |T \cup \{i\}|$. Hence, we have that

$$|B| \geq |(N\backslash(T \cup \{i\})) \cap L_j| - 1 = k + 1 - |(T \cup \{i\}) \cap L_j| - 1$$

$$= k + |(T \cup \{i\}) \cap R_j| - |T \cup \{i\}| \geq |(T \cup \{i\}) \cap R_j| > |T \cap R_j|,$$

where the first inequality follows from $k \geq m = |(T \cup \{i\})|$. The strict inequality follows from $i \in T \cup \{i\}$ and $i \in R_j$.

Case 2: $o_j \in T \cup \{i\}$.

We assumed that each element of $\cup_{m-1 \leq p \leq k} G_n(p)$ is perfectly covered by A. This implies that $M(T \cup \{i\}, \{j, l\})$ is perfectly covered for all $l \in N\backslash(T \cup \{i, j\})$. Hence, $M(T \cup \{i\}, \{j, l\})$ is perfectly covered for all $l \in (N\backslash(T \cup \{i\})) \cap L_j$. Let $l \in (N\backslash(T \cup \{i\})) \cap L_j$ and let $\tau \in A$ be a perfect covering of $M(T \cup \{i\}, \{j, l\})$.

Because $o_j \in T \cup \{i\}$ it follows that $l \neq o_j$. This implies, using (Q1), that $l \notin R_j$. Hence, $\tau(p) \in T \cup \{i\}$ for all $p \leq |T|+1$, $\tau(|T|+2) = j$ and $\tau(|T|+3) = l$. It follows that $\tau \in B$. We conclude that $|B| \geq |(N\backslash(T \cup \{i\})) \cap L_j|$. It also holds that $|(N\backslash(T \cup \{i\})) \cap L_j| + |(T \cup \{i\}) \cap L_j| = k+1$. Because of $o_j \in T \cup \{i\}$, (Q1) and (Q2) it holds that $|(T \cup \{i\}) \cap L_j| + |(T \cup \{i\}) \cap R_j| = |T \cup \{i\}| + 1$. Hence, we have that

$$|B| \geq |(N\backslash(T \cup \{i\})) \cap L_j| = k + 1 - |(T \cup \{i\}) \cap L_j|$$

$$= k + 1 + |(T \cup \{i\}) \cap R_j| - (|T \cup \{i\}| + 1) \geq |(T \cup \{i\}) \cap R_j| > |T \cap R_j|,$$

where the first inequality follows from $k \geq m = |(T \cup \{i\})|$. The strict inequality follows from $i \in T \cup \{i\}$ and $i \in R_j$.

Using the same argument as in the proof of Theorem 3 we can now obtain a perfect complete set of size $G_n(k)$. □

Theorem 3 and Theorem 4 give the formula for the minimum number of marginal vectors needed to characterize convexity. For $3 \leq n \leq 10$ these numbers are presented in Table 3. Note that M_n is relatively small for large n. It holds that g_n converges to zero exponentially fast, whereas the convergence of g_n in Van Velzen et al.(2002) is rather slow.

n	3	4	5	6	7	8	9	10
$n!$	6	24	120	720	5040	40320	362880	3628800
$\frac{n!}{2}$	3	12	60	360	2520	20160	181440	1814400
M_n	3	12	30	90	210	560	1260	3150
g_n	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{72}$	$\frac{1}{288}$	$\frac{1}{1152}$

Table 3: The minimum cardinality of complete sets.

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