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On the Estimation Error in Mean-Variance Efficient Portfolio Weights

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Abstract

This paper derives the asymptotic covariance matrix of estimated mean-variance efficient portfolio weights, both for gross returns (without a riskfree asset available) and for excess returns (in excess of the riskfree rate). When returns are assumed to be normally distributed, we obtain simple formulas for the covariance matrices. The results show that the estimation error increases as the risk aversion underlying the portfolio decreases and as the (asymptotic) slope or Sharpe ratio of the mean-variance frontier increases. For the tangency portfolio there is an additional estimation risk because the location of the tangency portfolio is not known beforehand. The empirical analysis of efficient portfolios based on the G7 countries indicates that the estimation error can be big in practice. It also shows that the standard errors that assume normality are usually very close to the standard errors that do not assume normality in returns, except for portfolios close to the Global Minimum Variance portfolio.

JEL: G11, G15

Keywords: Portfolio Choice, Estimation Risk, Mean-Variance Analysis

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1 Introduction

Mean-variance efficient (MVE) portfolios play an important role for both finance practitioners and academics. A well-known problem in mean-variance analysis is that the necessary parameters have to be estimated from the data, inducing sampling noise in the MVE portfolio weights. Huberman & Kandel (1987), Jobson & Korkie (1989), Gibbons, Ross & Shanken (1989) and Britten-Jones (1999) have shown how regression analysis can be used to make statistical inferences about the efficiency of certain portfolios. In particular, Britten-Jones (1999) has shown how an OLS regression that uses a constant as the dependent variable and excess returns as the independent variables, yields the efficient portfolio weights and can be used as a basis for statistical tests regarding these portfolio weights.

This paper uses a more direct approach to analyze the estimation error in MVE portfolio weights. We start from the distribution of the parameters that underlie the efficient portfolio weights: the means and covariances of the asset returns. The derivatives of the efficient portfolio weights with respect to the asset returns' means and covariances enable us to derive the limit distribution of the efficient portfolio weights. If we assume that returns are normally distributed, we obtain simple formulas for the covariance matrix of the estimated efficient portfolio weights.

For gross returns, with no riskfree asset available, the covariance matrix of the estimated efficient portfolio weights depends in a straightforward way on the covariance matrix of the asset returns, the efficient set constants, and the risk aversion that underlies the portfolio choice. We also show that the Global Minimum Variance (GMV) portfolio plays a special role in the covariance matrix of the efficient portfolio weights. From the results it follows that the estimation error increases as the risk aversion decreases and as the asymptotic slope of the MVE frontier increases. Our empirical analysis of portfolios based on the G7 countries illustrates that the estimation error is indeed very big for portfolios associated with low risk aversions, making the estimated portfolio weights imprecise. For instance when the risk aversion is 5, the standard errors for the G7 country weights are about 0.25, whereas for the GMV portfolio they are only about 0.05. We also find that for a risk aversion of 5, more than 80 percent of the estimation error is due to estimation error in the means rather than in the covariances.

For excess returns, the covariance matrix of the efficient portfolio weights turns out to be a straightforward function of the covariance matrix of the asset returns, the risk aversion underlying the portfolio, and the squared Sharpe ratio of the portfolio. When the risk aversion is specified beforehand, the standard errors we find coincide with the ones suggested by Britten-Jones. Focussing on the tangency portfolio, without specifying the agent's risk aversion beforehand, there is an additional estimation error. This estimation error results from the fact that the risk aversion associated with the tangency portfolio is not known beforehand, but has to be estimated as well. Again, our empirical analysis for the G7 countries shows that the estimation error in the efficient portfolio weights

can be big. For the tangency portfolio of the G7 countries, the standard errors are between 0.22 and 0.44.

The framework outlined in the paper also applies when returns are not normally distributed. However, in that case, no simple formulas for the covariance matrix of the portfolio weights are obtained. The empirical analysis suggests that the difference in estimation error when assuming normality or not are very big in most cases, except for portfolios close to the GMV portfolio.

The plan of this paper is as follows. Section 2 gives an outline of the approach used in the paper. Section 3 derives the covariance matrix for portfolios based on gross returns, i.e., when there is not a riskfree asset available. Here we also illustrate the standard errors associated with portfolios based on the G7 countries for various levels of risk aversion. Section 4 derives the covariance matrix for portfolios based on excess returns (in excess of the riskfree rate). Here we distinguish between the case where the risk aversion is known and the case of the tangency portfolio, where the implied risk aversion has to be estimated. Finally, Section 5 provides a summary and some concluding remarks.

2 The limiting distribution of MVE portfolio weights: outline

In this section we sketch the general approach to derive the limiting distribution of Mean-Variance Efficient (MVE) portfolio weights. We first consider the case where the investor can invest in K risky assets only, without a riskfree asset available. The gross returns on the assets are given by the K -dimensional vector R_t , with $R_{i,t} = (P_{i,t} + D_{i,t} - P_{i,t-1})/P_{i,t-1}$. The vector of expected returns is denoted by $E[R_t] = \mu$ and the $K \times K$ covariance matrix of the returns is denoted by $Var[R_t] = \Sigma$. An investor with a mean-variance utility function and risk aversion parameter γ will choose his portfolio by solving the problem

$$\begin{aligned} \max_w w' \mu - \frac{1}{2} \gamma w' \Sigma w, \\ \text{s.t. } w' \iota = 1, \end{aligned} \tag{1}$$

where ι is K -vector containing only ones. The resulting optimal portfolio is given by

$$w_0 = \gamma^{-1} \Sigma^{-1} (\mu - \eta \iota), \tag{2}$$

where η is the zero-beta rate associated with the portfolio w_0 . The zero-beta rate η is also the Lagrange multiplier for the portfolio constraint $w' \iota = 1$, i.e., the portfolio weights have to sum to one.

In characterizing mean-variance efficient portfolios that satisfy (2) it is useful to define the efficient set constants:

$$A = \iota' \Sigma^{-1} \iota, \tag{3a}$$

$$B = \iota' \Sigma^{-1} \mu, \tag{3b}$$

$$C = \mu' \Sigma^{-1} \mu. \tag{3c}$$

Using these constants it is straightforward for instance to show that the zero-beta rate η can be written as a function of γ : $\eta = (B - \gamma)/A$.¹

In practice we do not know the actual expected returns and covariances, but we have to estimate them from the data. In general, if returns are independently and identically distributed with means and covariances given by μ and Σ , and assuming the relevant moments exist, then we have for the limiting distribution of the estimated means $\hat{\mu}$ and covariances $\hat{\Sigma}$:

$$\sqrt{T} \begin{pmatrix} \hat{\mu} - \mu \\ \text{vec}(\hat{\Sigma} - \Sigma) \end{pmatrix} \overset{asy}{\sim} N \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Phi_{\mu\mu} & \Phi_{\mu\Sigma} \\ \Phi_{\Sigma\mu} & \Phi_{\Sigma\Sigma} \end{pmatrix} \right), \quad (4)$$

where the *vec* operator stacks the columns of a $K \times K$ matrix into a $K^2 \times 1$ vector. Here $\Phi_{\mu\mu}$ is the $K \times K$ covariance matrix of the estimated mean returns $\hat{\mu}$, $\Phi_{\Sigma\Sigma}$ is a $K^2 \times K^2$ matrix, containing the (co)variances of the elements of $\hat{\Sigma}$, and $\Phi_{\mu\Sigma}$ is a $K \times K^2$ matrix containing the covariances between the elements of $\hat{\mu}$ and $\hat{\Sigma}$. For instance, if σ_{ij} is the ij -th element of Σ , then the covariance between $\hat{\sigma}_{ij}$ and $\hat{\sigma}_{kl}$ is

$$\begin{aligned} \text{Cov}[\hat{\sigma}_{ij}, \hat{\sigma}_{kl}] &= E[\varepsilon_{it}\varepsilon_{jt}\varepsilon_{kt}\varepsilon_{lt}] - \sigma_{ij}\sigma_{kl}, \\ \varepsilon_t &= R_t - \mu. \end{aligned} \quad (5)$$

This covariance would then be the element on the $(i \times K + j)$ -th row and the $(k \times K + l)$ -th column of $\Phi_{\Sigma\Sigma}$. Similarly, the covariance between $\hat{\sigma}_{ij}$ and $\hat{\mu}_k$ is

$$\text{Cov}[\hat{\mu}_k, \hat{\sigma}_{ij}] = E[\varepsilon_{it}\varepsilon_{jt}\varepsilon_{kt}] - \sigma_{ij}\mu_k,$$

which would be the element on the k -th row and the $(i \times K + j)$ -th column of $\Phi_{\mu\Sigma}$. The covariance matrix $\Phi_{\mu\mu}$ is simply the covariance matrix Σ of the returns themselves.

The procedure that we will follow in this paper is in itself straightforward. From the central limit theorem it follows that we can derive the limiting distribution of the estimated portfolio weights \hat{w} , using (4) and the derivatives of w_0 with respect to μ and Σ . Thus, for different portfolios, we will derive the vectors of derivatives $\partial w/\partial\mu'$ and $\partial w/\partial\text{vec}(\Sigma)'$, where the derivatives are evaluated in the optimal portfolio weights. The limit distribution of the estimated portfolio weights \hat{w} then follows from combining these derivatives with (4):

$$\sqrt{T}(\hat{w} - w_0) \overset{asy}{\sim} N(0, \Omega_w) \quad (6a)$$

$$\Omega_w = (\partial w/\partial\mu') \Phi_{\mu\mu} (\partial w/\partial\mu')' + (\partial w/\partial\text{vec}(\Sigma)') \Phi_{\Sigma\Sigma} (\partial w/\partial\text{vec}(\Sigma)')' \quad (6b)$$

$$+ 2(\partial w/\partial\mu') \Phi_{\mu\Sigma} (\partial w/\partial\text{vec}(\Sigma)')' \quad (6c)$$

The important step in deriving the limiting distribution of \hat{w} is therefore to obtain the various derivatives $\partial w/\partial\mu'$ and $\partial w/\partial\text{vec}(\Sigma)'$, which together with the estimated covariance matrix Φ will give the asymptotic distribution of the

¹This follows from premultiplying (2) with ι' and noting that $\iota'w_0 = 1$.

efficient portfolio weights. As we will illustrate in Section 4, the same procedure can be applied in case there is a riskfree asset, except that in that case we will focus on the distribution of excess returns rather than gross returns.

If, as a special case, we assume that returns are normally distributed, it turns out that we can obtain simple expressions for the covariance matrix of the efficient portfolio weights. When returns are normally distributed,

$$R_t \sim N(\mu, \Sigma), \quad (7)$$

the limit distribution of the estimated means and covariances $\hat{\mu}$ and $\hat{\Sigma}$ simplifies a great deal and is given by (see, e.g., Hamilton (1994)):

$$\sqrt{T} \begin{pmatrix} \hat{\mu} - \mu \\ \text{vec}(\hat{\Sigma} - \Sigma) \end{pmatrix} \overset{asy}{\sim} N \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma \otimes \Sigma \end{pmatrix} \right). \quad (8)$$

Because the covariance matrix of $\text{vec}(\hat{\Sigma})$ now depends on Σ itself, the covariance matrix of \hat{w} can usually be expressed in simple terms. The covariance matrix of \hat{w} now follows from

$$\begin{aligned} \sqrt{T}(\hat{w} - w_0) &\overset{asy}{\sim} N(0, \Omega_w), \quad (9a) \\ \Omega_w &= (\partial w / \partial \mu')' \Sigma (\partial w / \partial \mu') + (\partial w / \partial \text{vec}(\Sigma))' (\Sigma \otimes \Sigma) (\partial w / \partial \text{vec}(\Sigma))' \end{aligned} \quad (9b)$$

3 MVE portfolios without a riskfree asset

In case we construct MVE portfolios without a riskfree asset, it turns out that the Global Minimum Variance (GMV) portfolio plays a special role. When considering estimation errors, the GMV portfolio is an interesting portfolio in itself, since it only depends on the covariances of asset returns and not on their expected returns. Thus, the estimation error in the GMV portfolio is caused by uncertainty in covariances only and not by uncertainty in mean returns. Therefore, we will first focus on the estimation error in the GMV portfolio before turning to the estimation error in efficient portfolios in general.

3.1 The estimation error in the GMV portfolio

The GMV portfolio only aims to minimize portfolio variance and results as a limiting case from (2) when $\gamma \rightarrow \infty$. The GMV portfolio is given by

$$w_g = \frac{1}{\mathbf{1}' \Sigma^{-1} \mathbf{1}} \Sigma^{-1} \mathbf{1} = \frac{1}{A} \Sigma^{-1} \mathbf{1}. \quad (10)$$

Since this portfolio only depends on the covariance matrix Σ and not on the expected returns μ , we only have to take into account the derivatives of w_g with respect to the elements of Σ . Using the results in Magnus & Neudecker (1988)

e.g., we find in Appendix A for the derivatives of the elements of w_g with respect to the elements of Σ :

$$\frac{\partial w_g}{\partial \text{vec}(\Sigma)'} = \frac{1}{A} \{(\iota' \otimes I) - w_g(\iota' \otimes \iota')\} (\Sigma^{-1} \otimes \Sigma^{-1}). \quad (11)$$

In the general case, substituting these derivatives in (6), the limit distribution of \hat{w}_g is given by

$$\sqrt{T}(\hat{w}_g - w_g) \overset{asy}{\rightsquigarrow} N(0, \Omega_g), \quad (12a)$$

$$\text{with } \Omega_g = \left(\frac{\partial w_g}{\partial \text{vec}(\Sigma)'} \right) \Phi_{\Sigma\Sigma} \left(\frac{\partial w_g}{\partial \text{vec}(\Sigma)'} \right)'. \quad (12b)$$

When normality is assumed, the resulting expression for the covariance matrix Ω_g gives more insights. Substituting (11) in (9), we obtain for the limit distribution of \hat{w}_g :

$$\sqrt{T}(\hat{w}_g - w_g) \overset{asy}{\rightsquigarrow} N(0, \Omega_g), \quad (13a)$$

$$\text{with } \Omega_g = \frac{1}{A} (\Sigma^{-1} - Aw_g w_g'). \quad (13b)$$

The details of this derivation are given in Appendix A. Equation (13b) presents the first main result of the paper. The covariance matrix Ω_g gives a simple expression for the covariance matrix of the GMV portfolio that only depends on the covariance matrix of the returns Σ , the efficient set constant A and the GMV portfolio w_g itself. Notice that $1/A$ is equal to the variance of the GMV portfolio returns.

Using the limit distribution of \hat{w}_g either from (12) or from (13) it is then straightforward to construct confidence bounds for the portfolio weights in w_g . Knowing the limiting distribution of \hat{w}_g we can perform all kinds of hypothesis tests on the estimated portfolio weights. The only caveat in performing hypothesis tests is that the matrix Ω_g is not of full rank, because of the restriction that $w_g' \iota = 1$. This means that the rank of Ω_g is actually only $K - 1$. If we want to test hypotheses of the form $Qw_g = q$, where Q is an $N \times K$ restriction matrix and q is an $N \times 1$ vector, we can have at most $N = K - 1$ restrictions.

3.2 The estimation error in efficient portfolios

Having established the limit distribution of the GMV portfolio, we next move on to the limit distribution of estimated efficient portfolios \hat{w} in general. Starting from (2) it follows that we have to take into account the sampling error in both $\hat{\mu}$ and $\hat{\Sigma}$. Taking the risk aversion of the investor, γ , as given, we can in addition use the result $\eta = (B - \gamma)/A$. Therefore, the portfolio weights are a function of γ , μ , and Σ only, and each MVE portfolio is uniquely defined by γ . Following the procedure outlined above, we first find the derivatives of the

efficient portfolio weights w_0 with respect to the elements of both μ and Σ . It is shown in Appendix B that these derivatives are given by

$$\frac{\partial w_0}{\partial \mu'} = \gamma^{-1} (I - w_g \iota') \Sigma^{-1}, \quad (14)$$

and

$$\frac{\partial w_0}{\partial \text{vec}(\Sigma)'} = \left\{ \gamma^{-1} ((\mu' \otimes I) - w_g (\mu' \otimes \iota')) - \left(\gamma^{-1} \frac{B}{A} - \frac{1}{A} \right) ((\iota' \otimes I) - w_g (\iota' \otimes \iota')) \right\} (\Sigma^{-1} \otimes \Sigma^{-1}). \quad (15)$$

These derivatives can be substituted in (6) to obtain, with the estimate of Φ , the limit distribution of \hat{w} .

Again, as with the Global Minimum Variance portfolio, more intuitive results can be obtained when it is assumed that returns are normally distributed. Combining (14) and (15) with (9), we get for the limit distribution of \hat{w} :²

$$\sqrt{T} (\hat{w} - w_0) \overset{asy}{\sim} N(0, \Omega_w), \quad (16a)$$

$$\text{with } \Omega_w = \left\{ \frac{1}{A} + \gamma^{-2} (1 + (AC - B^2)/A) \right\} (\Sigma^{-1} - Aw_g w_g') \quad (16b)$$

Equation (16b) presents the second main result of the paper. Notice that the covariance matrix Ω_w can also be written as

$$\Omega_w = \{1 + \gamma^{-2} (A + (AC - B^2))\} \Omega_g.$$

This formula highlights the special role of the GMV portfolio. Since the risk aversion γ , A , and $AC - B^2$ are all positive numbers, it follows that the estimation error in individual portfolio weights is smallest for the GMV portfolio and increases as the risk aversion underlying the chosen portfolio decreases. This is natural since the effect of expected returns on the estimated portfolios increases as the risk aversion decreases and the estimation error in means is known to be more important than the estimation error in covariances (see, e.g., DeRoos, TerHorst, & Werker (2003)).

It is also important to note that $\sqrt{(AC - B^2)/A}$ is the slope of the asymptote of the Mean-Variance Frontier (in mean-standard deviation space), or the asymptotic Sharpe ratio (defining excess returns as returns in excess of the zero-beta rate). Therefore, as the frontier is steeper, the estimation error for portfolios associated with lower risk aversions is bigger.

Equation (16b) is an easy to implement formula to derive the covariance matrix of estimated efficient portfolio weights, and therefore of the standard errors of the estimated portfolio weights, provided that returns are normally distributed.

²Details are given in Appendix B

3.3 Empirical application: Estimation error in international portfolios

To illustrate the estimation error of portfolios at different points of the Mean-Variance Frontier, we will construct portfolios from the MSCI indices of the G7 countries. We use monthly returns for the total return indices (including dividends) for the period January 1975 until December 2000. Table 1 presents the summary statistics for these returns, as well as the efficient set constants A , B , and C . Means and standard deviations are reported both for gross returns and for returns in excess of the 1-month Eurodollar rate. Similarly, for the correlations reported in Table 1, the correlations above the diagonal are for gross returns whereas below the diagonal they are for excess returns. Notice that excess returns are slightly more risky than gross returns. The excess returns will be the focus of Section 4.

Table 2 presents the estimated efficient portfolio weights along with the asymptotic standard errors for various levels of the risk aversion parameter γ . The most left column of Table 2 shows the GMV portfolio weights. The next two columns show the asymptotic standard errors, either without or with assuming normally distributed returns. The GMV has the largest weight in the US market, which is also the market with the lowest standard deviation. Except for Germany and Japan all other positions are smaller than 10 percent in absolute value, with small short positions in France and the UK. The standard errors for these weights are all around five percent, implying that the GMV weights are estimated fairly precisely. These standard errors also mean that the weights for Germany, Italy, Japan and the US are significantly different from zero, whereas for the other three markets they are less than 1.5 standard errors away from zero. These conclusions hold when the standard errors are calculated assuming normally distributed returns as well as when this assumption is not made. As the two columns show, the two standard errors are similar in magnitude, although the differences can be as big as 20 percent (for Canada) or even 40 percent (for the UK) of the reported standard errors.

Looking at the other columns in Table 2, portfolio weights and associated standard errors for lower risk aversions are reported. Thus, as we move to the right, the importance of expected returns increases. The table shows that for lower risk aversions, the estimated portfolio weights become very imprecise. When the risk aversion γ is 10, the standard errors are about three times the ones of the GMV portfolio. In this case only the weight for the US (74 percent) is significantly different from zero. When the risk aversion decreases to 5, the standard errors are in the order of magnitude of 25 percent.

When the risk aversion decreases, the difference between the two standard errors, with or without assuming normality of returns, also decreases as the table illustrates. Except for the UK the differences between the two standard errors are now less than five percent and usually less than one percentage point. The fact that the difference in standard errors is smaller for lower risk aversions follows from the fact that non-normalities only show up in the asymptotic covariance matrix of the elements of $\hat{\Sigma}$ and not of $\hat{\mu}$. Therefore, the biggest

differences occur in the GMV portfolio and the differences decrease as the risk aversion γ decreases.

Thus, Table 2 illustrates the familiar effect that estimation error in the means is very big and therefore that estimated portfolio weights, especially for low risk aversions, can be very imprecise. To see the effect that estimation error in the means has relative to the total estimation error reported in Table 2, Table 3 reports the standard errors of the efficient portfolio weights assuming there is no uncertainty in the estimated covariances, i.e., assuming that $\widehat{\Sigma} = \Sigma$. In terms of the limit distribution in (4) this implies that $\Phi_{\mu\Sigma}$ and $\Phi_{\Sigma\Sigma}$ are equal to zero.

Since the GMV portfolio is not affected by estimation error in mean returns by construction, Table 3 only considers the two portfolios based on risk aversions γ of 5 and 10. For both risk aversion the table shows the standard errors of the portfolio weights neglecting uncertainty in $\widehat{\Sigma}$ as well as the percentage of the variance of the portfolio weights that is caused by estimation error in the means only.³ Table 3 clearly illustrates the importance of estimation error in the means relative to estimation error in the variances. Except for the UK weights, in all cases at least 70 percent of the total estimation error is due to the estimation error in the means. For the lowest risk aversion, $\gamma = 5$, this is even more than 80 percent and usually around 90 percent.

Notice from Table 2 that the portfolios considered here do not contain very extreme positions, except for the short position in Canada. Also, the expected portfolio returns and risks are not very different for the three portfolios. Even for these fairly realistic portfolios we find that estimation error in the portfolio weights rapidly increases as we move away from the GMV portfolio and that this is largely due to the estimation error in the mean returns.

4 MVE portfolios with a riskfree asset: excess returns

In case we construct an efficient portfolio including a riskfree asset, the analysis simplifies since we can work with returns in excess of the riskfree rate, $R_{f,t-1}$, and thereby substitute out the portfolio constraint. Defining excess returns as $r_{i,t} = R_{i,t} - R_{f,t-1}$, we denote the K -dimensional excess return vector as r_t , the vector of expected excess returns as μ_r and the covariance matrix of the excess returns as Σ_{rr} . Solving the problem in (1) in terms of excess returns, without the portfolio constraint, the optimal portfolio for the K risky assets equals

$$w_r = \gamma^{-1} \Sigma_{rr}^{-1} \mu_r. \quad (17)$$

In order to derive the limit distribution of the estimated portfolio weights \widehat{w}_r , we now assume that excess returns are distributed with mean μ_r and covariance matrix Σ_{rr} . With abuse of notation, it follows that we can use (4) and (8) again, replacing μ with μ_r and Σ with Σ_{rr} .

³This percentage is the squared standard error in Table 3 divided by the squared standard error in Table 2.

4.1 The estimation error in case of excess returns

The derivatives of w_r with respect to the elements of μ_r and Σ_{rr} are given by

$$\frac{\partial w_r}{\partial \mu_r'} = \gamma^{-1} \Sigma_{rr}^{-1}, \quad (18)$$

and

$$\frac{\partial w_r}{\partial \text{vec}(\Sigma_{rr})'} = \gamma^{-1} (\mu_r' \otimes I) (\Sigma_{rr}^{-1} \otimes \Sigma_{rr}^{-1}). \quad (19)$$

It is straightforward to combine these derivatives with the distribution of $\hat{\mu}_r$ and $\text{vec}(\hat{\Sigma}_{rr})$ to obtain the limit distribution of \hat{w}_r . Assuming normally distributed returns, we obtain (see Appendix C):

$$\sqrt{T}(\hat{w}_r - w_r) \overset{asy}{\rightsquigarrow} N(0, \Omega_r), \quad (20a)$$

$$\text{with } \Omega_r = \gamma^{-2} (1 + \mu_r' \Sigma_{rr}^{-1} \mu_r) \Sigma_{rr}^{-1}. \quad (20b)$$

Focusing on the asymptotic covariance matrix of \hat{w}_r , Ω_r , we see that this has a simple structure. The covariance matrix Ω_r of the estimated portfolio weights is now proportional to the inverse of the covariance matrix of the excess returns, Σ_{rr} . The proportionality is a function of γ^{-2} as it was with the gross returns in Section 2, and of the term $\mu_r' \Sigma_{rr}^{-1} \mu_r$, which is the squared Sharpe ratio of the efficient portfolio. Since the squared Sharpe ratio is always positive, we have again that the estimation error in portfolio weights increases as the risk aversion decreases. The asymptotic covariance matrix Ω_r is similar to the one for gross returns, Ω_w in (16b), except that the GMV portfolio does not play any role now. This is natural, because in case of excess returns the GMV portfolio is the riskfree asset with no variance. Notice that since we work in excess returns now, without the portfolio constraint that $w_r' \iota = 1$, Ω_r is of full rank, and the caveat that we encountered with the gross returns in Section 3 does not apply here.

A special case arises for the portfolio of excess returns for which $w_r' \iota = 1$, since this is the tangency portfolio w_τ . When focusing on the tangency portfolio, we do not specify the risk aversion beforehand, but choose it in such a way that the portfolio weights sum to one. Thus,

$$\begin{aligned} 1 &= \iota' w_\tau = \iota' \gamma_\tau^{-1} \Sigma_{rr}^{-1} \mu_r \\ \Leftrightarrow \gamma_\tau &= \iota' \Sigma_{rr}^{-1} \mu_r. \end{aligned}$$

The fact that we focus on the tangency portfolio implies that γ_τ has to be estimated as well, inducing additional estimation risk. Therefore, in deriving the limit distribution of w_τ , we also have to take into account the estimation error in γ_τ . Allowing for this additional estimation error, the derivatives of w_τ with respect to the elements of μ_r and Σ_{rr} are:

$$\frac{\partial w_\tau}{\partial \mu_r'} = \gamma_\tau^{-1} \{I - w_\tau \iota'\} \Sigma_{rr}^{-1}, \quad (21)$$

and

$$\frac{\partial w_\tau}{\partial \text{vec}(\Sigma_{rr})'} = \gamma_\tau^{-1} \{(\mu_r' \otimes I) - w_\tau (\mu_r' \otimes \iota')\} (\Sigma_{rr}^{-1} \otimes \Sigma_{rr}^{-1}). \quad (22)$$

Assuming normally distributed returns again, these derivatives combined with the distribution of $\hat{\mu}_r$ and $\text{vec}(\hat{\Sigma}_{rr})$ imply for the limit distribution of \hat{w}_τ :

$$\sqrt{T}(\hat{w}_\tau - w_\tau) \overset{asy}{\sim} N(0, \Omega_\tau), \quad (23a)$$

$$\text{with } \Omega_\tau = \gamma_\tau^{-2} (1 + \mu_r' \Sigma_{rr}^{-1} \mu_r) (I - w_\tau \iota') \Sigma_{rr}^{-1} (I - \iota w_\tau'). \quad (23b)$$

The covariance matrix of w_τ is very similar to the covariance matrix of w_r , except for the premultiplication and postmultiplication of Σ_{rr}^{-1} by $(I - \iota w_\tau')$. The multiplication with $(I - \iota w_\tau')$ reflects the additional estimation error that results from the fact that the location of the tangency portfolio, γ_τ , has to be estimated as well. Notice that because (23) is derived with the constraint that $\omega_r' \iota = 1$, the rank of the covariance matrix Ω_τ is only $K - 1$.

4.2 Estimation error in international portfolios: excess returns

The estimation error in case of excess returns is illustrated for the G7 countries in Table 4. The first three columns show the estimated portfolio weights and the associated standard errors (without and with assuming normality) for a risk aversion γ of 10 whereas the last three columns show the same results for a risk aversion of 5.

Comparing the standard errors for the two levels of risk aversion in Table 4, we see - as follows from (20b) - that the standard errors for $\gamma = 5$ are twice the standard errors for $\gamma = 10$. Thus, independent of the level of risk aversion, the ratio of the estimated portfolio weights to the standard errors is constant. This is consistent with the standard errors suggested by Britten-Jones (1999), which are based on the ratio of the estimated portfolio weight and the t -value of the Britten-Jones regression of 1 on the excess returns. As in Table 2, we see in Table 4 that the standard errors assuming normality are usually close to the standard errors without assuming normality. However, for the UK the difference is sizable again, about 20 percent of the reported standard errors.

The first column of Table 5 shows the estimated tangency portfolio for the G7 countries. When the risk aversion is 4.02, the portfolio weights sum exactly to one, implying that an agent with a risk aversion of 4.02 invests 100 percent of his wealth in risky assets (i.e., the G7 countries) and nothing in the riskfree asset. The next columns show the standard errors σ_r that takes the risk aversion $\gamma = 4.02$ as a given. In case we assume normally distributed returns, the standard errors as reported in the third column of Table 5 coincide with the standard errors suggested by Britten-Jones (1999). The last two columns report the standard errors σ_τ that take into account that the risk aversion $\gamma = 4.02$ has to be estimated, i.e., adjusted for the constraint that for the tangency portfolio the weights have to sum to one. Although the standard errors σ_r and σ_τ are

comparable, the estimates show that in three out of seven countries (Canada, UK and US) the differences between σ_τ and σ_τ are about 10 percent. This illustrates that it is important to take the portfolio constraint for the tangency portfolio into account. The fact that we do not know the location of the tangency portfolio, i.e., we do not know which risk aversion is associated with it, induces additional estimation error which is nontrivial.

The standard errors for the tangency weights are big. It is only for Italy and Japan that they are smaller than 0.25, but there the estimated weights are close to zero. For Canada and the UK the estimated weights are about one standard error away from zero and it is only for the US that the estimated weight is significantly different from zero. This is due to the large estimated weight for the US, since the US also has the biggest standard error (0.44).

5 Summary and conclusions

This paper shows how to derive the asymptotic covariance matrix of estimated MVE portfolio weights for various cases. First, assuming normality, we obtain simple expressions for the covariance matrix in case there is no riskfree asset (gross returns) and in case there is a risk free asset available (excess returns). For the case of gross returns, we start out with the GMV portfolio, where the asymptotic covariance matrix of the GMV portfolio weights depends on the variance of this portfolio, the covariance matrix of the asset returns and the GMV portfolio itself. For MVE portfolios associated with lower risk aversions, the asymptotic covariance matrix of the portfolio weights is proportional to the covariance matrix of the GMV portfolio, where the constant of proportionality is always bigger than one and is a function of the risk aversion and the asymptotic slope of the MVE Frontier. Therefore, the estimation error is smallest for the GMV portfolio. We illustrate this for the G7 countries, where the standard errors for the weights in the GMV portfolio are about 5 percent, increasing to about 25 percent for a MVE portfolio based on a risk aversion of 5.

For excess returns, we find that the covariance matrix of the efficient portfolio weights also depends on the covariance matrix of the excess returns, the risk aversion and the Sharpe ratio of the tangency portfolio (i.e., the slope of the frontier). For the tangency portfolio we show that there is additional estimation error because of the fact that the location of the tangency portfolio (i.e., the risk aversion that makes sure that the weights sum to one) has to be estimated. For the G7 countries, the standard errors of the tangency portfolio weights are found to be between 0.22 and 0.44.

The empirical illustration shows that relaxing the normality assumption usually yields similar standard errors as in the case normality is assumed, except for portfolios close to the GMV portfolio. However, in specific cases the differences between the standard errors with or without assuming normalities can be sizable, even for portfolios based on low risk aversions.

A The covariance matrix of the GMV portfolio weights

To find the asymptotic covariance matrix for the GMV portfolio, start from (10) and the derivatives of A and $\Sigma^{-1}\iota$ with respect to the elements of Σ , which are:

$$\frac{\partial A}{\partial \text{vec}(\Sigma)'} = (\iota' \otimes \iota') (\Sigma^{-1} \otimes \Sigma^{-1})$$

and

$$\frac{\partial \Sigma^{-1}\iota}{\partial \text{vec}(\Sigma)'} = (\iota' \otimes I) (\Sigma^{-1} \otimes \Sigma^{-1})$$

Using these, we can apply the rules of differentiation to obtain the derivatives of w_g with respect to the elements of Σ :

$$\begin{aligned} \frac{\partial w_g}{\partial \text{vec}(\Sigma)'} &= \frac{1}{A} (\iota' \otimes I) (\Sigma^{-1} \otimes \Sigma^{-1}) - \frac{1}{A^2} \Sigma^{-1}\iota (\iota' \otimes \iota') (\Sigma^{-1} \otimes \Sigma^{-1}) \\ &= \frac{1}{A} \{(\iota' \otimes I) - w_g (\iota' \otimes \iota')\} (\Sigma^{-1} \otimes \Sigma^{-1}). \end{aligned}$$

When returns are normally distributed, applying these to the covariance matrix of $\text{vec}(\hat{\Sigma}) = (\Sigma \otimes \Sigma)$ gives us:

$$\begin{aligned} \Omega_g &= \frac{\partial w_g}{\partial \text{vec}(\Sigma)'} (\Sigma \otimes \Sigma) \frac{\partial w_g'}{\partial \text{vec}(\Sigma)} \\ &= \frac{1}{A^2} ((\iota' \otimes I) - w_g (\iota' \otimes \iota')) (\Sigma^{-1} \otimes \Sigma^{-1}) (\Sigma \otimes \Sigma) (\Sigma^{-1} \otimes \Sigma^{-1}) ((\iota \otimes I) - (\iota \otimes \iota) w_g') \\ &= \frac{1}{A} (\Sigma^{-1} - A w_g w_g'), \end{aligned}$$

which is the first result in the paper.

B The covariance matrix of MVE portfolio weights: gross returns

This appendix shows the necessary steps to derive (16). We start from (2), substituting $\eta = (B - \gamma)/A$:

$$\begin{aligned} w_0 &= \gamma^{-1} \Sigma^{-1} (\mu - \eta \iota) \\ &= \gamma^{-1} \Sigma^{-1} \mu - \gamma^{-1} \frac{B - \gamma}{A} \Sigma^{-1} \iota \\ &= \gamma^{-1} \Sigma^{-1} \mu - \gamma^{-1} B w_g + w_g. \end{aligned}$$

In order to derive the derivatives in (14) and (??) we first have to find these derivatives for the various elements of w_0 :

$$\begin{aligned}\frac{\partial \gamma^{-1} \Sigma^{-1} \mu}{\partial \mu'} &= \gamma^{-1} \Sigma^{-1}, \\ \frac{\partial \gamma^{-1} \Sigma^{-1} \mu}{\partial \text{vec}(\Sigma)'} &= \gamma^{-1} (\mu' \otimes I) (\Sigma^{-1} \otimes \Sigma^{-1}).\end{aligned}$$

and for B :

$$\begin{aligned}\frac{\partial B}{\partial \mu'} &= \iota' \Sigma^{-1}, \\ \frac{\partial B}{\partial \text{vec}(\Sigma)'} &= (\mu' \otimes \iota') (\Sigma^{-1} \otimes \Sigma^{-1}).\end{aligned}$$

The derivatives of w_g with respect to $\text{vec}(\Sigma)$ are given in Section 2.1. Combining these subresults with the rules of differentiation then yields the derivatives in (14) and (15). If we assume normally distributed returns, combining these derivatives with (9) in turn gives us for the first part of Ω_w :

$$\begin{aligned}\frac{\partial w}{\partial \mu'} \Sigma \frac{\partial w'}{\partial \mu} &= \gamma^{-2} (I - w_g \iota') \Sigma^{-1} \Sigma \Sigma^{-1} (I - \iota w'_g) \\ &= \gamma^{-2} (\Sigma^{-1} - A w_g w'_g),\end{aligned}$$

and for the second part:

$$\begin{aligned}& \frac{\partial w}{\partial \text{vec}(\Sigma)'} (\Sigma^{-1} \otimes \Sigma^{-1}) \frac{\partial w'}{\partial \text{vec}(\Sigma)} \\ &= \left\{ \gamma^{-1} ((\mu' \otimes I) - w_g (\mu' \otimes \iota')) - \left(\frac{B}{A\gamma} - \frac{1}{A} \right) ((\iota' \otimes I) - w_g (\iota' \otimes \iota')) \right\} \times \\ & \quad \left\{ \gamma^{-1} ((\Sigma^{-1} \mu \otimes \Sigma^{-1}) - (\Sigma^{-1} \mu \otimes \Sigma^{-1} \iota) w'_g) - \left(\frac{B}{A\gamma} - \frac{1}{A} \right) ((\Sigma^{-1} \iota \otimes \Sigma^{-1}) - (\Sigma^{-1} \iota \otimes \Sigma^{-1} \iota) w'_g) \right\} \\ &= \gamma^{-2} \left(\frac{AC - B^2}{A} \right) (\Sigma^{-1} - A w_g w'_g) + \frac{1}{A} (\Sigma^{-1} - A w_g w'_g).\end{aligned}$$

Combined, these two parts give the expression for the covariance matrix Ω_w in (16b).

C The covariance matrix of MVE portfolio weights: excess returns

In case of excess returns, the relevant derivatives are obtained as:

$$\frac{\partial w_r}{\partial \mu'_r} = \gamma^{-1} \Sigma_{rr}^{-1},$$

and

$$\frac{\partial w_r}{\partial \text{vec}(\Sigma_{rr})'} = \gamma^{-1} (\mu_r' \otimes I) (\Sigma_{rr}^{-1} \otimes \Sigma_{rr}^{-1}).$$

In case of normally distributed returns, it is straightforward to combine these with (6) and obtain Ω_r :

$$\begin{aligned} \Omega_r &= (\partial w_r / \partial \mu_r') \Sigma_{rr} (\partial w_r / \partial \mu_r')' + (\partial w_r / \partial \text{vec}(\Sigma_{rr})') (\Sigma_{rr} \otimes \Sigma_{rr}) (\partial w_r / \partial \text{vec}(\Sigma_{rr})')' \\ &= \gamma^{-2} \Sigma_{rr}^{-1} \Sigma_{rr} \Sigma_{rr}^{-1} + \gamma^{-2} (\mu_r' \otimes I) (\Sigma_{rr}^{-1} \otimes \Sigma_{rr}^{-1}) (\Sigma_{rr} \otimes \Sigma_{rr}) (\Sigma_{rr}^{-1} \otimes \Sigma_{rr}^{-1}) (\mu_r \otimes I) \\ &= \gamma^{-2} \Sigma_{rr}^{-1} + \gamma^{-2} (\mu_r' \Sigma_{rr}^{-1} \mu_r \otimes \Sigma_{rr}^{-1}) = \gamma^{-2} (1 + \mu_r' \Sigma_{rr}^{-1} \mu_r) \Sigma_{rr}^{-1}. \end{aligned}$$

For the tangency portfolio, we know that $w_r' \iota = 1$, and therefore, $\gamma_\tau = \iota' \Sigma^{-1} \mu_r$. Therefore, we have to apply the rules of differentiation and also take derivatives of γ_τ with respect to μ_r and Σ_{rr} . First,

$$\frac{\partial \gamma_\tau}{\mu_r'} = \iota' \Sigma_{rr}^{-1}.$$

Similarly,

$$\frac{\partial \gamma_\tau}{\partial \text{vec}(\Sigma_{rr})'} = (\mu_r' \otimes \iota') (\Sigma_{rr}^{-1} \otimes \Sigma_{rr}^{-1}).$$

Using these additional results we get for the derivative of the tangency portfolio with respect to μ_r and Σ_{rr} :

$$\frac{\partial w_\tau}{\partial \mu_r'} = \gamma_\tau^{-1} \Sigma_{rr}^{-1} - \gamma_\tau^{-1} w_\tau \iota' \Sigma_{rr}^{-1},$$

giving for the first part of Ω_τ :

$$\begin{aligned} &(\gamma_\tau^{-1} \Sigma_{rr}^{-1} - \gamma_\tau^{-1} w_\tau \iota' \Sigma_{rr}^{-1}) \Sigma_{rr} (\gamma_\tau^{-1} \Sigma_{rr}^{-1} - \gamma_\tau^{-1} \Sigma_{rr}^{-1} \iota w_\tau') \\ &= \gamma_\tau^{-2} (I - w_\tau \iota') \Sigma_{rr}^{-1} (I - \iota w_\tau'). \end{aligned}$$

Next, we have:

$$\frac{\partial w_\tau}{\partial \text{vec}(\Sigma)} = \gamma_\tau^{-1} \{(\mu_r' \otimes I) - w_\tau (\mu_r' \otimes \iota')\} (\Sigma_{rr}^{-1} \otimes \Sigma_{rr}^{-1}),$$

giving for the second part of Ω_τ :

$$\begin{aligned} &(\gamma_\tau^{-1} (\mu_r' \otimes I) - \gamma_\tau^{-1} w_\tau (\mu_r' \otimes \iota')) (\Sigma_{rr}^{-1} \otimes \Sigma_{rr}^{-1}) (\gamma_\tau^{-1} (\mu_r \otimes I) - \gamma_\tau^{-1} (\mu_r \otimes \iota) w_\tau') \\ &= \gamma_\tau^{-2} (\mu_r' \Sigma_{rr}^{-1} \mu_r) (I - w_\tau \iota') \Sigma_{rr}^{-1} (I - \iota w_\tau'). \end{aligned}$$

Combining the two parts, we obtain for Ω_τ :

$$\Omega_\tau = \gamma_\tau^{-2} (1 + \mu_r' \Sigma_{rr}^{-1} \mu_r) (I - w_\tau \iota') \Sigma_{rr}^{-1} (I - \iota w_\tau').$$

D References

- Britten-Jones, M., 1999, "The Sampling Error in Estimates of Mean-Variance Efficient Portfolio Weights", *Journal of Finance* 54, 655-672.
- DeRoos, F.A., TerHorst, J.R., and Werker, B.J.M., 2003, "Incorporating Estimation Risk in Portfolio Choice", *Working Paper*, Tilburg University.
- Gibbons, M., Ross, S., and Shanken, J., 1989, "A Test of the Efficiency of a Given Portfolio", *Econometrica* 57, 1121-1152.
- Hamilton, J.D., 1994, *Time Series Analysis*, Princeton University Press.
- Huberman, G., and Kandel, S., 1987, "Mean Variance Spanning", *Journal of Finance* 42, 873-888.
- Jobson, J.D., and Korkie, B., 1989, "A Performance Interpretation of Multivariate Tests of Intersection, Spanning and Asset Pricing", *Journal of Financial and Quantitative Analysis* 24, 185-204.
- Magnus, J.R., and Neudecker, H., 1988, *Matrix Differential Calculus*, Wiley.

Table 1: Summary statistics

The table presents summary statistics for both gross returns and excess returns on the MSCI indices for the G7 countries. All returns are monthly US Dollar-based returns. The first two rows present the monthly means and standard deviations in percentages for gross returns. The next two rows show similar statistics for the excess returns. The correlation matrix shows the correlations between the gross returns in the upper right part, whereas it shows the correlations between the excess returns in the lower left part. Finally, the bottom row presents the efficient set constants. The sample period is January 1975 until December 2000.

	Can	Fra	Ger	Ita	Jap	UK	US
Mean (gross)	1.10	1.46	1.25	1.18	1.18	1.64	1.32
Stdev (gross)	5.63	6.66	5.94	7.64	6.70	6.85	4.32
Mean (excess)	0.45	0.81	0.60	0.54	0.53	0.99	0.67
Stdev (excess)	5.65	6.69	5.97	7.65	6.71	6.86	4.34
Correlations (excess \ gross)							
Can		0.443	0.336	0.302	0.301	0.494	0.702
Fra	0.447		0.615	0.463	0.410	0.542	0.456
Ger	0.341	0.617		0.396	0.345	0.446	0.389
Ita	0.304	0.464	0.398		0.361	0.349	0.249
Jap	0.303	0.413	0.348	0.362		0.366	0.295
UK	0.496	0.544	0.449	0.350	0.368		0.502
US	0.704	0.460	0.395	0.251	0.299	0.504	
Efficient Set Constants:	<i>A</i> :	666.8	<i>B</i> :	8.40	<i>C</i> :	0.116	

Table 2: Standard errors for portfolio weights without a riskfree asset
The table shows the MVE portfolio weights and the associated standard errors (in brackets) for the GMV portfolios and for portfolios based on risk aversions 5 and 10. Standard errors in square brackets assume normally distributed returns. The bottom two rows show the mean portfolio return and the standard deviation of the portfolio return for each portfolio. Results are for Dollar-based returns for the period January 19975 until December 2000.

	GMV			$\gamma = 10$			$\gamma = 5$		
	\hat{w}_g	σ_g	σ_g [norm]	\hat{w}	σ_w	σ_w [norm]	\hat{w}	σ_w	σ_w [norm]
Can	0.014	(0.069)	[0.057]	-0.161	(0.160)	[0.158]	-0.336	(0.301)	[0.301]
Fra	-0.054	(0.041)	[0.048]	0.021	(0.132)	[0.133]	0.097	(0.254)	[0.252]
Ger	0.184	(0.051)	[0.047]	0.118	(0.147)	[0.131]	0.052	(0.267)	[0.250]
Ita	0.089	(0.029)	[0.033]	0.062	(0.095)	[0.093]	0.036	(0.179)	[0.176]
Jap	0.145	(0.039)	[0.037]	0.103	(0.108)	[0.102]	0.061	(0.200)	[0.193]
UK	-0.017	(0.058)	[0.042]	0.113	(0.162)	[0.117]	0.243	(0.285)	[0.221]
US	0.641	(0.071)	[0.066]	0.743	(0.186)	[0.183]	0.846	(0.351)	[0.347]
\bar{R}_p	1.26%			1.37%			1.47%		
σ_p	3.87%			4.01%			4.38%		

Table 3: Standard errors of portfolio weights caused by estimation error in the means only

The table shows the standard errors MVE portfolios based on risk aversions $\gamma = 5$ and $\gamma = 10$ respectively. The reported standard errors σ_μ assume that there is estimation error in the means only and not in the covariances. The columns "% Total" show the fraction of the total variance of the estimated portfolio weights attributed to the estimation error in the means, i.e., $\sigma_{\mu,i}^2/\sigma_{w,i}^2$. Results are for Dollar-based returns for the period January 1975 until December 2000.

	$\gamma = 10$		$\gamma = 5$	
	σ_μ	% Total	σ_μ	% Total
Can	(0.147)	84.4%	(0.294)	95.3%
Fra	(0.123)	86.4%	(0.246)	93.8%
Ger	(0.122)	69.1%	(0.244)	83.5%
Ita	(0.086)	81.9%	(0.172)	92.0%
Jap	(0.094)	75.9%	(0.189)	88.8%
UK	(0.108)	44.6%	(0.216)	57.8%
US	(0.170)	83.7%	(0.339)	93.6%

Table 4: Standard errors for portfolio weights with a riskfree asset

The table shows various MVE efficient portfolios and the associated standard errors (in brackets), based on excess returns. The first three columns show the estimated weights and standard errors for a level of risk aversion $\gamma = 10$. The last three columns show the weights and standard errors for a risk aversion $\gamma = 5$. Standard errors in square brackets assume normally distributed returns. The bottom two rows show the mean excess portfolio return and the standard deviation of the excess portfolio return for each portfolio. Results are for Dollar-based returns for the period January 1975 until December 2000.

	$\gamma = 10$			$\gamma = 5$		
	\hat{w}_r	σ_r	σ_r [norm]	\hat{w}_r	σ_r	σ_r [norm]
Can	-0.169	(0.149)	[0.150]	-0.339	(0.298)	[0.299]
Fra	0.053	(0.126)	[0.126]	0.106	(0.252)	[0.251]
Ger	0.008	(0.131)	[0.127]	0.015	(0.263)	[0.254]
Ita	0.010	(0.090)	[0.089]	0.020	(0.180)	[0.177]
Jap	0.017	(0.102)	[0.098]	0.033	(0.203)	[0.197]
UK	0.124	(0.138)	[0.110]	0.248	(0.277)	[0.220]
US	0.361	(0.194)	[0.197]	0.721	(0.388)	[0.394]
\bar{r}_p	0.35%			0.70%		
σ_p	1.87%			3.75%		

Table 5: Standard errors for the tangency portfolio weights

The table shows portfolio weights and the associated standard errors (in brackets), for the tangency portfolio. The first three columns show the estimated weights and standard errors for a level of risk aversion $\gamma = 4.02$, that yields the tangency portfolio and takes the risk aversion as given. Standard errors in square brackets assume normally distributed returns. The last three columns show the weights and standard errors for the same tangency portfolio, taking into account that the γ_τ has to be estimated. The bottom two rows show the mean excess portfolio return and the standard deviation of the excess portfolio return for each portfolio. Results are for Dollar-based returns for the period January 1975 until December 2000.

	$\gamma = 4.02, \text{tan}$			γ_τ		
	\hat{w}_r	σ_r	σ_r [norm]	\hat{w}_τ	σ_τ	σ_τ [norm]
Can	-0.421	(0.371)	[0.372]	-0.421	(0.392)	[0.405]
Fra	0.131	(0.314)	[0.312]	0.131	(0.319)	[0.319]
Ger	0.019	(0.326)	[0.316]	0.019	(0.326)	[0.314]
Ita	0.025	(0.223)	[0.220]	0.025	(0.222)	[0.219]
Jap	0.041	(0.252)	[0.245]	0.041	(0.249)	[0.242]
UK	0.308	(0.344)	[0.274]	0.308	(0.347)	[0.298]
US	0.896	(0.482)	[0.490]	0.896	(0.442)	[0.439]
\bar{r}_p	0.87%			0.87%		
σ_p	4.46%			4.46%		