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PROJECT GAMES

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Project games

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Abstract:

This paper studies situations in which a project consisting of several activities is not executed as planned. It is divided into three parts. The first part analyzes the case where the activities may be delayed; this possibly induces a delay on the project as a whole with additional costs. Associated delayed project games are defined and are shown to have a nonempty core. The second part considers the case where the activities may be expedited; this possibly induces an expedition of the project as a whole creating profits. Corresponding expedited project games are introduced and are shown to be convex. The third and last part studies situations where some activities may be delayed and some activities may be expedited. Related project games are defined and shown to have a nonempty core.

Keywords: Project planning, delay, expedition, cooperative game, convexity.

JEL classification: C71

1 Introduction

A project consists of a set of activities, for which the interconnections are known, being completed over a period of time and intended to achieve a particular aim. Before the project is realized the time needed to execute each of the activities is estimated and thus, in particular, a planned duration of the project can be determined. In practice, the estimated duration and the duration after realization (or real duration) of an activity may differ and as result the real duration of the project may not coincide with its planned duration. In many real-life situations, if a project is delayed some additional costs arise due to this delay. Moreover, if the project is expedited some extra reward may be obtained. How to allocate the costs (rewards) due to the delay (expedition) of the project among the activities that have caused this difference in duration? Moreover, even if the real duration of the project is as planned, those activities that are delayed might compensate those that have been expedited in avoiding a delay of the project.

In the literature, the focus has been on projects where activities may be delayed but not expedited. Bergantiños and Sánchez (2002) propose two rules to allocate the total delay of the project. They propose to allocate first the total delay among the paths in the project and then, in a second step, the delay assigned to each path is attributed to the activities in the path. Branzêi, Ferrari, Fragnelli and Tijs (2002) analyze the problem of sharing the total delay of a project within the framework of taxation. Their proposal is to consider an associated taxation problem (and associated rules) where the total delay of the project (total tax) has to be allocated among the different activities where the maximal ability to pay for each activity is given by that activity's own delay. By doing so, the underlying structure of the project, and especially the slack of the various paths, is ignored. Finally, Castro and Tejada (2005), also in the same delayed project setting, propose a parameterized family of rules stemming from the cost sharing literature. A common aspect in these three papers is that game theoretical aspects are only indirectly present in analyzing the allocation problem related to project situations.

In this article we will not only provide a direct approach based on cooperative game theory to analyze allocation issues in delayed project situations but we also consider the opposite setting where activities may be expedited but not delayed. Moreover, we will also analyze the mixed case where some activities may be delayed and some activities may be expedited. Throughout we will assume that the associated reward and cost functions are linear with respect to the difference of the planned and real project times. For a better understanding of the rather technical general problem where some activities are delayed and some are expedited, we will separately study the situations where all activities are either delayed or expedited. Moreover, in case activities can not be delayed but only expedited stronger results can be obtained. Another reason to treat delayed project problems separately is the direct connection to the usual setting in the literature. Hence, three different models are studied: delayed project problems (activities are possibly delayed), expedited project problems (activities are possibly expedited), and project problems (activities may be delayed or expedited). It is shown that (general) project games have a nonempty core, while expedited project games are convex.

Section 2 introduces the definitions and terminology on projects. Section 3 studies delayed project problems. An associated delayed project game is introduced where the worth of a coalition measures the maximum contribution of the coalition to the total delay of the project caused by those paths in which the coalition is involved. It is shown that delayed project games have a nonempty core. In Section 4 we study expedited project problems. First, we note that the total expedition can be divided in several parts depending on the slack of the paths of the planned project. Besides, we distinguish between several levels of expedition that can be claimed by a specific set of paths. Using such a "peeling of" approach, we define expedited project games by applying ideas from bankruptcy games recursively to the various levels of the total expedition in an interrelated way. Although expedited project games are not necessarily (strategically equivalent to) bankruptcy games, they turn out to be convex. Section 5 studies general project problems. We define an associated project game where the underlying ideas of section 3 and 4 are combined. It is shown that project games have a nonempty core.

2 Project situations

A project consists of a set of activities for which the inter-connections are known. These activities are completed over a period of time and intended to achieve a particular aim. Let N denote the set of activities of a project. Given an activity $i \in N$, let P_i denote the set of *predecessors* of i, i.e. the set of activities that have to be processed before i can start. Analogously, let F_i be the set of *followers* of i, i.e. the set of activities that need i to be completed before starting. A *project* will be defined as a collection of *ordered* subsets of N or paths, $\{N_1, \ldots, N_m\}$, where a bijection $\sigma_t : \{1, \ldots |N_t|\} \to N_t$ describes the order in N_t, $t \in \{1, \ldots, m\}$, satisfying the following conditions:

(i) $N = \bigcup_{t=1}^{m} N_t$;

(ii) $F_{\sigma_t(|N_t|)} = \emptyset$, $P_{\sigma_t(1)} = \emptyset$, and $P_{\sigma_t(r)} = \{\sigma_t(1), \ldots, \sigma_t(r-1)\}$ for every $t \in \{1, \ldots, m\}$ and every $r \in \{2, \ldots, |N_t|\}$;

(iii) for $t, u \in \{1, \ldots, m\}$, if $i, j \in N_t \cap N_u$ with $\sigma_t^{-1}(i) < \sigma_t^{-1}(j)$, then $\sigma_u^{-1}(i) < \sigma_u^{-1}(j)$.

A project is called a *parallel project* if $\{N_1, \ldots, N_m\}$ is a partition of N. Throughout there will be no specific need to explicitly keep track of the ordering. Therefore, $\sigma_1, \ldots, \sigma_m$ are suppressed from the notations.

Note that a project can be represented by a directed graph where the set of arcs corresponds to the set of activities. In order to avoid multiple arcs dummy activities are introduced in the graph (a dummy activity is an activity that does not consume neither time nor resources). Dummy activities will be represented by a discontinuous arc.

Example 2.1. Table 1 gives the set of activities of a project with their corresponding predecessors.

Activity	Predecessors
A	
B	
C	A,B

Table 1: Predecessors of activities.

Here, the set of activities is $N = \{A, B, C\}$ and the collection of paths is $\{N_1, N_2\}$, with $N_1 = \{A, C\}$, and $N_2 = \{B, C\}$. The graphical representation of this project is given in Figure 1.

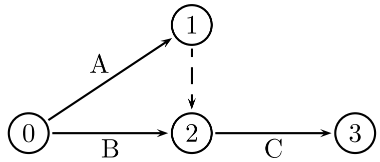

Figure 1: Representation of the project given in Table 1.

□

Associated to a project $\{N_1, \ldots, N_m\}$ there is a function $l : N \to [0, \infty)$ with $l(i)$ denoting the length or *duration* of activity $i \in N$. Given a project $\{N_1, \ldots, N_m\}$ and a duration function l, we define the *duration of a path* N_t, as the sum of the duration of its activities, i.e. as $\sum_{i \in N_t} l(i)$. The *duration of the project* according to l, $D(l)$, is the maximum duration of its paths, i.e. $D(l) = \max_{1 \leq t \leq m} \left\{ \sum_{i \in N_t} l(i) \right\}$. The *slack of* N_t is the maximum time that the activities of N_t can be delayed without altering the duration of the project, i.e. $\text{slack}(N_t, l) = D(l) - \sum_{i \in N_t} l(i)$. We say that a path is *critical* if it has slack zero.

Example 2.2. Consider the project given in Example 2.1 and let $l : N \to [0, \infty)$ be given by $l(A) = 3$, $l(B) = 5$, and $l(C) = 2$. Table 2 summarizes the duration and slack of the paths. Note that $D(l) = 7$.

N_t	Duration	slack(N_t)
AC	5	2
BC	7	0

Table 2: Duration and slack of the paths.

□

Throughout we will use a fixed notation for two specific duration functions. We will denote by $p : N \to [0, \infty)$ the function representing the *planned* or estimated *time* of the activities and by $r : N \to [0, \infty)$ the function giving the *real time* of the activities after the realization of the project. We define the *delay function* $d : N \to [0, \infty)$ as $d(i) = (r(i) - p(i))_+ (:= \max\{r(i) - p(i), 0\})$, i.e. d(i) represents the delay of activity i. Analogously, we define the *expedition function* $e : N \to [0, \infty)$ as $e(i) = (p(i) - r(i))_+$, i.e. e(i) represents the expedition of activity i.

In the following sections we will study three kind of situations. Section 3 is devoted to delayed project problems where $r \geq p$. In Section 4 expedited project problems are studied, where $r \leq p$. Finally, Section 5 analyzes the general situation.

3 Delayed project games

In this section we will study those project situations where activities may be delayed but not expedited, which possibly causes the real duration of the project to be larger than the planned duration. A cost is associated to the delay of the project which will be assumed linear w.r.t. the total delay of the project. Due to the linearity of the cost function, we can identify the total cost with the total delay of the project. We will analyze the corresponding allocation problem with techniques from cooperative TU games.

Before starting our discussion we will recall some basic concepts from game theory. A cooperative *cost game* in characteristic function form is an ordered pair (N, c) where N is a finite set (the set of players) and $c : 2^N \to \mathbb{R}$ represents the maximum amount of cost chargeable to the different coalitions (or subsets of players) satisfying $c(\emptyset) = 0$. The *core* of a cost game (N, c) is defined by

$$\text{Core}(c) = \{x \in \mathbb{R}^N | \sum_{i \in N} x_i = c(N), \sum_{i \in S} x_i \leq c(S) \text{ for all } S \in 2^N\},$$

i.e. the core is the set of allocations of $c(N)$ to which no coalition can reasonably object. An important subclass of cost games with nonempty core is the class of concave games. A game (N, c) is said to be *concave* if

$$c(T) + c(S) \geq c(T \cup S) + c(T \cap S) \tag{3.1}$$

for every $S, T \subset N$.

Now, we start our study on delayed project problems. We define *delayed project problems* as those project problems where the planned time of the activities was underestimated, i.e. the real time of an activity is at least its planned time. Hence, a delayed project problem can be described by $(\{N_1, \ldots, N_m\}, p, r)$ with $p \leq r$. To a delayed project problem we associate a delayed project game where the set of players is the set of activities and the cost of a coalition is the maximal contribution of the coalition to the delay of the project caused by those paths where members of S are involved. Formally, given a delayed project problem $(\{N_1, \ldots, N_m\}, p, r)$ we define the associated cost game (N, c) by

$$c(S) = \max_{t \in \mathcal{P}(S)} \left\{ \max \left\{ \sum_{i \in N_t} r(i), D(p) \right\} - \max \left\{ \sum_{i \in N_t \setminus S} r(i) + \sum_{i \in N_t \cap S} p(i), D(p) \right\} \right\}$$

for every $S \subset N$, where $\mathcal{P}(S) = \{t \in \{1, \ldots, m\} | \ N_t \cap S \neq \emptyset\}$ represents the set of paths in which S is involved. Note that $c(N)$ equals the total delay of the project.

Example 3.1. Consider the project given in Example 2.1 and let $p : N \to [0, \infty)$ be given by $p(A) = 3$, $p(B) = 5$, and $p(C) = 2$ and $r : N \to [0, \infty)$ by $r(A) = 6$, $r(B) = 9$, and $r(C) = 3$. Table 3 gives the duration of the paths according to the planned and real times.

N_t	Duration w.r.t. p	Duration w.r.t. r
AC	5	9
BC	7	12

Table 3: Duration of the paths.

Hence, the planned duration of the project is $D(p) = 7$ while the real duration is $D(r) = 12$. Therefore, there is a total delay of 5 units of time. The value of the associated delayed project game, (N, c), for coalition $\{A, B\}$ is obtained as follows: $\mathcal{P}(\{A, B\}) = \{1, 2\}$, the contribution of $\{A, B\}$ to the delay caused by N_1 is $\max\{r(A) + r(C), D(p)\} - \max\{p(A) + r(C), D(p)\} = \max\{6 + 3, 7\} - \max\{3 + 3, 7\} = 2$ and the contribution of $\{A, B\}$ to the delay caused by N_2 is $\max\{r(B) + r(C), D(p)\} - \max\{p(B) + r(C), D(p)\} = \max\{9+3, 7\} - \max\{5+3, 7\} = 4$, then $c(\{A, B\}) = \max\{2, 4\} = 4$. All values of the game are: $c(\{A\}) = 2$, $c(\{B\}) = 4$, $c(\{C\}) = 1$, $c(\{A, B\}) = 4$, $c(\{A, C\}) = 2$, $c(\{B, C\}) = 5$, $c(N) = 5$. □

Next, we will give an alternative expression of the coalitional values in a delayed project game.

Lemma 3.1. Let $(\{N_1, \ldots, N_m\}, p, r)$ be a delayed project problem and let (N, c) be the associated delayed project game. Then,

$$c(S) = \max_{t \in \mathcal{P}(S)} \left\{ \min \left\{ \sum_{i \in N_t \cap S} d(i), \left(\sum_{i \in N_t} d(i) - \text{slack}(N_t, p) \right)_+ \right\} \right\}$$

for every $S \subset N$.

Proof: Let $S \subset N$. Then,

$$\begin{aligned}
c(S) &= \max_{t \in \mathcal{P}(S)} \left\{ \max\left\{ \sum_{i \in N_t} r(i), D(p) \right\} - \max\left\{ \sum_{i \in N_t \setminus S} r(i) + \sum_{i \in N_t \cap S} p(i), D(p) \right\} \right\} \\
&= \max_{t \in \mathcal{P}(S)} \left\{ \left(\sum_{i \in N_t} r(i) - D(p) \right)_+ - \left(\sum_{i \in N_t \setminus S} r(i) + \sum_{i \in N_t \cap S} p(i) - D(p) \right)_+ \right\} \\
&= \max_{t \in \mathcal{P}(S)} \left\{ \left(\sum_{i \in N_t} (r(i) - p(i)) + \sum_{i \in N_t} p(i) - D(p) \right)_+ - \left(\sum_{i \in N_t \setminus S} (r(i) - p(i)) + \sum_{i \in N_t} p(i) - D(p) \right)_+ \right\} \\
&= \max_{t \in \mathcal{P}(S)} \left\{ \left(\sum_{i \in N_t} d(i) - \text{slack}(N_t, p) \right)_+ - \left(\sum_{i \in N_t \setminus S} d(i) - \text{slack}(N_t, p) \right)_+ \right\} \\
&= \max_{t \in \mathcal{P}(S)} \left\{ \min \left\{ \sum_{i \in N_t \cap S} d(i), \left(\sum_{i \in N_t} d(i) - \text{slack}(N_t, p) \right)_+ \right\} \right\}
\end{aligned}$$

The first four equalities are straightforward. The last equality is proven below. We distinguish between three cases:

Case 1: $\left(\sum_{i\in N_t} d(i) - \text{slack}(N_t,p)\right)_+ = 0$. Then $\left(\sum_{i\in N_t\setminus S} d(i) - \text{slack}(N_t,p)\right)_+ = 0$ and

$$\min\left\{\sum_{i\in N_t\cap S} d(i), \left(\sum_{i\in N_t} d(i) - \text{slack}(N_t,p)\right)_+\right\} = 0.$$

Case 2: $\left(\sum_{i\in N_t} d(i) - \text{slack}(N_t,p)\right)_+ > 0$, $\left(\sum_{i\in N_t\setminus S} d(i) - \text{slack}(N_t,p)\right)_+ = 0$. Then, $\sum_{i\in N_t\setminus S} d(i) - \text{slack}(N_t,p) \leq 0$ and hence $\sum_{i\in N_t} d(i) - \text{slack}(N_t,p) \leq \sum_{i\in N_t\cap S} d(i)$. Therefore,

$$\left(\sum_{i\in N_t} d(i) - \text{slack}(N_t,p)\right)_+ = \min\left\{\sum_{i\in N_t\cap S} d(i), \left(\sum_{i\in N_t} d(i) - \text{slack}(N_t,p)\right)_+\right\}.$$

Case 3: $\left(\sum_{i\in N_t} d(i) - \text{slack}(N_t,p)\right)_+ > 0$, $\left(\sum_{i\in N_t\setminus S} d(i) - \text{slack}(N_t,p)\right)_+ > 0$. Then, $\sum_{i\in N_t\setminus S} d(i) - \text{slack}(N_t,p) > 0$ and hence $\sum_{i\in N_t} d(i) - \text{slack}(N_t,p) > \sum_{i\in N_t\cap S} d(i)$. Therefore,

$$\sum_{i\in N_t\cap S} d(i) = \min\left\{\sum_{i\in N_t\cap S} d(i), \left(\sum_{i\in N_t} d(i) - \text{slack}(N_t,p)\right)_+\right\}.$$

□

Next, we will show that delayed project games have a nonempty core. To do this, we will first study delayed parallel project games. The following result states that subgames obtained by restricting delayed parallel project games to the paths of the project are concave.

Lemma 3.2. Let $(\{N_1,\ldots,N_m\},p,r)$ be a delayed parallel project problem and let (N,c) be the associated delayed project game. Then, $(N_t, c_{|N_t})$ is concave for any $t\in\{1,\ldots,m\}$.

Proof: Let $t\in\{1,\ldots,m\}$. According to Lemma 3.1, $c_{|N_t}(S) = \min\{\sum_{i\in S} d_i, \left(\sum_{i\in N_t} d(i) - \text{slack}(N_t,p)\right)_+\}$ for every $S\subset N_t$. Let $S,T\subset N_t$. We will show that

$$c_{|N_t}(S) + c_{|N_t}(T) \geq c_{|N_t}(S\cup T) + c_{|N_t}(S\cap T).$$

We will distinguish between two cases.

Case 1: $c_{|N_t}(S) = \sum_{i\in S} d_i$ and $c_{|N_t}(T) = \sum_{i\in T} d_i$. Hence, $c_{|N_t}(S\cap T) = \sum_{i\in S\cap T} d_i$ and

$$\begin{aligned} c_{|N_t}(S) + c_{|N_t}(T) &= \sum_{i\in S} d_i + \sum_{i\in T} d_i = \sum_{i\in S\cup T} d_i + \sum_{i\in S\cap T} d_i \\ &\geq \min\left\{\sum_{i\in S\cup T} d_i,\ \left(\sum_{i\in N_t} d(i) - \text{slack}(N_t,p)\right)_+\right\} + \sum_{i\in S\cap T} d_i \\ &= c_{|N_t}(S\cup T) + c_{|N_t}(S\cap T). \end{aligned}$$

Case 2: $c_{|N_t}(S) = \left(\sum_{i\in N_t} d(i) - \text{slack}(N_t,p)\right)_+$. Hence, $c_{|N_t}(S\cup T) = \left(\sum_{i\in N_t} d(i) - \text{slack}(N_t,p)\right)_+$ and

$$\begin{aligned}
c_{|N_t}(S) + c_{|N_t}(T) &= \left(\sum_{i\in N_t} d(i) - \text{slack}(N_t,p)\right)_+ + \min\left\{\sum_{i\in T} d_i,\ \left(\sum_{i\in N_t} d(i) - \text{slack}(N_t,p)\right)_+\right\} = \\
&\geq \left(\sum_{i\in N_t} d(i) - \text{slack}(N_t,p)\right)_+ + \min\left\{\sum_{i\in S\cap T} d_i,\ \left(\sum_{i\in N_t} d(i) - \text{slack}(N_t,p)\right)_+\right\} \\
&= c_{|N_t}(S\cup T) + c_{|N_t}(S\cap T).
\end{aligned}$$

□

The next example illustrates that games arising from delayed parallel project problems need not be concave.

Example 3.2. Consider the delayed parallel project problem $(\{N_1, N_2\}, p, r)$ with $N_1 = \{\text{A,B}\}$, $N_2 = \{\text{C,D,E}\}$; $p = (3,5,2,1,3)$; and $r = (4,7,5,4,5)$. The project is represented in Figure 2. Let (N,c) be the associated delayed parallel project game. Let $S = \{\text{A,B,D}\}$ and $T = \{\text{A,B,C}\}$. Then, $c(S) + c(T) = 3 + 3 = 6 < 9 = 6 + 3 = c(S\cup T) + c(S\cap T)$ and the game is not concave.

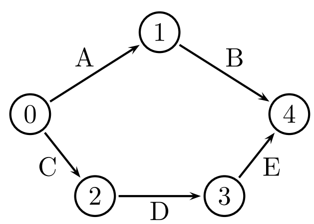

Figure 2: Representation of the parallel project in Example 3.2.

□

Lemma 3.3. Delayed project games associated to delayed parallel project problems have a nonempty core.

Proof: Let $(\{N_1,\ldots,N_m\}, p, r)$ be a delayed parallel project problem and let (N,c) be the associated game. Let N_t be a critical path in the realization. Hence, $c(N_t) = \sum_{i\in N_t} r(i) - D(p) = D(r) - D(p) = c(N)$. By Lemma 3.2, there exists $y \in \text{Core}(c_{|N_t})$. Let $x \in \mathbb{R}^N$ defined as

$$x_i = \begin{cases} y_i & \text{if } i \in N_t, \\ 0 & \text{if } i \in N\setminus N_t. \end{cases}$$

for every $i \in N$. We will show that $x \in \text{Core}(c)$. First, we will show efficiency.

$$\sum_{i\in N} x_i = \sum_{i\in N_t} y_i = c(N_t) = c(N)$$

where the second equality holds because $y \in \text{Core}(c_{|N_t})$ and $c(N_t) = c_{|N_t}(N_t)$, and the third one because N_t is critical in the realization.

Next, we will show stability. If $S \subset N \setminus N_t$, then $\sum_{i\in S} x_i = 0 \le c(S)$ because c is non-negative. Let $S \subset N$, $S \cap N_t \neq \emptyset$. Hence,

$$\sum_{i\in S} x_i = \sum_{i\in S\cap N_t} y_i \le c_{|N_t}(S\cap N_t) = c(S\cap N_t) \le c(S)$$

where the first inequality holds because $y \in \text{Core}(c_{|N_t})$ and the second because $c(S) = \max_{u\in\mathcal{P}(S)}\{c(S\cap N_u)\}$. $\square$

Theorem 3.4. Delayed project games have a nonempty core.

Proof: Let $(\{N_1, \dots, N_m\}, p, r)$ be a delayed project problem and let (N, c) be the associated delayed project game. Let $(\{N_1^*, \dots, N_m^*\}, p^*, r^*)$ be the delayed project problem defined as follows: $N_t^* = \{i^t |\ i \in N_t\}$, with $p^*(i^t) = p(i)$ and $r^*(i^t) = r(i)$ for all $i \in N$. Note that $(\{N_1^*, \dots, N_m^*\}, p^*, r^*)$ is a delayed parallel project problem with $N^* = \bigcup_{t=1}^m N_t^*$. Let (N^*, c^*) be the associated delayed project game. One readily verifies that, $c(S) = c^*(S^*)$ for every $S \subset N$ with $S^* := \bigcup_{t=1}^m \{i^t |\ i \in N_t \cap S\} \subset N^*$.

By Lemma 3.3, there exists a $y \in \text{Core}(c^*)$. Let $x \in \mathbb{R}^N$ defined as $x_i = \sum_{t=1}^m y_{i^t}$. We will show that $x \in \text{Core}(c)$. Efficiency holds by construction of x, because $y \in \text{Core}(c^*)$ and $c(N) = c^*(N^*)$. Next, we will show stability. Let $S \subset N$. Then,

$$\sum_{i\in S} x_i = \sum_{i\in S}\sum_{t=1}^{m} y_{i^t} = \sum_{i\in S^*} y_i \le c^*(S^*) = c(S)$$

$\square$

4 Expedited project games

This section analyzes project situations in which activities may be expedited but not delayed. Consequently, the duration of the project after realization may be shorter than the planned duration. A reward is associated to the expedition of the project which will be assumed to be linear w.r.t. the total expedition of the project. Again, due to the linearity of the reward function, we will identify the total reward with the total expedition of the project.

Contrary to the previous section we are now in a reward setting. For this reason we will briefly overview concepts from reward games for our later purposes. A cooperative *reward game* in characteristic function form is an ordered pair (N, v) where N is a finite set of players and $v : 2^N \to \mathbb{R}$ is the function representing the worth of each coalition, which satisfies $v(\emptyset) = 0$. The *core* of a game (N, v) is defined by

$$\text{Core}(v) = \{x \in \mathbb{R}^N | \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N\},$$

i.e. the core is the set of efficient allocations of $v(N)$ to which no coalition can reasonably object. An important subclass of games with nonempty core is the class of convex games (see Shapley (1971)). A game (N, v) is said to be *convex* if

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)^1 \tag{4.1}$$

for every $i \in N$ and every $S \subset T \subset N \setminus \{i\}$.

We define *expedited project problems* as those project situations where the planned time of the activities was overestimated, i.e. the real time of an activity is at most its planned time. Hence, an expedited project problem can be described by a 3-tuple $(\{N_1, \ldots, N_m\}, p, r)$ with $p \geq r$. We first illustrate this kind of problem by means of an example.

Example 4.1. Consider the expedited project problem $(\{N_1, N_2, N_3\}, p, r)$ with $N_1 = \{\text{A,B}\}$, $N_2 = \{\text{C}\}$, and $N_3 = \{\text{D}\}$; $p = (6, 4, 9, 7)$; and $r = (3, 2, 4, 6)$. This project is represented in Figure 3.

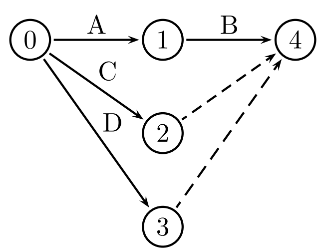

Figure 3: Representation of the project in Example 4.1.

The total duration of the planned project is 10 units of time, while the real duration of the project after realization is 6 units of time. Hence, the total expedition of the project is 4 units of time. Table 4 gives the duration and slack of the paths according to the plan.

[1]This is equivalent to the dual of Equation (3.1): $v(T) + v(S) \leq v(T \cup S) + v(T \cap S)$ for every $S, T \subset N$.

N_t	Duration	slack(N_t, p)
AB	10	0
C	9	1
D	7	3

Table 4: Duration and slack of the paths according to the planned times.

Note that the project can not be expedited if the critical path N_1 according to plan is not expedited. Next, we will analyze how this total expedition is obtained. First, suppose that only the activities in N_1 act according to the realization while the activities in paths N_2 and N_3 act according to the plan. Then, the project is expedited just 1 unit of time, while path N_2 becomes critical, and path N_3 has an slack of 2 units of time in the new situation. Hence, path N_1 is responsible by itself of 1 unit of time of the total expedition. Second, suppose that the activities both in N_1 and N_2 act according to realization while the activities in N_3 act according to plan. Then, N_3 becomes critical and there is an additional expedition of 2 units of time. Hence, both paths N_1 and N_2 are needed for and responsible of 2 units of time of the total expedition. Finally, suppose that all the activities act according to realization, then there is an additional expedition of 1 unit of time for which all paths N_1, N_2, and N_3 are responsible. The contribution of the paths to the total expedition of the project during the different phases is summarized in Table 5. Note that the sum of the first row gives the total expedition of the project. This kind of "peeling of" into levels of expedition will play a prominent role in the definition of expedited project games below.

	Phase 1	Phase 2	Phase 3
N_1	1	2	1
N_2	0	2	1
N_3	0	0	1

Table 5: Contribution of the paths to the total expedition of the project.

$\square$

Let $(\{N_1, \ldots, N_m\}, p, r)$ be an expedited project problem. We denote by I_1 the set (of indices) of critical paths according to the planned time. Formally,

$$I_1 = \{t \in \{1, \ldots, m\} |\ \text{slack}(N_t, p) = 0\}.$$

Recursively we define for $k \geq 2$,

$$I_k = \left\{ t \in \{1, \ldots, m\} \setminus \bigcup_{l=1}^{k-1} I_l |\ \text{slack}(N_t, p) \leq \text{slack}(N_u, p) \text{ for all } u \in \{1, \ldots, m\} \setminus \bigcup_{l=1}^{k-1} I_l \right\},$$

i.e. I_k corresponds to all paths that would be critical in the (sub)project where all the paths in $I_1, \ldots, I_{k-1}$ were not present. We will denote by slack(I_k) the slack of the paths in I_k according to the planned time, i.e. slack$(I_k) = $ slack(N_t, p) for each $t \in I_k$. Let h be such that slack$(I_h) < D(p) - D(r) \leq$ slack(I_{h+1}). For $k = 1, \ldots, h$, we define E^k as the level of expedition for which all paths in $I_1, \ldots, I_k$ are needed:

$$E^k = \begin{cases} \text{slack}(I_{k+1}) - \text{slack}(I_k) & \text{if } 1 \leq k < h; \\ D(p) - D(r) - \text{slack}(I_h) & \text{if } k = h. \end{cases}$$

Note that $\sum_{k=1}^{h} E^k = D(p) - D(r)$.

To an expedited project problem we associate an expedited project game. The set of players is the set of activities. The worth of a coalition is the sum over all $k \in \{1, \ldots, h\}$ of those specific parts of the level of expedition E^k for which the activities outside the coalition that are in paths of $\bigcup_{l=1}^{k} I_l$ can not be held responsible for anymore at that phase. Formally, given an expedited project problem $(\{N_1, \ldots, N_m\}, p, r)$ we define the associated game (N, v), where v is defined by

$$v(S) = \sum_{k=1}^{h} \left(E^k - w^k(S)\right) \tag{4.2}$$

for every $S \subset N$, where for all $k \in \{1, \ldots, h\}$, $w^k(S)$ is recursively defined by

$$w^k(S) = \min \left\{ \sum_{i \in (\bigcup_{l=1}^{k} N_{I_l}) \setminus S} e(i) - \sum_{l=1}^{k-1} w^l(S),\ E^k \right\}, \tag{4.3}$$

with $N_{I_l} = \bigcup_{t \in I_l} N_t$. Here, $w^k(S)$ represents the part of the level of expedition E^k that players in S maximally would have to concede to players in the paths corresponding to $\bigcup_{l=1}^{k} I_l$ outside S, taking into account earlier concessions from the previous phases. Note that w^k is non-negative. Moreover, $v(N)$ equals the total expedition of the project because $w^k(N) = 0$ for any $k \in \{1, \ldots, h\}$.

Example 4.2. Consider the expedited project problem given in Example 4.1. Recall that the problem was given by $(\{N_1, N_2, N_3\}, p, r)$ with $N_1 = \{\text{A,B}\}$, $N_2 = \{\text{C}\}$, and $N_3 = \{\text{D}\}$; $p = (6, 4, 9, 7)$; and $r = (3, 2, 4, 6)$. Here, $D(p) = 10$ and $D(r) = 6$. Hence, the total expedition is $D(p) - D(r) = 4$ units of time. Here, $e = (3, 2, 5, 1)$; $I_1 = \{1\}$, $I_2 = \{2\}$, and $I_3 = \{3\}$; and $E^1 = $ slack$(I_2) - $slack$(I_1) = 1$, $E^2 = $ slack$(I_3) - $slack$(I_2) = 2$, and $E^3 = D(p) - D(r) - $ slack$(I_3) = 1$. Consequently, for coalition $\{A, C\}$:

$$w^1(\{A, C\}) = \min \left\{e(B),\ E^1\right\} = \min\{2, 1\} = 1,$$

$$w^2(\{A, C\}) = \min \left\{e(B) - w^1(\{A, C\}),\ E^2\right\} = \min\{2 - 1, 2\} = 1,$$

$$w^3(\{A, C\}) = \min \left\{e(B) + e(D) - w^1(\{A, C\}) - w^2(\{A, C\}),\ E^3\right\} = \min\{2 + 1 - 1 - 1, 1\} = 1.$$

Let (N, v) be the associated expedited project game. Then,

$$v(\{A,C\}) = (E^1 - w^1(\{A,C\})) + (E^2 - w^2(\{A,C\})) + (E^3 - w^3(\{A,C\})) = (1-1)+(2-1)+(1-1) = 1.$$

All coalitional worths are given by: $v(\{A\}) = v(\{B\}) = v(\{C\}) = v(\{D\}) = 0$, $v(\{A,B\}) = v(\{A,C\}) = 1$, $v(\{A,D\}) = v(\{B,C\}) = v(\{B,D\}) = v(\{C,D\}) = 0$, $v(\{A,B,C\}) = 3$, $v(\{A,B,D\}) = 1$, $v(\{A,C,D\}) = 2$, $v(\{B,C,D\}) = 1$, $v(N) = 4$. □

Next, we introduce some extra notation. For $R \subset N$, $\mathcal{A}(R)$ denotes the set of "active levels of expedition" of R, i.e.

$$\mathcal{A}(R) := \{k \in \{1,\ldots,h\} |\ w^k(R) < E^k\}.$$

and $A(R)$ corresponds to the highest active level of expedition of R, i.e.

$$A(R) := \begin{cases} \max \mathcal{A}(R) & \text{if } \mathcal{A}(R) \neq \emptyset; \\ 0 & \text{if } \mathcal{A}(R) = \emptyset. \end{cases}.$$

For $j \in N$, $k(j)$ corresponds to the first level of expedition in which j is involved, i.e.

$$k(j) := \min\{k \in \{1,\ldots,h\} |\ j \in N_{I_k}\}.$$

For $R \subset N$ and $j \in N$, $\mathcal{A}(j,R)$ corresponds to the set of all active levels of expedition for R in which j is also involved, i.e.

$$\mathcal{A}(j,R) := \{k \in \{1,\ldots,h\} |\ k \geq k(j),\ k \in \mathcal{A}(R)\}$$

and $a(j,R)$ corresponds to the first active level of expedition of R in which j is also involved, i.e.

$$a(j,R) := \begin{cases} \min \mathcal{A}(j,R) & \text{if } \mathcal{A}(j,R) \neq \emptyset; \\ h+1 & \text{if } \mathcal{A}(j,R) = \emptyset. \end{cases}.$$

The following example illustrates the notation above.

Example 4.3. Consider the expedited project problem given in Examples 4.1 and 4.2. Let $R = \{A,C\}$ and $i = B$. Recall that $N_1 = \{\text{A,B}\}$, $N_2 = \{\text{C}\}$, and $N_3 = \{\text{D}\}$. Furthermore, $I_1 = \{1\}$, $I_2 = \{2\}$, and $I_3 = \{3\}$ and $E^1 = 1$, $E^2 = 2$, and $E^3 = 1$. Moreover,

$$w^1(\{A,C\}) = 1 = E^1,$$
$$w^2(\{A,C\}) = 1 < E^2,$$
$$w^3(\{A,C\}) = 1 = E^3.$$

Hence, $\mathcal{A}(R) = \{2\}$, $A(R) = 2$, $k(B) = 1$, $\mathcal{A}(i,R) = \{2\}$ and $a(i,R) = 2$. □

The basic message of the following lemma is that, at each level of expedition, the concession of a smaller coalition exceeds the concession of a larger coalition.

Lemma 4.1. Let $(\{N_1,\dots,N_m\},p,r)$ be an expedited project problem. Let $k\in\{1,\dots,h\}$. Then, for every $S\subset T\subset N$ the following three statements hold:

(i) $\sum_{i\in(\bigcup_{l=1}^{k}N_{I_l})\setminus S}e(i)-\sum_{l=1}^{k-1}w^l(S)\geq\sum_{i\in(\bigcup_{l=1}^{k}N_{I_l})\setminus T}e(i)-\sum_{l=1}^{k-1}w^l(T)$,

(ii) $w^k(S)\geq w^k(T)$,

(iii) $\mathcal{A}(S)\subset\mathcal{A}(T)$.

Proof: Obviously, (ii) is a direct consequence of (i) and (iii) follows immediately from (ii).

Let $S\subset T\subset N$. We will show (i) by induction on k. For $k=1$ it is obvious. Let $k=2$. Then

$$\begin{aligned}\sum_{i\in(N_{I_1}\cup N_{I_2})\setminus S}e(i)-w^1(S)&=\sum_{i\in(N_{I_1}\cup N_{I_2})\setminus S}e(i)-\min\{\sum_{i\in N_{I_1}\setminus S}e(i),\ E^1\}\\&=\max\{\sum_{i\in N_{I_2}\setminus(N_{I_1}\cup S)}e(i),\ \sum_{i\in(N_{I_1}\cup N_{I_2})\setminus S}e(i)-E^1\}\\&\geq\max\{\sum_{i\in N_{I_2}\setminus(N_{I_1}\cup T)}e(i),\ \sum_{i\in(N_{I_1}\cup N_{I_2})\setminus T}e(i)-E^1\}\\&=\sum_{i\in(N_{I_1}\cup N_{I_2})\setminus T}e(i)-\min\{\sum_{i\in N_{I_1}\setminus T}e(i),\ E^1\}\\&=\sum_{i\in(N_{I_1}\cup N_{I_2})\setminus T}e(i)-w^1(T)\end{aligned}$$

where the inequality holds because $S\subset T$.

Now, suppose that $k\geq3$ and suppose (i) is true for k-1. Then,

$$\begin{aligned}\sum_{i\in(\bigcup_{l=1}^{k}N_{I_l})\setminus S}e(i)-\sum_{l=1}^{k-1}w^l(S)&=\sum_{i\in(\bigcup_{l=1}^{k}N_{I_l})\setminus S}e(i)-\sum_{l=1}^{k-2}w^l(S)-\min\left\{\sum_{i\in(\bigcup_{l=1}^{k-1}N_{I_l})\setminus S}e(i)-\sum_{l=1}^{k-2}w^l(S),\ E^{k-1}\right\}\\&=\max\left\{\sum_{i\in N_{I_k}\setminus(\bigcup_{l=1}^{k-1}N_{I_l}\cup S)}e(i),\ \sum_{i\in N_{I_k}\setminus(\bigcup_{l=1}^{k-1}N_{I_l}\cup S)}e(i)+\sum_{i\in(\bigcup_{l=1}^{k-1}N_{I_l})\setminus S}e(i)-\sum_{l=1}^{k-2}w^l(S)-E^{k-1}\right\}\\&\geq\max\left\{\sum_{i\in N_{I_k}\setminus(\bigcup_{l=1}^{k-1}N_{I_l}\cup T)}e(i),\ \sum_{i\in N_{I_k}\setminus(\bigcup_{l=1}^{k-1}N_{I_l}\cup T)}e(i)+\sum_{i\in(\bigcup_{l=1}^{k-1}N_{I_l})\setminus T}e(i)-\sum_{l=1}^{k-2}w^l(T)-E^{k-1}\right\}\\&=\sum_{i\in(\bigcup_{l=1}^{k}N_{I_l})\setminus T}e(i)-\sum_{l=1}^{k-2}w^l(T)-\min\left\{\sum_{i\in(\bigcup_{l=1}^{k-1}N_{I_l})\setminus T}e(i)-\sum_{l=1}^{k-2}w^l(T),\ E^{k-1}\right\}\end{aligned}$$

$$= \sum_{i \in (\bigcup_{l=1}^{k} N_{I_l}) \setminus T} e(i) - \sum_{l=1}^{k-1} w^l(T)$$

where the inequality holds by induction together with $S \subset T$. $\square$

In the proof of the main convexity theorem we refer to some technical lemmas that can be found in the appendix.

Theorem 4.2. Expedited project games are convex.

PROOF: Let $i \in N$ and $S \subset T \subset N \setminus \{i\}$. It suffices to show that

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$$

By Equation (4.2) it holds

$$v(S \cup \{i\}) - v(S) = \sum_{l=1}^{h} [w^l(S) - w^l(S \cup \{i\})] \tag{4.4}$$

and

$$v(T \cup \{i\}) - v(T) = \sum_{l=1}^{h} [w^l(T) - w^l(T \cup \{i\})] \tag{4.5}$$

In order to show our result, we will distinguish between three cases.

Case 1: $\mathcal{A}(i, S \cup \{i\}) = \emptyset$. Then,

$$\sum_{l=1}^{h} \left[w^l(S) - w^l(S \cup \{i\})\right] = 0 \leq \sum_{l=1}^{h} \left[w^l(T) - w^l(T \cup \{i\})\right]$$

the equality holds by Lemma A.3 and the inequality holds by Lemma 4.1 (ii).

Case 2: $\mathcal{A}(i, S) \neq \emptyset$. Then, by Lemma 4.1 (iii), $\mathcal{A}(i, T) \neq \emptyset$. Using Lemma A.3,

$$\sum_{l=1}^{h} \left[w^l(S) - w^l(S \cup \{i\})\right] = e(i) = \sum_{l=1}^{h} \left[w^l(T) - w^l(T \cup \{i\})\right]$$

Case 3: $\mathcal{A}(i, S) = \emptyset$ and $\mathcal{A}(i, S \cup \{i\}) \neq \emptyset$. By Lemma 4.1 (iii), $\mathcal{A}(i, T \cup \{i\}) \neq \emptyset$. We will distinguish between two subcases.

Subcase 3.1: $\mathcal{A}(i, T) = \emptyset$. By Lemma A.1 (i) it holds $w^l(S) = w^l(S \cup \{i\})$ and $w^l(T) = w^l(T \cup \{i\})$ for every $l < k(i)$. Moreover, $w^l(S) = w^l(T) = E^l$ for every $l \geq k(i)$ since $\mathcal{A}(i, S) = \mathcal{A}(i, T) = \emptyset$. Then,

$$\begin{aligned} v(S \cup \{i\}) - v(S) &= \sum_{l=1}^{h} [E^l - w^l(S \cup \{i\})] - \sum_{l=1}^{h} [E^l - w^l(S)] \\ &= \sum_{l=1}^{h} [E^l - w^l(S \cup \{i\})] - \sum_{l=1}^{k(i)-1} [E^l - w^l(S)] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=k(i)}^{h} [E^l - w^l(S \cup \{i\})] \\
&\leq \sum_{l=k(i)}^{h} [E^l - w^l(T \cup \{i\})] \\
&= \sum_{l=1}^{h} [E^l - w^l(T \cup \{i\})] - \sum_{l=1}^{k(i)-1} [E^l - w^l(T)] \\
&= \sum_{l=1}^{h} [E^l - w^l(T \cup \{i\})] - \sum_{l=1}^{h} [E^l - w^l(T)] = v(T \cup \{i\}) - v(T)
\end{aligned}
$$

where the inequality holds by Lemma 4.1 (ii).

Subcase 3.2: $\mathcal{A}(i, T) \neq \emptyset$. By Lemma A.3,

$$
\begin{aligned}
\sum_{l=1}^{h} \left[w^l(S) - w^l(S \cup \{i\})\right] &= e(i) - \sum_{j \in (\bigcup_{l=1}^{A(S \cup \{i\})} N_{I_l}) \setminus S} e(j) + \sum_{l=1}^{A(S \cup \{i\})} w^l(S) \\
&\leq e(i) - w^{A(S \cup \{i\})}(S) \leq e(i) = \sum_{l=1}^{h} \left[w^l(T) - w^l(T \cup \{i\})\right]
\end{aligned}
$$

where the first inequality holds by Equation (4.3) and the second because w^k is non-negative for all $k \in \{1, \ldots, h\}$. □

5 Project games

In this section we will study general project situations in which some activities may have suffered a delay with respect to the planned time while other may have been expedited. The basis of analysis will be rewards where costs are considered to be negative rewards. We will assume that the reward function is linear in the difference between real duration and planned duration.

Let $\mathcal{D} = \{i \in N | \ d(i) > 0\}$ and $\mathcal{E} = \{i \in N | \ e(i) > 0\}$ denote the sets of delayed activities and expedited activities, respectively. Associated to a (general) project problem $(\{N_1, \ldots, N_m\}, p, r)$ we define a project game where the set of players is the set of activities and the worth of a coalition combines the underlying ideas from sections 3 and 4. In determining the worth of a coalition we will pessimistically assume that all delayed activities have indeed acted according to realization and that all expedited activities outside the coalition have acted according to plan. Then, if the expedition given by the expedited activities in the coalition itself is not enough to expedite the duration of the (planned) project, the worth of the coalition will be negative and will be determined along the lines of delayed project games. Otherwise, the worth of the coalition will be positive and determined along the lines of expedited project games. Formally, given a project problem $(\{N_1, \ldots, N_m\}, p, r)$ we define the associated game (N, u), where u is defined by

$$u(S) = \begin{cases} -c(S), & \text{if } D(p_{|N\setminus(\mathcal{D}\cup(\mathcal{E}\cap S))}, r_{|\mathcal{D}\cup(\mathcal{E}\cap S)}) \geq D(p); \\ v(S), & \text{if } D(p_{|N\setminus(\mathcal{D}\cup(\mathcal{E}\cap S))}, r_{|\mathcal{D}\cup(\mathcal{E}\cap S)}) < D(p). \end{cases} \tag{5.1}$$

for every $S \subset N$.

Let $D(p_{|N\setminus(\mathcal{D}\cup(\mathcal{E}\cap S))}, r_{|\mathcal{D}\cup(\mathcal{E}\cap S)}) \geq D(p)$. Then, $c(S)$ reflects the maximum delay a coalition can be held responsible for. Given a path N_t, coalition S can not be held responsible for more than its (positive) net delay nor for more than the net delay of the path as a consequence of the delay of activities in the path and the expedition of the activities within the coalition. Formally,

$$c(S) = \max_{t\in\mathcal{P}(S)} \left\{ \min \left\{ \left(\sum_{i\in N_t\cap S} d(i) - \sum_{i\in N_t\cap S} e(i) \right)_+, \left(\sum_{i\in N_t} d(i) - \sum_{i\in N_t\cap S} e(i) - \text{slack}(N_t, p) \right)_+ \right\} \right\}. \tag{5.2}$$

Note that Equation (5.2) applied to a coalition S with $D(p_{|N\setminus(\mathcal{D}\cup(\mathcal{E}\cap S))}, r_{|\mathcal{D}\cup(\mathcal{E}\cap S)}) < D(p)$ will give $c(S) = 0$. Moreover, if $D(r) \geq D(p)$ then $D(p_{|N\setminus(\mathcal{D}\cup(\mathcal{E}\cap S))}, r_{|\mathcal{D}\cup(\mathcal{E}\cap S)}) \geq D(p)$ for every $S \subset N$ and hence $u(S) = -c(S)$ for every $S \subset N$.

Next, consider the case $D(p_{|N\setminus(\mathcal{D}\cup(\mathcal{E}\cap S))}, r_{|\mathcal{D}\cup(\mathcal{E}\cap S)}) < D(p)$. In order to define $v(S)$ we need to introduce some notation. We denote by $\text{rslack}(N_t, p, r)$ the amount of *remaining slack* of a path w.r.t. the planned duration if only its delayed activities act according to realization, i.e. $\text{rslack}(N_t, p, r) = \text{slack}(N_t, p) - \sum_{i\in N_t} d(i)$. Note that $\text{rslack}(N_t, p, r)$ can be negative, meaning that the delayed activities have consumed all the initial slack and would produce a delay on the project as a whole of $-\text{rslack}(N_t, p, r)$ if the expedited activities had acted according to plan. We denote by J_1 the set of indexes of paths with remaining slack less than or equal to zero:

$$J_1 = \{t \in \{1, \ldots, m\} |\ \text{rslack}(N_t, p, r) \leq 0\}.$$

Recursively, we define for $k \geq 2$

$$J_k = \left\{ t \in \{1, \ldots, m\} \setminus \bigcup_{l=1}^{k-1} J_l |\ \text{rslack}(N_t, p, r) \leq \text{rslack}(N_u, p, r) \text{ for all } u \in \{1, \ldots, m\} \setminus \bigcup_{l=1}^{k-1} J_l \right\},$$

i.e. J_k contains all paths that would have smallest remaining slack if the paths in $J_1, \ldots, J_{k-1}$ where not present. Set $\text{rslack}(J_1) := 0$ and let $\text{rslack}(J_k)$ denote the remaining slack of the paths in J_k for $k \geq 2$, i.e. $\text{rslack}(J_k) = \text{rslack}(N_t, p, r)$ for each $t \in J_k$, $k \geq 2$. Let g be such that $\text{rslack}(J_g) < D(p) - D(r) \leq \text{rslack}(J_{g+1})$ if $D(p) - D(r) > 0$ and $g = 0$ otherwise. For $k = 1, \ldots, g$, we define F^k as the level of expedition that the paths in $J_1, \ldots, J_k$ can obtain by acting according to realization, i.e.

$$F^k = \begin{cases} \text{rslack}(J_{k+1}) - \text{rslack}(J_k) & \text{if } 1 \leq k < g; \\ D(p) - D(r) - \text{rslack}(J_g) & \text{if } k = g. \end{cases}$$

Note that $\sum_{k=1}^{g} F^k = D(p) - D(r)$.

Next, we define $v(S)$. By $v(S)$ we represent the sum over all $k = 1, \ldots, g$ of those specific parts of the corresponding level of expedition F^k for which expedited activities outside the coalition that are in paths of $\bigcup_{l=1}^{k} J_l$ can not be held responsible of. Formally,

$$v(S) = \sum_{k=1}^{g} \left(F^k - w^k(S)\right) \tag{5.3}$$

where $w^k(S)$ represents the part of the level of expedition F^k that players in S maximally would have to concede to players in $\bigcup_{l=1}^{k} J_l$ outside S, taking into account concessions from the previous phases. Formally,

$$w^k(S) = \min \left\{ \sum_{i \in (\bigcup_{l=1}^{k} N_{J_l}) \setminus S} e(i) - \sum_{l=1}^{k-1} w^l(S),\ F^k \right\}, \tag{5.4}$$

for all $k \in \{1, \ldots, g\}$, where $N_{J_l} = \bigcup_{t \in J_l} N_t$. Note that Equations (5.3) and (5.4) applied to a coalition S with $D(p_{|N \setminus (\mathcal{D} \cup (\mathcal{E} \cap S))}, r_{|\mathcal{D} \cup (\mathcal{E} \cap S)}) \geq D(p)$ will give $v(S) = 0$.

Example 5.1. Consider the project problem $(\{N_1, N_2\}, p, r)$ with $N_1 = \{\text{A,B}\}$ and $N_2 = \{\text{C,D}\}$. Let $p = (2, 5, 3, 6)$ and $r = (5, 3, 4, 3)$. The project is represented in Figure 4.

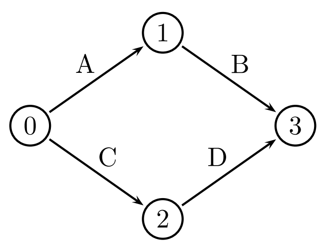

Figure 4: Representation of the project in Example 5.1.

Here, $D(p) = 9$, slack$(N_1, p) = 2$, slack$(N_2, p) = 0$, $D(r) = 8$, $d = (3, 0, 1, 0)$, $e = (0, 2, 0, 3)$, $\mathcal{D} = \{A, C\}$, $\mathcal{E} = \{B, D\}$, rslack$(N_1, p, r) = -1$, rslack$(N_2, p, r) = -1$, $J_1 = \{1, 2\}$, $g = 1$ and $F^1 = 1$. Note that the project has been expedited 1 unit of time and hence $u(N) = 1$. We will explain in detail how to compute the value of the associated project game for the coalitions $\{A, C, D\}$ and $\{B, C, D\}$.

First, consider $\{A, C, D\}$. Since $D(p_{|\{B\}}, r_{|\{A,C,D\}}) = 10 > 9 = D(p)$, we have $u(\{A, C, D\}) = -c(\{A, C, D\})$. In this case, $\mathcal{P}(\{A, C, D\}) = \{1, 2\}$. The maximum amount chargeable w.r.t. path N_1 is

$$\min\{(d(A) - e(A))_+,\ (d(A) + d(B) - e(A) - slack(N_1, p))_+\} = \min\{(3)_+,\ (3 - 2)_+\} = 1$$

and for path N_2 this equals

$$\min\{(d(C)+d(D)-e(C)-e(D))_+,\ (d(C)+d(D)-e(C)-e(D)-slack(N_2,p))_+\} = \min\{(1-3)_+,\ (1-3-0)_+\} = 0.$$

Hence, $c(\{A,C,D\}) = \max\{1,0\} = 1$. Therefore,

$$u(\{A,C,D\}) = -c(\{A,C,D\})) = -1.$$

Second, consider the coalition $\{B,C,D\}$. Since $D(r) = 8 < 9 = D(p)$ we have $u(\{B,C,D\}) = v(\{B,C,D\})$. In this case,

$$w^1(\{B,C,D\}) = \min\left\{e(A),\ F^1\right\} = \min\{0,1\} = 0,$$

and therefore $v(\{B,C,D\}) = F^1 - w^1(\{B,C,D\}) = 1 - 0 = 1$ and

$$u(\{B,C,D\}) = 1.$$

The complete project game is given by: $u(\{A\}) = -1$, $u(\{B\}) = 0$, $u(\{C\}) = -1$, $u(\{D\}) = 0$, $u(\{A,B\}) = 0$, $u(\{A,C\}) = u(\{A,D\}) = u(\{B,C\}) = -1$, $u(\{B,D\}) = 1$, $u(\{C,D\}) = 0$, $u(\{A,B,C\}) = -1$, $u(\{A,B,D\}) = 1$, $u(\{A,C,D\}) = -1$, $u(\{B,C,D\}) = 1$, $u(N) = 1$. □

Theorem 5.1. Project games have a nonempty core.

PROOF: Let $(\{N_1,\ldots,N_m\},p,r)$ be a project problem and let (N,u) be the associated project game. By the remarks above, it holds $u(S) = v(S) - c(S)$ for *every* $S \subset N$. Hence, it suffices to show that (N,v) and (N,c) have nonempty cores.

Note that the game (N,v) has a similar structure as an expedited project game (J_k is replaced by I_k, F^k is substituted by E^k, and g is replaced by h). Moreover, the explicit definition of I_k, E^k, and h is not relevant for the proof of Theorem 4.2. Therefore, it can be shown that (N,v) is convex according to the same line of reasoning. Hence, (N,v) has a nonempty core.

Next, we will show that (N,c) has a nonempty core. Let $S \subset N$, then

$$\begin{aligned}
c(S) &= \max_{t\in\mathcal{P}(S)}\left\{\min\left\{\left(\sum_{i\in N_t\cap S} d(i) - \sum_{i\in N_t\cap S} e(i)\right)_+,\left(\sum_{i\in N_t} d(i) - \sum_{i\in N_t\cap S} e(i) - \text{slack}(N_t,p)\right)_+\right\}\right\}\\
&= \max_{t\in\mathcal{P}(S)}\left\{\min\left\{\max\left\{\sum_{i\in N_t\cap S} d(i), \sum_{i\in N_t\cap S} e(i)\right\},\max\left\{\sum_{i\in N_t} d(i) - \text{slack}(N_t,p), \sum_{i\in N_t\cap S} e(i)\right\}\right\} - \sum_{i\in N_t\cap S} e(i)\right\}\\
&= \max_{t\in\mathcal{P}(S)}\left\{\max\left\{0,\min\left\{\sum_{i\in N_t\cap S} d(i),\left(\sum_{i\in N_t} d(i) - \text{slack}(N_t,p)\right)_+\right\} - \sum_{i\in N_t\cap S} e(i)\right\}\right\}
\end{aligned}$$

First, suppose that the project is parallel. Then, for $t \in \{1, \dots, m\}$ and $S \subset N_t$ it holds

$$c_{|N_t}(S) = \max\left\{0, \min\left\{\sum_{i\in S} d(i), \left(\sum_{i\in N_t} d(i) - \text{slack}(N_t, p)\right)_+\right\} - \sum_{i\in S} e(i)\right\}$$

Hence, $c_{|N_t}$ is the maximum of the zero(-sub)game with (a game which is strategically equivalent to) a concave game according to Lemma 3.2.Therefore, $(N_t, c_{|N_t})$ has a nonempty core. Subsequently, following the same argument as in the proof of Lemma 3.3, we can explicitly provide a core element for delayed parallel project games. By applying the same translation technique as in the proof of Theorem 3.4, it follows that (N, c) has a nonempty core in general. □

Appendix

Lemma A.1. Let $(\{N_1, \dots, N_m\}, p, r)$ be an expedited project problem. Let $i \in N$ and $S \subset N \setminus \{i\}$. Then, the following statements hold.

(i) $w^k(S \cup \{i\}) = w^k(S)$ for all $k < k(i)$.

(ii) Let $\mathcal{A}(i, S \cup \{i\}) = \emptyset$. Then, $w^k(S \cup \{i\}) = w^k(S)$ for all $k \geq k(i)$ and consequently $A(S \cup \{i\}) = A(S)$.

(iii) Let $\mathcal{A}(i, S) \neq \emptyset$. Then $w^k(S \cup \{i\}) = w^k(S)$ for all $k > a(i, S)$ and consequently $A(S \cup \{i\}) = A(S)$.

Proof: (i) follows readily by definition of w^k and (ii) by definition of $\mathcal{A}(i, S \cup \{i\})$ and the fact that $\mathcal{A}(S) \subset \mathcal{A}(S \cup \{i\})$. Next, we will show (iii). Let $\mathcal{A}(i, S) \neq \emptyset$. It is sufficient to show that

$$w^k(S \cup \{i\}) = w^k(S) \text{ for all } k > a(i, S) \tag{A.1}$$

From the definition of "active level" and Equation (4.3) we have

$$w^{a(i,S)}(S) = \sum_{j \in (\bigcup_{l=1}^{a(i,S)} N_{I_l}) \setminus S} e(j) - \sum_{l=1}^{a(i,S)-1} w^l(S),$$

or equivalently

$$\sum_{l=1}^{a(i,S)} w^l(S) = \sum_{j \in (\bigcup_{l=1}^{a(i,S)} N_{I_l}) \setminus S} e(j). \tag{A.2}$$

By Lemma 4.1 (iii), $a(i, S) \in \mathcal{A}(S \cup \{i\})$. Then

$$w^{a(i,S)}(S \cup \{i\}) = \sum_{j \in (\bigcup_{l=1}^{a(i,S)} N_{I_l}) \setminus S \cup \{i\}} e(j) - \sum_{l=1}^{a(i,S)-1} w^l(S \cup \{i\})$$

$$= \sum_{j \in (\bigcup_{l=1}^{a(i,S)} N_{I_l}) \setminus S} e(j) - e(i) - \sum_{l=1}^{a(i,S)-1} w^l(S \cup \{i\}),$$

or equivalently

$$\sum_{l=1}^{a(i,S)} w^l(S \cup \{i\}) = \sum_{j \in (\bigcup_{l=1}^{a(i,S)} N_{I_l}) \setminus S} e(j) - e(i). \tag{A.3}$$

We will show (A.1) by induction on k. First,

$$\begin{aligned}
w^{a(i,S)+1}(S \cup \{i\}) &= \min \left\{ \sum_{j \in (\bigcup_{l=1}^{a(i,S)+1} N_{I_l}) \setminus (S \cup \{i\})} e(j) - \sum_{l=1}^{a(i,S)} w^l(S \cup \{i\}),\ E^{a(i,S)+1} \right\} \\
&= \min \left\{ \sum_{j \in (\bigcup_{l=1}^{a(i,S)+1} N_{I_l}) \setminus S} e(j) - e(i) - \left(\sum_{j \in (\bigcup_{l=1}^{a(i,S)} N_{I_l}) \setminus S} e(j) - e(i) \right),\ E^{a(i,S)+1} \right\} \\
&= \min \left\{ \sum_{j \in (\bigcup_{l=1}^{a(i,S)+1} N_{I_l}) \setminus S} e(j) - \left(\sum_{j \in (\bigcup_{l=1}^{a(i,S)} N_{I_l}) \setminus S} e(j) \right),\ E^{a(i,S)+1} \right\} \\
&= \min \left\{ \sum_{j \in (\bigcup_{l=1}^{a(i,S)+1} N_{I_l}) \setminus (S \cup \{i\})} e(j) - \sum_{l=1}^{a(i,S)} w^l(S),\ E^{a(i,S)+1} \right\} = w^{a(i,S)+1}(S)
\end{aligned}$$

where the second equality holds by Equation (A.3), and the fourth equality holds by Equation (A.2).

Let $k > a(i,S)+1$ and suppose (A.1) holds for all levels from $a(i,S)+1$ to $k-1$,

$$\begin{aligned}
w^k(S \cup \{i\}) &= \min \left\{ \sum_{j \in (\bigcup_{l=1}^{k} N_{I_l}) \setminus (S \cup \{i\})} e(j) - \sum_{l=1}^{k-1} w^l(S \cup \{i\}),\ E^k \right\} \\
&= \min \left\{ \sum_{j \in (\bigcup_{l=1}^{k} N_{I_l}) \setminus (S \cup \{i\})} e(j) - \sum_{l=1}^{a(i,S)} w^l(S \cup \{i\}) - \sum_{l=a(i,S)+1}^{k-1} w^l(S \cup \{i\}),\ E^k \right\} \\
&= \min \left\{ \sum_{j \in (\bigcup_{l=1}^{k} N_{I_l}) \setminus S} e(j) - e(i) - \left(\sum_{j \in (\bigcup_{l=1}^{a(i,S)} N_{I_l}) \setminus S} e(j) - e(i) \right) - \sum_{l=a(i,S)+1}^{k-1} w^l(S \cup \{i\}),\ E^k \right\} \\
&= \min \left\{ \sum_{j \in (\bigcup_{l=1}^{k} N_{I_l}) \setminus S} e(j) - \left(\sum_{j \in (\bigcup_{l=1}^{a(i,S)} N_{I_l}) \setminus S} e(j) \right) - \sum_{l=a(i,S)+1}^{k-1} w^l(S),\ E^k \right\} \\
&= \min \left\{ \sum_{j \in (\bigcup_{l=1}^{k} N_{I_l}) \setminus S} e(j) - \sum_{l=1}^{k-1} w^l(S),\ E^k \right\} = w^k(S)
\end{aligned}$$

where the third equality holds by Equation (A.3), the fourth equality holds by induction, and the fifth one by Equation (A.2). $\square$

The following result provides an explicit expression for the sum of all concessions for a coalition S.

Lemma A.2. Let $(\{N_1,\dots,N_m\}, p, r)$ be an expedited project problem. Let $S \subset N$. Then,

$$\sum_{l=1}^{h} w^l(S) = \sum_{i\in(\bigcup_{l=1}^{A(S)} N_{I_l})\setminus S} e(i) + \sum_{l=A(S)+1}^{h} E^l$$

PROOF: For $\mathcal{A}(S)=\emptyset$, the statement is obvious. Suppose $\mathcal{A}(S)\neq\emptyset$. Then

$$\begin{aligned}\sum_{l=1}^{h} w^l(S) &= \sum_{l=1}^{A(S)-1} w^l(S) + w^{A(S)}(S) + \sum_{l=A(S)+1}^{h} w^l(S)\\ &= \sum_{l=1}^{A(S)-1} w^l(S) + \sum_{i\in(\bigcup_{l=1}^{A(S)} N_{I_l})\setminus S} e(i) - \sum_{l=1}^{A(S)-1} w^l(S) + \sum_{l=A(S)+1}^{h} E^l\\ &= \sum_{i\in(\bigcup_{l=1}^{A(S)} N_{I_l})\setminus S} e(i) + \sum_{l=A(S)+1}^{h} E^l\end{aligned}$$

where the second equality holds because $A(S)\in\mathcal{A}(S)$ and by definition of $w^{A(S)}(S)$. □

Lemma A.3. Let $(\{N_1,\dots,N_m\}, p, r)$ be an expedited project problem. Let $i\in N$ and $S\subset N\setminus\{i\}$. Then,

$$\sum_{l=1}^{h}\left[w^l(S)-w^l(S\cup\{i\})\right] = \begin{cases} 0 & \text{if } \mathcal{A}(i,S\cup\{i\})=\emptyset;\\ e(i) & \text{if } \mathcal{A}(i,S)\neq\emptyset;\\ e(i) - \sum_{j\in(\bigcup_{l=1}^{A(S\cup\{i\})} N_{I_l})\setminus S} e(j) + \sum_{l=1}^{A(S\cup\{i\})} w^l(S) & \text{if } \mathcal{A}(i,S\cup\{i\})\neq\emptyset \text{ and } \mathcal{A}(i,S)=\emptyset.\end{cases}$$

PROOF: (i) Let $\mathcal{A}(i,S\cup\{i\})=\emptyset$. By Lemma 4.1 (iii), $\mathcal{A}(i,S)=\emptyset$. Hence, by Lemma A.1 (i) and A.1 (ii) we find that $\sum_{l=1}^{h}\left[w^l(S)-w^l(S\cup\{i\})\right]=0$.

(ii) Let $\mathcal{A}(i,S)\neq\emptyset$. Then

$$\sum_{l=1}^{h} w^l(S) = \sum_{j\in(\bigcup_{l=1}^{A(S)} N_{I_l})\setminus S} e(j) + \sum_{l=A(S)+1}^{h} E^l = \sum_{j\in(\bigcup_{l=1}^{A(S)} N_{I_l})\setminus S} e(j) + \sum_{l=A(S)+1}^{h} w^l(S) \tag{A.4}$$

where the first equality holds by Lemma A.2 and the second by definition of $A(S)$. Moreover

$$\begin{aligned}\sum_{l=1}^{h} w^l(S\cup\{i\}) &= \sum_{j\in(\bigcup_{l=1}^{A(S\cup\{i\})} N_{I_l})\setminus(S\cup\{i\})} e(j) + \sum_{l=A(S\cup\{i\})+1}^{h} E^l\\ &= \sum_{j\in(\bigcup_{l=1}^{A(S)} N_{I_l})\setminus S} e(j) - e(i) + \sum_{l=A(S)+1}^{h} w^l(S)\end{aligned}$$

$$= \sum_{l=1}^{h} w^l(S) - e(i),$$

where the first equality holds by Lemma A.2, the second equality holds by Lemma A.1 (iii) and the third one by equation (A.4). Hence,

$$\sum_{l=1}^{h} \left[w^l(S) - w^l(S \cup \{i\}) \right] = e(i).$$

(iii) Let $\mathcal{A}(i, S \cup \{i\}) \neq \emptyset$ and $\mathcal{A}(i, S) = \emptyset$[1]. Then

$$\sum_{l=1}^{h} w^l(S \cup \{i\}) = \sum_{j \in (\bigcup_{l=1}^{A(S\cup\{i\})} N_{I_l}) \setminus (S \cup \{i\})} e(j) + \sum_{l=A(S\cup\{i\})+1}^{h} E^l = \sum_{j \in (\bigcup_{l=1}^{A(S\cup\{i\})} N_{I_l}) \setminus S} e(j) - e(i) + \sum_{l=A(S\cup\{i\})+1}^{h} w^l(S).$$

where the first equality holds by Lemma A.2 and the second by the fact that $A(S) \leq A(S \cup \{i\})$. Hence,

$$\sum_{l=1}^{h} \left[w^l(S) - w^l(S \cup \{i\}) \right] = e(i) - \sum_{j \in (\bigcup_{l=1}^{A(S\cup\{i\})} N_{I_l}) \setminus S} e(j) + \sum_{l=1}^{A(S\cup\{i\})} w^l(S).$$

□

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[1]In fact $\mathcal{A}(i, S) = \emptyset$ can be omitted here. In particular, part (ii) offers a further elaboration if $\mathcal{A}(i, S) \neq \emptyset$. In this case, $-\sum_{j \in (\bigcup_{l=1}^{A(S\cup\{i\})} N_{I_l}) \setminus S} e(j) + \sum_{l=1}^{A(S\cup\{i\})} w^l(S) = 0$.