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LINEAR QUADRATIC GAMES: AN OVERVIEW

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Linear Quadratic Differential Games: an overview.

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Abstract

In this paper we review some basic results on linear quadratic differential games. We consider both the cooperative and non-cooperative case. For the non-cooperative game we consider the open-loop and (linear) feedback information structure. Furthermore the effect of adding uncertainty is considered. The overview is based on [9]. Readers interested in detailed proofs and additional results are referred to this book.

Keywords: linear-quadratic games, Nash equilibrium, affine systems, solvability conditions, Riccati equations.
Jel-codes C61, C72,C73.

1 Introduction

Many situations in economics and management are characterized by multiple decision makers/players and enduring consequences of decisions. The theory which conceptualizes problems of this kind is dynamic games. Dynamic game theory tries to arrive at appropriate models describing the process. Depending on the specific problem this model sometimes can be used by an individual decision maker to optimize his performance. In other cases it may serve as a starting point to introduce new communication lines which may help to improve upon the outcome of the current process. Furthermore it is possible by the introduction as "nature" as an additional player in games, which is trying to work against the other decision makers in a process, to analyze the robustness of strategies of players w.r.t. worst-case scenarios.

Examples of dynamic games in economics and management science can be found e.g. in [8], [19] and [27].

In this paper we consider a special class of dynamic games. We study games where the process can be modeled by a set of linear differential equations and the preferences are formalized by quadratic utility functions. The so-called linear quadratic differential games. These games are very popular in literature and a recent exposition (and additional references) of this theory can be found in [9]. The popularity of these games is caused on the one hand by practical considerations. To some extent these kinds of differential games are analytically and numerically solvable. On the other hand this linear quadratic setting naturally appears if the decision makers' objective is to minimize the effect of a small perturbation of their optimally controlled nonlinear process. By solving a linear quadratic control problem, and using the optimal actions implied by this problem, players can avoid most of the additional cost incurred by this perturbation.

In a dynamic game, information available to the players at the time of their decisions plays an important role and, therefore, has to be specified before one can analyze these kind of games appropriately. We will distinguish two cases: the open-loop and the feedback information case, respectively. In the open-loop information case it is assumed that all players know just the initial state of the process and the model structure. More specifically, it is assumed that players simultaneously determine their actions for the whole planning horizon of the process before it starts. Next they submit their actions to some authority who then enforces these plans as binding commitments. So players can not react on any deviations occurring during the evolution of the process. In the feedback information case it is assumed that all players can observe at every point in time the current state of the process and determine their actions based on this observation.

In this paper we will review some main results for linear quadratic differential games. Both the case that players cooperate in order to achieve their objectives as the case that players do not cooperate with eachother are considered. The reason that players do not cooperate may be caused by individual motivations or for physical reasons. We will take these reasons for granted. In case the players do not cooperate it seems reasonable that all players individually will try to play actions which are optimal for them. That is, for actions they can not improve upon themselves. If there exists a set of actions such that none of the players has an incentive to deviate from his action (or stated otherwise, given the actions of the other players his choice of action is optimal), we call such a set of actions a Nash[1] equilibrium of the game. In general, a game may either have none, one or more than one Nash equilibrium. This leads on the one hand to the question under which conditions these different situations will occur and on the other hand, in case there is more than one equilibrium solution, whether there are additional motivations to prefer one equilibrium outcome to another.

As already indicated above linear quadratic differential games have been studied a lot in the past. We will review some basic results and algorithms to compute equilibria. As well for the cooperative case as the open-loop and feedback information case. The outline of the rest of the paper is as follows. Section 2 considers the cooperative case, section 3 the open-loop information case and section 4 the feedback information case. Section 5 recalls some results for the case that the system is corrupted by noise. Finally section 6 reviews some extensions that can be found elsewhere in literature.

For ease of exposition we will just deal with the two-player case. Throughout this paper we will assume that each player has a (quadratic) cost function (1) he wants to minimize given by

$$J_i(u_1,u_2)=\int_0^T\{x^T(t)Q_ix(t)+u_i^T(t)R_{ii}u_i(t)+u_j^T(t)R_{ij}u_j(t)\}dt+x^T(T)Q_{T,i}x(T),\ i=1,2,\ j\neq i. \quad (1)$$

Here the matrices Q_i, R_{ii} and $Q_{T,i}$ are assumed to be symmetric and R_{ii} positive definite (denoted as $R_{ii}>0$). Sometimes some additional positive definiteness assumptions are made w.r.t. the matrices Q_i and $Q_{T,i}$. In the minimization a state variable $x(t)$ occurs. This is a dynamic variable that can be influenced by both players. Its dynamics are described by

$$\dot{x}(t)=Ax(t)+B_1u_1+B_2u_2,\ x(0)=x_0, \quad (2)$$

where A and $B_i,\ i=1,2$, are constant matrices, and u_i is a vector of variables which can be manipulated by player i. The objectives are possibly conflicting. That is, a set of policies u_1 which

[1]This after J.F. Nash who proved in a number of papers from 1950-1953 the existence of such equilibria.

is optimal for player one, may have rather negative effects on the evolution of the state variable x from another player's point of view.

2 The Cooperative Game

In this section we assume that players can communicate and can enter into binding agreements. Furthermore it is assumed that they cooperate in order to achieve their objectives. However, no side-payments take place. Moreover, it is assumed that every player has all information on the state dynamics and cost functions of his opponents and all players are able to implement their decisions. Concerning the strategies used by the players we assume that there are no restrictions. That is, every $u_i(.)$ may be chosen arbitrarily from a set $\mathcal{U}$ which is chosen such that we get a well-posed problem (in particular it is chosen such that the differential equation 2 has a unique solution for every initial state).

By cooperation, in general, the cost one specific player incurs is not uniquely determined anymore. If all players decide, e.g., to use their control variables to reduce the cost of player 1 as much as possible, a different minimum is attained for player 1 than in case all players agree to help collectively a different player in minimizing his cost. So, depending on how the players choose to "divide" their contol efforts, a player incurs different "minima". Consequently, in general, each player is confronted with a whole set of possible outcomes from which somehow one outcome (which in general does not coincide with a player's overall lowest cost) is cooperatively selected. Now, if there are two strategies γ_1 and γ_2 such that every player has a lower cost if strategy γ_1 is played, then it seems reasonable to assume that all players will prefer this strategy. We say that the solution induced by strategy γ_1 *dominates* in that case the solution induced by the strategy γ_2. So, dominance means that the outcome is better for all players. Proceeding in this line of thinking, it seems reasonable to consider only those cooperative outcomes which have the property that if a different strategy than the one corresponding with this cooperative outcome is chosen, then at least one of the players has higher costs. Or, stated differently, to consider only solutions that are such that they can not be improved upon by all players simultaneously. This motivates the concept of Pareto efficiency.

Definition 2.1 A set of actions $(\hat{u_1}, \hat{u_2})$ is called *Pareto efficient* if the set of inequalities

$$J_i(u_1, u_2) \leq J_i(\hat{u_1}, \hat{u_2}),\ i = 1, 2,$$

where at least one of the inequalities is strict, does not allow for any solution $(u_1, u_2) \in \mathcal{U}$. The corresponding point $(J_1(\hat{u_1}, \hat{u_2}), J_2(\hat{u_1}, \hat{u_2})) \in \mathbb{R}^2$ is called a *Pareto solution*. The set of all Pareto solutions is called the *Pareto frontier*. □

A Pareto solution is therefore never dominated, and for that reason called an *undominated* solution. Typically there is always more than one Pareto solution, because dominance is a property which generally does not provide a total ordering.

It turns out that if we assume $Q_i \geq 0,\ i = 1, 2$, in our cost functions (1) that there is a simple characterization for all Pareto solutions in our cooperative linear quadratic game. Below we will adopt the notation $u := (u_1, u_2)$ and

$$\mathcal{A} := \{\alpha = (\alpha_1, \alpha_2) | \alpha_i \geq 0 \text{ and } \sum_{i=1}^{2} \alpha_i = 1\}.$$

Theorem 2.2 *Assume $Q_i \geq 0$. Let $\alpha_i > 0,\ i = 1, 2$, satisfy $\sum_{i=1}^{2} \alpha_i = 1$. If $\hat{u} \in \mathcal{U}$ is such that*

$$\hat{u} \in \arg\min_{u \in \mathcal{U}} \{\sum_{i=1}^{2} \alpha_i J_i(u)\},$$

then $\hat{u}$ is Pareto efficient.
Moreover, if $\mathcal{U}$ is convex, then for all Pareto efficient $\hat{u}$ there exist $\alpha \in \mathcal{A}$, such that

$$\hat{u} \in \arg\min_{u \in \mathcal{U}} \{\sum_{i=1}^{2} \alpha_i J_i(\gamma)\}, \qquad \square$$

.

Theorem 2.2 shows that to find all cooperative solutions for the linear quadratic game one has to solve a regular linear quadratic optimal control problem which depends on a parameter α. The existence of a solution for this problem is related to the existence of solutions of Riccati equations. In Lancaster et al. [22, Section 11.3], it is shown that if the parameters appearing in an algebraic Riccati equation are, e.g., differentiable functions of some parameter α (or, more general, depend analytically on a parameter α), and the maximal solution exists for all α in some open set V, then this maximal solution of the Riccati equation will be a differentiable function of this parameter α too on V (c.q., depend analytically on this parameter α too). Since in the linear quadratic case the parameters depend linearly on α, this implies that in the infinite horizon case the corresponding Pareto frontier will be a smooth function of α (provided the maximal solution exists for all α). A similar statement holds for the finite planning horizon case. In case for all $\alpha \in V$ the cooperative linear quadratic differential game has a solution or, equivalently, the corresponding Riccati differential equations have a solution, then it follows in fact directly (see also Perko [26, Theorem 2.3.2] for a precise statement and proof) that the solution of the Riccati differential equation is a differentiable function of α, since all parameters in this Riccati differential equation are differentiable functions of α.

The Pareto frontier does not always have to be a one-dimensional surface in $\mathbb{R}^2$, like in the above corollary. This is, e.g., already illustrated in the two-player case when both players have the same cost function. In that case the Pareto frontier reduces to a single point in $\mathbb{R}^2$.

Example 2.3 Consider the following differential game on government debt stabilization (see van Aarle et al. [1]). Assume that government debt accumulation, $\dot{d}(t)$ is the sum of interest payments on government debt, $rd(t)$, and primary fiscal deficits, $f(t)$, minus the seignorage (i.e. the issue of base money) $m(t)$. So,

$$\dot{d}(t) = rd(t) + f(t) - m(t), d(0) = d_0.$$

Here $d(t)$, $f(t)$ and $m(t)$ are expressed as fractions of GDP and r represents the rate of interest on outstanding government debt minus the growth rate of output. The interest rate $r > 0$ is assumed to be exogenous. Assume that fiscal and monetary policies are controlled by different institutions, the fiscal authority and the monetary authority, respectively, which have different objectives. The objective of the fiscal authority is to minimize a sum of time profiles of the primary fiscal deficit, base-money growth and government debt

$$J_1 = \int_0^\infty e^{-\delta t}\{f^2(t) + \eta m^2(t) + \lambda d^2(t)\}dt.$$

The parameters, η and λ express the relative priority attached to base-money growth and government debt by the fiscal authority. The monetary authorities are assumed to choose the growth of base money such that a sum of time profiles of base-money growth and government debt is minimized. That is

$$J_2 = \int_0^\infty e^{-\delta t}\{m^2(t) + \kappa d^2(t)\}dt.$$

Here $1/\kappa$ can be interpreted as a measure for the conservatism of the central bank w.r.t. the money growth. Furthermore all variables are normalized such that their targets are zero, and all parameters are positive.

Introducing $\tilde{d}(t) := e^{-\frac{1}{2}\delta t}d(t)$, $\tilde{m} := e^{-\frac{1}{2}\delta t}m(t)$ and $\tilde{f} := e^{-\frac{1}{2}\delta t}f(t)$ the above model can be rewritten as

$$\dot{\tilde{d}}(t) = (r - \frac{1}{2}\delta)\tilde{d}(t) + \tilde{f}(t) - \tilde{m}(t),\ \tilde{d}(0) = d_0.$$

Where the cost functions of both players are

$$J_1 = \int_0^\infty \{\tilde{f}^2(t) + \eta\tilde{m}^2(t) + \lambda\tilde{d}^2(t)\}dt$$

and

$$J_2 = \int_0^\infty \{\tilde{m}^2(t) + \kappa\tilde{d}^2(t)\}dt.$$

If both the monetary and fiscal authority agree to cooperate in order to reach their goals, by Theorem 2.2 the set of all Pareto solutions is obtained by considering the simultaneous minimization of

$$\begin{aligned} J_c(\alpha) &:= \alpha J_1 + (1-\alpha)J_2 \\ &= \int_0^\infty \{\alpha\tilde{f}^2(t) + \beta_1\tilde{m}^2(t) + \beta_2\tilde{d}^2(t)\}dt, \end{aligned}$$

where $\beta_1 = 1+\alpha(-1+\eta)$ and $\beta_2 = \kappa+\alpha(\lambda-\kappa)$. This cooperative game problem can be reformulated as the minimization of

$$J_c(\alpha) = \int_0^\infty \{\beta_2\tilde{d}^2(t) + [\tilde{f}\ \tilde{m}]\begin{bmatrix} \alpha & 0 \\ 0 & \beta_1 \end{bmatrix}\begin{bmatrix} \tilde{f} \\ \tilde{m} \end{bmatrix}\}dt,$$

subject to

$$\dot{\tilde{d}}(t) = (r - \frac{1}{2}\delta)\tilde{d}(t) + [1\ \ -1]\begin{bmatrix} \tilde{f} \\ \tilde{m} \end{bmatrix},\ \tilde{d}(0) = d_0.$$

In Figure 1 we plotted the set of Pareto solutions in case $\eta = 0.1$; $\lambda = 0.6$; $\kappa = 0.5$; $r = 0.06$ and $\delta = 0.04$. $\square$

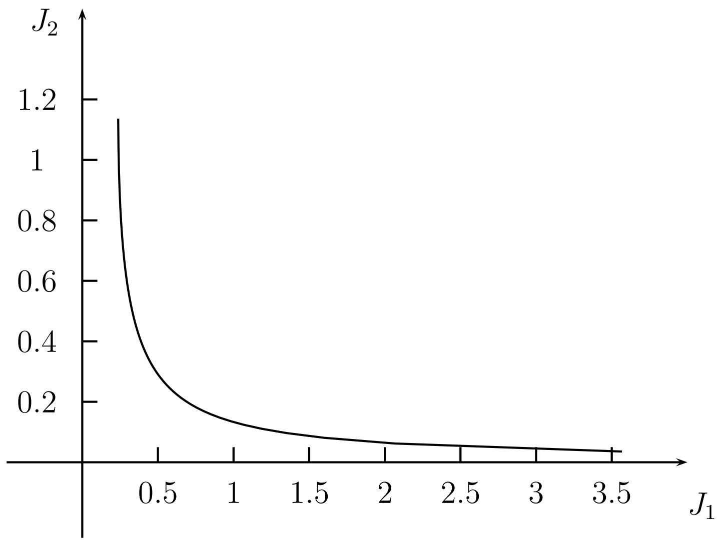

Figure 1: Pareto frontier Example 2.3 if $\eta = 0.1$; $\lambda = 0.6$; $\kappa = 0.5$; $r = 0.06$ and $\delta = 0.04$.

As Theorem 2.2 already indicates, in general, there are a lot of Pareto solutions. This raises the question which one is the "best". By considering this question we enter the arena of what is called bargaining theory.

This theory has its origin in two papers by Nash [24] and [25]. In these papers a bargaining problem is defined as a situation in which two (or more) individuals or organizations have to agree on the choice of one specific alternative from a set of alternatives available to them, while having conflicting interests over this set of alternatives. Nash proposes in [25] two different approaches to the bargaining problem, namely the *axiomatic* and the *strategic* approach. The axiomatic approach lists a number of desirable properties the solution must have, called the *axioms*. The strategic approach on the other hand, sets out a particular bargaining procedure and asks what outcomes would result from rational behavior by the individual players.

So, bargaining theory deals with the situation in which players can realize -through cooperation- other (and better) outcomes than the one which becomes effective when they do not cooperate. This non-cooperative outcome is called the *threatpoint*. The question is to which outcome the players may possibly agree.

In Figure 2 a typical bargaining game is sketched (see also Figure 1). The inner part of the ellipsoid marks out the set of possible outcomes, the *feasible set* S, of the game. The point d is the threatpoint. The edge P is the set of individually rational Pareto-optimal outcomes.

We assume that if the agents unanimously agree on a point $x = (J_1, J_2) \in S$, they obtain x. Otherwise they obtain d. This presupposes that each player can enforce the threatpoint, when he does not agree with a proposal. The outcome x the players will finally agree on is called the solution of the bargaining problem. Since the solution depends on as well the feasible set S as the threatpoint d, it will be written as $F(S, d)$. Notice that the difference for player i between the solution and the threatpoint, $J_i - d_i$, is the reduction in cost player i incurs by accepting the solution. In the sequel we will call this difference the utility gain for player i. We will use the notation $J := (J_1, J_2)$ to denote a point in S and $x \succ y (x \prec y)$ to denote the *vector inequality*, i.e. $x_i > y_i (x_i < y_i)$, $i = 1, 2$. In axiomatic bargaining theory a number of solutions have been proposed. In Thomson [28] a survey is given on this theory. We will present here the three most commonly used solutions: the Nash

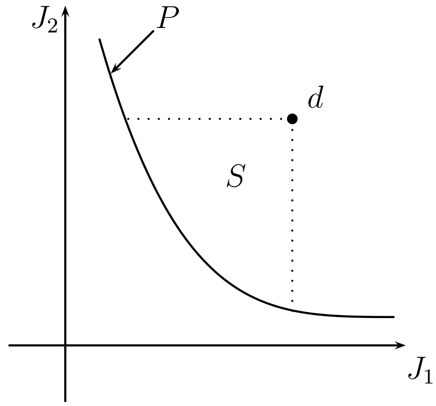

Figure 2: The bargaining game.

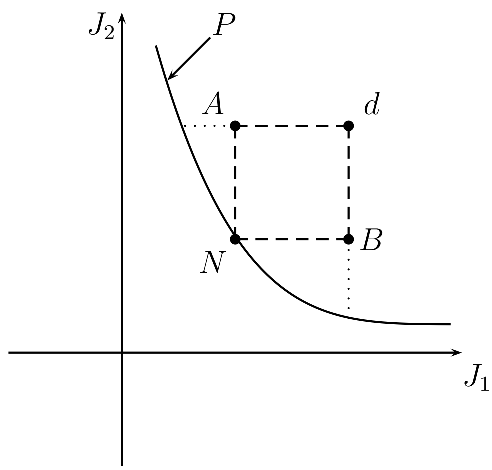

Figure 3: The Nash Bargaining solution $N(S,d)$.

bargainig solution, the Kalai-Smorodinsky solution and the Egalitarian solution.

The *Nash bargaining solution*, $N(S,d)$, selects the point of S at which the product of utility gains from d is maximal. That is,

$$N(S,d) = \arg\max_{J \in S} \prod_{i=1}^{N} (J_i - d_i), \text{ for } J \in S \text{ with } J \preceq d.$$

In Figure 3 we sketched the N solution. Geometrically, the Nash Bargaining solution is the point on the edge of S (that is a part of the Pareto frontier) which yields the largest rectangle (N, A, B, d).

The Kalai-Smorodinsky solution, $K(S,d)$, sets utility gains from the threatpoint proportional to the player's most optimistic expectations. For each agent, the most optimistic expectation is defined as the lowest cost he can attain in the feasible set subject to the constraint that no agent incurs a cost higher than his coordinate of the threatpoint. Defining the *ideal point* as

$$I(S,d) := \max\{J_i \mid J \in S,\ J \preceq d\},$$

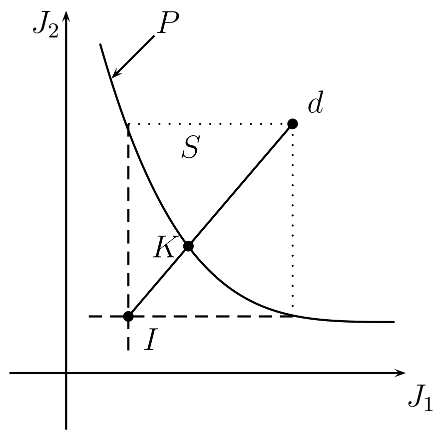

Figure 4: The Kalai-Smorodinsky solution $K(S, d)$.

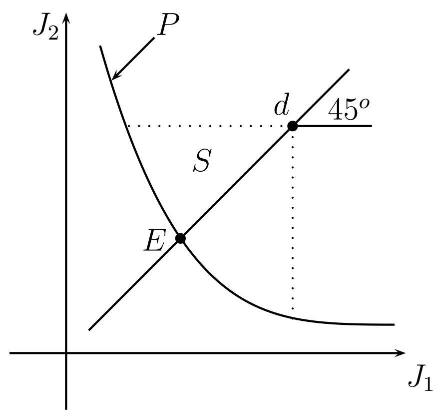

Figure 5: The Egalitarian solution $E(S, d)$.

the *Kalai-Smorodinsky solution* is then

$$K(S, d) := \text{ maximal point of } S \text{ on the segment connecting } d \text{ to } I(S, d).$$

In Figure 4 the Kalai-Smorodinsky solution is sketched for the two-player case. Geometrically, it is the intersection of the Pareto frontier P with the line which connects the threatpoint and the ideal point. The components of the ideal point are the minima each player can reach when the other player is fully altruistic under cooperation.

Finally, the *Egalitarian solution*, $E(S, d)$, represents the idea that gains should be equal divided between the players. Formal,

$$E(S, d) := \text{maximal point in } S \text{ for which } E_i(S, d) - d_i = E_j(S, d) - d_j, \; i, j = 1, \cdots, N.$$

Again, we sketched this solution for the two-player case. In Figure 5 we observe that geometrically this Egalitarian solution is obtained as the intersection point of the 45^o-line through the threatpoint d with the Pareto frontier P.

Notice that in particular in contexts where interpersonal comparisons of utility is inappropriate or impossible, the first two bargaining solutions still make sense.

As already mentioned above these bargaining solutions can be motivated using an "axiomatic approach". In this case some people prefer to speak of an arbitration scheme instead of a bargaining game. An arbiter draws up the reasonable axioms and depending on these axioms, a solution results.

Algorithms to calculate the first two solutions numerically are outlined in [9]. The calculation of the Egalitarian solution requires the solution of one non-linear constrained equations problem. The involved computer time to calculate this E-solution approximately equals that of calculating the N-solution.

3 The Open-Loop Game

In the rest of this paper we consider the case that players do not cooperate in order to realize their goals. In this section we will be dealing with the *open-loop* information structure. That is, the case where every player knows at time $t \in [0, T]$ just the initial state x_0 and the model structure. This scenario can be interpreted as that the players simultaneously determine their actions, next submit their actions to some authority who then enforces these plans as binding commitments. For the game (1,2) we will study the set of Nash equilibria. Formal a Nash equilibrium is defined as follows.

Definition 3.1 An admissible set of actions (u_1^*, u_2^*) is a *Nash equilibrium* for the game (1,2), if for all admissible (u_1, u_2) the following inequalities hold:

$$J_1(u_1^*, u_2^*) \leq J_1(u_1, u_2^*) \text{ and } J_2(u_1^*, u_2^*) \leq J_2(u_1^*, u_2). \qquad \square$$

Here admissibility is meant in the sense that $u_i(.)$ belongs to some restricted set, where this set depends on the information players have on the game, the set of strategies the players like to use to control the system, and the system (2) must have a unique solution.

So, the Nash equilibrium is defined such that it has the property that there is no incentive for any unilateral deviation by any one of the players. Notice that in general one cannot expect to have a unique Nash equilibrium. Moreover, it is easily verified that whenever a set of actions (u_1^*, u_2^*) is a Nash equilibrium for a game with cost functions $J_i,\ i = 1, 2$, these actions also constitute a Nash equilibrium for the game with cost functions $\alpha_i J_i,\ i = 1, 2$, for every choice of $\alpha_i > 0$.

Using the shorthand notation $S_i := B_i R_{ii}^{-1} B_i^T$, we have the following theorem in case the planning horizon T is finite.

Theorem 3.2 *Consider matrix*

$$M := \begin{bmatrix} A & -S_1 & -S_2 \\ -Q_1 & -A^T & 0 \\ -Q_2 & 0 & -A^T \end{bmatrix}. \tag{3}$$

Assume that the two Riccati differential equations,

$$\dot{K}_i(t) = -A^T K_i(t) - K_i(t)A + K_i(t)S_i K_i(t) - Q_i,\ K_i(T) = Q_{iT},\ i = 1, 2, \tag{4}$$

have a symmetric solution $K_i(.)$ on $[0,T]$.
Then, the two-player linear quadratic differential game (1-2) has an open-loop Nash equilibrium for every initial state x_0 if and only if matrix

$$H(T) := [I\ 0\ 0]e^{-MT}\begin{bmatrix} I \\ Q_{1T} \\ Q_{2T} \end{bmatrix} \tag{5}$$

is invertible. Moreover, if for every x_0 there exists an open-loop Nash equilibrium then the solution is unique. The unique equilibrium actions as well as the associated state trajectory can be calculated from the linear two-point boundary value problem

$$\dot{y}(t) = My(t), \text{ with } Py(0) + Qy(T) = [x_0^T\ 0\ 0]^T. \tag{6}$$

Here

$$P = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 & 0 & 0 \\ -Q_{1T} & I & 0 \\ -Q_{2T} & 0 & I \end{bmatrix}.$$

Denoting $[y_0^T(t),\ y_1^T(t),\ y_2^T(t)]^T := y(t)$, with $y_0 \in \mathbb{R}^n$, and $y_i \in \mathbb{R}^{m_i}$, $i = 1,2$, the state and equilibrium actions are

$$x(t) = y_0(t) \text{ and } u_i(t) = -R_{ii}^{-1}B_i^T y_i(t),\ i = 1,2,$$

respectively. □

Assumption (4) is equivalent to the statement that for both players a with this game problem associated linear quadratic control problem should be solvable on $[0,T]$. That is, the optimal control problem that arises in case the action of his opponent(s) would be known must be solvable for each player. Generically one may expect that, if there exists an open-loop Nash equilibrium, the set of Riccati differential equations (4) will have a solution on the closed interval $[0,T]$ (see [9, p.269]).

Next consider the set of coupled asymmetric Riccati-type differential equations:

$$\dot{P}_1 = -A^T P_1 - P_1 A - Q_1 + P_1 S_1 P_1 + P_1 S_2 P_2;\ P_1(T) = Q_{1T} \tag{7}$$

$$\dot{P}_2 = -A^T P_2 - P_2 A - Q_2 + P_2 S_2 P_2 + P_2 S_1 P_1;\ P_2(T) = Q_{2T} \tag{8}$$

Existence of a Nash equilibrium is closely related to the existenc of a solution of the above set of coupled Riccati differential equations. The next result holds.

Theorem 3.3 *The following statements are equivalent:*

- **a)** *For all $T \in [0,t_1)$ there exists for all x_0 a unique open-loop Nash equilibrium for the two-player linear quadratic differential game (1-2).*
- **b)** *The next two conditions hold on $[0,t_1)$.*
    1. *$H(t)$ is invertible for all $t \in [0,t_1)$.*

2. *The two Riccati differential equations (4) have a solution* $K_i(0,T)$ *for all* $T \in [0, t_1)$.

c) *The next two conditions hold on* $[0, t_1)$.

1. *The set of coupled Riccati differential equations (7,8) has a solution* $(P_1(0,T), P_2(0,T))$ *for all* $T \in [0, t_1)$.
2. *The two Riccati differential equations (4) have a solution* $K_i(0,T)$ *for all* $T \in [0, t_1)$.

Moreover, if either one of the above conditions is satisfied the equilibrium is unique. The set of equilibrium actions is in that case given by:

$$u_i^*(t) = -R_{ii}^{-1} B_i^T P_i(t) \Phi(t,0) x_0, \;\; i = 1, 2.$$

Here $\Phi(t,0)$ *satisfies the transition equation*

$$\dot{\Phi}(t,0) = (A - S_1 P_1 - S_2 P_2)\Phi(t,0); \;\; \Phi(t,t) = I. \qquad \square$$

Note that there are situations where the set of Riccati differential equations (7,8) does not have a solution, whilst there does exist an open-loop Nash equilibrium for the game.

With

$$P := \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}; \; D := \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix}; \; S := [S_1 \; S_2]\,; \text{ and } Q := \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix},$$

the set of coupled Riccati equations (7,8) can be rewritten as the non-symmetric matrix Riccati differential equation

$$\dot{P} = -DP - PA + PSP - Q; \;\; P^T(T) = [Q_{1T}, \; Q_{2T}].$$

The solution of such a Riccati differential equation can be obtained by solving a set of linear differential equations. In particular, if this linear system of differential equations (9), below, can be analytically solved we also obtain an analytic solution for (P_1, P_2) (see e.g. [2]). Due to this relationship it is possible to compute solutions of (7,8) in an efficient reliable way using standard computer software packages like, e.g., MATLAB. We have the following result.

Proposition 3.4 *The set of coupled Riccati differential equations (7,8) has a solution on* $[0,T]$ *if and only if the set of linear differential equations*

$$\begin{bmatrix} \dot{U}(t) \\ \dot{V}_1(t) \\ \dot{V}_2(t) \end{bmatrix} = M \begin{bmatrix} U(t) \\ V_1(t) \\ V_2(t) \end{bmatrix}; \; \begin{bmatrix} U(T) \\ V_1(T) \\ V_2(T) \end{bmatrix} = \begin{bmatrix} I \\ Q_{1T} \\ Q_{2T} \end{bmatrix} \qquad (9)$$

has a solution on $[0,T]$, *with* $U(.)$ *nonsingular.*
Moreover, if (9) has an appropriate solution $(U(.), V_1(.), V_2(.))$, *the solution of (7,8) is obtained as* $P_i(t) := V_i(t)U^{-1}(t), \; i = 1, 2.$ $\square$

Next we consider the infinite planning horizon case. That is, the case that the performance criterion player $i = 1, 2$, likes to minimize is:

$$\lim_{T\to\infty} J_i(x_0, u_1, u_2, T) \tag{10}$$

where

$$J_i = \int_0^T \{x^T(t)Q_i x(t) + u_i^T(t)R_{ii}u_i(t) + u_j^T(t)R_{ij}u_j(t)\}dt,$$

subject to the familiar dynamic state equation (2). We assume that the matrix pairs $(A, B_i),\ i = 1, 2$, are stabilizable. So, in principle, each player is capable to stabilize the system on his own.

Since we only like to consider those outcomes of the game that yield a finite cost to both players and the players are assumed to have a common interest in stabilizing the system, we restrict ourselves to functions belonging to the set

$$\mathcal{U}_s(x_0) = \left\{u \in L_2 \mid J_i(x_0, u) \text{ exists in } \mathbb{R} \cup \{-\infty, \infty\}, \lim_{t\to\infty} x(t) = 0\right\}.$$

A similar remark as for the finite planning horizon case applies here. That is, the restriction to this set of control functions requires some form of communication between the players.
To find conditions under which this game has a unique equilibrium the next algebraic Riccati equations play a fundamental role.

$$0 = A^T P_1 + P_1 A + Q_1 - P_1 S_1 P_1 - P_1 S_2 P_2, \tag{11}$$
$$0 = A^T P_2 + P_2 A + Q_2 - P_2 S_2 P_2 - P_2 S_1 P_1; \tag{12}$$

and

$$0 = A^T K_i + K_i A - K_i S_i K_i + Q_i,\ i = 1, 2. \tag{13}$$

Notice that the equations (13) are just the ordinary Riccati equations that appear in the regulator control problem.

Definition 3.5 A solution (P_1, P_2) of the set of algebraic Riccati equations (11,12) is called

a. *stabilizing*, if $\sigma(A - S_1 P_1 - S_2 P_2) \subset \mathbb{C}^-$;

b. *strongly stabilizing* if

i. it is a stabilizing solution, and

ii.

$$\sigma\left(\begin{bmatrix} -A^T + P_1 S_1 & P_1 S_2 \\ P_2 S_1 & -A^T + P_2 S_2 \end{bmatrix}\right) \subset \mathbb{C}_0^+. \tag{14}$$

Here $\mathbb{C}^-$ denotes the left open half of the complex plane and $\mathbb{C}_0^+$ its complement. □

The next theorem gives an answer under which conditions the set of algebraic Riccati equations has a unique strongly stabilizing solution. One of these conditions is that a certain subspace should be a graph subspace. A subspace $V = \text{Im}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, with $X_i \in \mathbb{R}^{n\times n}$, which has the additional property that X_1 is invertible is called a *graph subspace* (since it can be "visualized" as the graph of the map: $x \to X_2 X_1^{-1} x$).

Theorem 3.6

1. *The set of algebraic Riccati equations (11,12) has a strongly stabilizing solution* (P_1, P_2) *if and only if matrix* M *has an* n*-dimensional stable graph subspace and* M *has* $2n$ *eigenvalues (counting algebraic multiplicities) in* $\mathbb{C}_0^+$.

2. *If the set of algebraic Riccati equations (11,12) has a strongly stabilizing solution, then it is unique.*

Then we have the following main result (see also [11]).

Theorem 3.7 *The linear quadratic differential game (2,10) has a unique open-loop Nash equilibrium for every initial state if and only if*

1. *The set of coupled algebraic Riccati equations (11,12) has a strongly stabilizing solution, and*

2. *the two algebraic Riccati equations (13) have a stabilizing solution.*

Moreover, the unique equilibrium actions are given by

$$u_i^*(t) = -R_{ii}^{-1} B_i^T P_i \Phi(t,0) x_0, \; i = 1,2. \tag{15}$$

Here $\Phi(t,0)$ *satisfies the transition equation*

$$\dot{\Phi}(t,0) = (A - S_1 P_1 - S_2 P_2)\Phi(t,0); \; \Phi(t,t) = I. \qquad \square$$

To calculate the unique equilibrium numerically one can use the next algorithm.

Algorithm 3.8

Step 1: Calculate the eigenstructure of $H_i := \begin{bmatrix} A & -S_i \\ -Q_i & -A^T \end{bmatrix}$.
If $H_i, \; i = 1,2$, has an n-dimensional stable graph subspace, then proceed. Otherwise goto Step 5.

Step 2: Calculate matrix $M := \begin{bmatrix} A & -S_1 & -S_2 \\ -Q_1 & -A^T & 0 \\ -Q_2 & 0 & -A^T \end{bmatrix}$. Next calculate the spectrum of M. If the number of eigenvalues having a strict negative real part (counted with algebraic multiplicities) differs from n, goto Step 5.

Step 3: Calculate the n-dimensional M-invariant subspace for which $\mathrm{Re}\,\lambda < 0$ for all $\lambda \in \sigma(M)$. That is, calculate the subspace consisting of the union of all (generalized) eigenspaces associated with each of these eigenvalues. Let $\mathcal{P}$ denote this invariant subspace. Calculate $3n \times n$ matrices X, Y and Z such that $\mathrm{Im} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \mathcal{P}$.
Verify whether matrix X is invertible. If X is not invertible goto Step 5.

Step 4: Denote $P_1 := YX^{-1}$ and $P_2 := ZX^{-1}$. Then

$$u_i^*(t) := -R_{ii}^{-1} B_i^T P_i e^{A_{cl}t} x_0$$

is the unique open-loop Nash equilibrium strategy for every initial state of the game. Here $A_{cl} := A - S_1P_1 - S_2P_2$. The spectrum of the corresponding closed-loop matrix A_{cl} equals $\sigma(M|_{\mathcal{P}})$. The involved cost for player i is $x_0^T M_i x_0$, where M_i is the unique solution of the Lyapunov equation:

$$A_{cl}^T M_i + M_i A_{cl} + Q_i + P_i^T S_i P_i + P_j B_j^T R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j P_j = 0.$$

Step 5: End of algorithm. □

Step 1 in the algorithm verifies whether the two algebraic Riccati equations (13) have a stabilizing solution. Step 2 and 3 verify whether matrix M has an n-dimensional stable graph subspace. Finally, Step 4 determines the unique equilibrium.

The next example illustrates the algorithm.

Example 3.9

1. Consider the system

$$\dot{x}(t) = -2x(t) + u_1(t) + u_2(t),\ x(0) = x_0;$$

and cost functions

$$J_1 = \int_0^\infty \{x^2(t) + u_1^2(t)\}dt \text{ and } J_2 = \int_0^\infty \{4x^2(t) + u_2^2(t)\}dt.$$

Then,

$$M = \begin{bmatrix} -2 & -1 & -1 \\ -1 & 2 & 0 \\ -4 & 0 & 2 \end{bmatrix}.$$

The eigenvalues of M are $\{-3, 2, 3\}$. An eigenvector corresponding with the eigenvalue -3 is $[5,\ 1,\ 4]^T$.

So, according Theorem 3.6 item 1, the with this game corresponding set of algebraic Riccati equations (11,12) has a strongly stabilizing solution. Furthermore, since $q_i > 0,\ i = 1, 2$, the two algebraic Riccati equations (13) have a stabilizing solution. Consequently, this game has a unique open-loop Nash equilibrium for every initial state x_0.

The equilibrium actions and cost are

$$u_1^*(t) = \frac{-1}{5}x^*(t),\ u_2^*(t) = \frac{-4}{5}x^*(t), \text{ where } \dot{x}^*(t) = -3x^*(t) \text{ and } J_1 = \frac{13}{75}x_0^2,\ J_2 = \frac{58}{75}x_0^2.$$

2. Reconsider the game from item 1, but with the system dynamics replaced by

$$\dot{x}(t) = 2x(t) + u_1(t) + u_2(t),\ x(0) = x_0.$$

The with this game corresponding matrix M has the eigenvalues $\{-3, -2, 3\}$. Since M has two stable eigenvalues, it follows from Theorem 3.6 item 1 that the with this game corresponding set of algebraic Riccati equations (11,12) does not have a strongly stabilizing solution. So, see Theorem 3.7, the game does not have for every initial state a unique open-loop Nash equilibrium.

It can be shown (see [9]) that in this example for every initial state there are actually an infinite number of open-loop Nash equilibria. □

4 The Feedback Game

It is often argued that weak time consistency is a minimal requirement for the credibility of an equilibrium solution. That is, if the advertised equilibrium action of, say, player one is not weakly time consistent, player one would have an incentive to deviate from this action during the course of the game. For the other players, knowing this, it is therefore rational to incorporate this defection of player one into their own actions, which would lead to a different equilibrium solution. On the other hand, the property that the equilibrium solution does not have to be adapted by the players during the course of the game, although the system evolutes not completely as expected beforehand, is generally experienced as a very nice property. Since the open-loop Nash equilibria in general do not have this property, the question arises whether there exist strategy spaces Γ_i such that if we look for Nash equilibria within these spaces, the equilibrium solutions do satisfy this strong time consistency property.

Since the system we consider is linear, it is often argued that the equilibrium actions should be a linear function of the state too. This argument implies that we should consider either a refinement of the feedback Nash equilibrium concept, or, strategy spaces that only contain functions of the above mentioned type. The first option amounts to consider only those feedback Nash equilibria which permit a linear feedback synthesis as being relevant. For the second option one has to consider the strategy spaces defined by

$$\Gamma_i^{lfb} := \{u_i(0,T)|u_i(t) = F_i(t)x(t) \text{ where } F_i(.) \text{ is a piecewise continuous function, } i = 1,2\},$$

and consider Nash equilibrium actions (u_1^*, u_2^*) within the strategy space $\Gamma_1^{lfb} \times \Gamma_2^{lfb}$.

It turns out that both equilibrium concepts yield the same characterization of these equilibria for the linear quadratic differential game, which will be presented below in Theorem 4.5. Therefore, we will define just one equilibrium concept here.

Definition 4.1 The set of control actions $u_i^*(t) = F_i^*(t)x(t)$ constitute a *linear feedback Nash equilibrium* solution if both

$$J_1(u_1^*, u_2^*) \leq J_1(u_1, u_2^*) \text{ and } J_2(u_1^*, u_2^*) \leq J_1(u_1^*, u_2),$$

for all $u_i \in \Gamma_i^{lfb}$. □

Remark 4.2 In the sequel, with some abuse of notation, sometimes the pair $(F_1^*(t), F_2^*(t))$ will be called a (linear) feedback Nash equilibrium. □

Similar as for open-loop Nash equilibria, it turns out that linear feedback Nash equilibria can be explicitly determined by solving a set of coupled Riccati equations.

Theorem 4.3 *The two-player linear quadratic differential game (1,2) has for every initial state a linear feedback Nash equilibrium if and only if the next set of coupled Riccati differential equations* *has a set of symmetric solutions* K_1, K_2 *on* $[0,T]$

$$\begin{aligned}
\dot{K}_1(t) &= -(A - S_2K_2(t))^T K_1(t) - K_1(t)(A - S_2K_2(t)) + K_1(t)S_1K_1(t) - Q_1 - K_2(t)S_{21}K_2(t), \\
K_1(T) &= Q_{1T} \qquad (16) \\
\dot{K}_2(t) &= -(A - S_1K_1(t))^T K_2(t) - K_2(t)(A - S_1K_1(t)) + K_2(t)S_2K_2(t) - Q_2 - K_1(t)S_{12}K_1(t), \\
K_2(T) &= Q_{2T}. \qquad (17)
\end{aligned}$$

Moreover, in that case there is a unique equilibrium. The equilibrium actions are

$$u_i^*(t) = -R_{ii}^{-1} B_i^T K_i(t) x(t), \; i = 1, 2.$$

The cost incurred by player i *is* $x_0^T K_i(0) x_0, \; i = 1, 2.$ □

Next we consider the infinite planning horizon case. Like in the open-loop case we consider the minimization of the performance criterion (10). In line with our motivation for the finite-planning horizon, it seems reasonable to study Nash equilibria within the class of linear time-invariant state feedback policy rules. Therefore we shall restrain our set of permitted controls to the constant linear feedback strategies. That is, to $u_i = F_i x$, with $F_i \in \mathbb{R}^{m_i \times n}, \; i = 1, 2$, and where (F_1, F_2) belongs to the set

$$\mathcal{F} := \{F = (F_1, F_2) \mid A + B_1F_1 + B_2F_2 \text{ is stable}\}.$$

The stabilization constraint is imposed to ensure the finiteness of the infinite-horizon cost integrals that we will consider. This assumption can also be justified from the supposition that one is studying a perturbed system which is temporarily out of equilibrium. In that case it is reasonable to expect that the state of the system remains close to the origin. Obviously the stabilization constraint is a bit unwieldy since it introduces dependence between the strategy spaces of the players. So, it presupposes that there is at least the possibility of some coordination between both players. This coordination assumption seems to be more stringent in this case than for the equilibrium concepts we introduced before. However, the stabilization constraint can be motivated from the supposition that both players have a first priority in stabilizing the system. Whether this coordination actually takes place, depends on the outcome of the game. Only in case the players have the impression that their actions are such that the system becomes unstable, they will coordinate their actions in order to realize this meta-objective and adapt their actions accordingly. Probably for most games the equilibria without this stabilization constraint coincide with the equilibria of the game if one does consider this additional stabilization constraint. That is, the stabilization constraint will be in most cases not active. But there are games where it does play a role.

To make sure that our problem setting makes sense, we assume that the set $\mathcal{F}$ is non-empty. A necessary and sufficient condition for this to hold is that the matrix pair $(A, \begin{bmatrix} B_1, & B_2 \end{bmatrix})$ is stabilizable.

Summarizing, we define the concept of a linear feedback Nash equilibrium on an infinite-planning horizon as follows.

Definition 4.4 $(F_1^*, F_2^*) \in \mathcal{F}$ is called a *stationary linear feedback Nash equilibrium* if the following inequalities hold:

$$J_1(x_0, F_1^*, F_2^*) \leq J_1(x_0, F_1, F_2^*) \text{ and } J_2(x_0, F_1^*, F_2^*) \leq J_2(x_0, F_1^*, F_2)$$

for each x_0 and for each state feedback matrix $F_i, \; i = 1, 2$ such that (F_1^*, F_2) and $(F_1, F_2^*) \in \mathcal{F}$. □

Unless stated differently, the phrases "stationary" and "linear" in the notion of stationary linear feedback Nash equilibrium are dropped. This, since it is clear from the context here which equilibrium concept we are dealing with.

Next, consider the set of coupled algebraic Riccati equations

$$0 = -(A - S_2K_2)^TK_1 - K_1(A - S_2K_2) + K_1S_1K_1 - Q_1 - K_2S_{21}K_2, \quad (18)$$

$$0 = -(A - S_1K_1)^TK_2 - K_2(A - S_1K_1) + K_2S_2K_2 - Q_2 - K_1S_{12}K_1. \quad (19)$$

Theorem 4.5 below states that feedback Nash equilibria are completely characterized by *stabilizing solutions* of (18,19). That is, by solutions (K_1, K_2) for which the closed-loop system matrix $A - S_1K_1 - S_2K_2$ is stable.

Theorem 4.5 *Let (K_1, K_2) be a stabilizing solution of (18,19) and define $F_i^* := -R_{ii}^{-1}B_i^TK_i$ for $i = 1, 2$. Then (F_1^*, F_2^*) is a feedback Nash equilibrium. Moreover, the cost incurred by player i by playing this equilibrium action is $x_0^TK_ix_0,\ i = 1, 2$.*
Conversely, if (F_1^*, F_2^*) is a feedback Nash equilibrium, there exists a stabilizing solution (K_1, K_2) of (18,19) such that $F_i^* = -R_{ii}^{-1}B_i^TK_i$. □

Theorem 4.5 shows that all infinite-planning horizon feedback Nash equilibria can be found by solving a set of coupled algebraic Riccati equations. Solving the system (18,19) is in general a difficult problem. To get some intuition for the solution set we next consider the scalar two-player game, where players are not interested in the control actions pursued by the other player. In that case it is possible to derive some analytic results. In particular it can be shown that in this game never more than three equilibria occur. Furthermore a complete characterization of parameters which give rise to either 0, 1, 2, or 3 equilibria is possible.

So, consider the game.

$$J_i(x_0, u_1, u_2) = \int_0^\infty \{q_ix^2(t) + r_iu_i^2\}dt,\ i = 1, 2, \quad (20)$$

subject to the dynamical system

$$\dot{x}(t) = ax(t) + b_1u_1(t) + b_2u_2(t),\ x(0) = x_0. \quad (21)$$

The associated relevant algebraic Riccati equations are obtained from (18,19) by substitution of $R_{21} = R_{12} = 0$, $A = a$, $B_i = b_i$, $Q_i = q_i$, $R_{ii} = r_i$ and $s_i := b_i^2/r_i$, $i = 1, 2$, into these equations. By Theorem 4.5 then a pair of control actions $f_i^* := -\frac{b_i}{r_i}k_i,\ i = 1, 2$, constitute a feedback Nash equilibrium if and only if the next equations have a solution $x_i = k_i,\ i = 1, 2:$

$$s_1x_1^2 + 2s_2x_1x_2 - 2ax_1 - q_1 = 0 \quad (22)$$

$$s_2x_2^2 + 2s_1x_1x_2 - 2ax_2 - q_2 = 0 \quad (23)$$

$$a - s_1x_1 - s_2x_2 < 0. \quad (24)$$

Geometrically, the equations (22) and (23) represent two hyperbolas in the (x_1, x_2) plane, whereas the inequality (24) divides this plane into a "stable" and an "anti-stable" region. So, all feedback Nash equilibria are obtained as the intersection points of both hyperbolas in the "stable" region. Example 4.6 illustrates the situation.

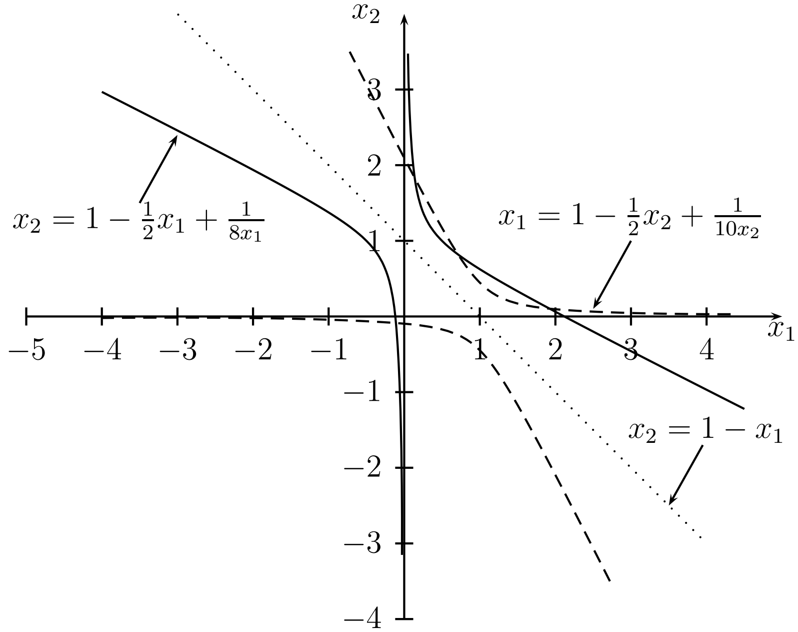

Figure 6: A game with three feedback Nash equilibria: $a = b_i = r_i = 1, q_1 = \frac{1}{4}, q_2 = \frac{1}{5}$.

Example 4.6 Consider $a = b_i = r_i = 1,\ i = 1, 2,\ q_1 = \frac{1}{4}$ and $q_2 = \frac{1}{5}$. Then the hyperbola describing (22,23) are

$$\begin{aligned} x_2 &= 1 - \frac{1}{2}x_1 + \frac{1}{8x_1}, \text{ and} \\ x_1 &= 1 - \frac{1}{2}x_2 + \frac{1}{10x_2}, \end{aligned}$$

respectively.
Both hyperbola, as well as the "stability-separating" line $x_2 = 1 - x_1$, are plotted in Figure 6. From the plot we see that both hyperbola have three intersection points in the stable region. So, the game has three feedback Nash equilibria. □

Next introduce $\sigma_i := s_i q_i$ and for all $x > 0$, satisfying $x^2 \geq \sigma_1$, the functions

$$f_1(x) = x - \sqrt{x^2 - \sigma_1} - \sqrt{x^2 - \sigma_2} \quad (25)$$

$$f_2(x) = x + \sqrt{x^2 - \sigma_1} - \sqrt{x^2 - \sigma_2} \quad (26)$$

$$f_3(x) = x - \sqrt{x^2 - \sigma_1} + \sqrt{x^2 - \sigma_2} \quad (27)$$

$$f_4(x) = x + \sqrt{x^2 - \sigma_1} + \sqrt{x^2 - \sigma_2}. \quad (28)$$

The number of equilibria coincides with the total number of intersection points all of the functions $f_i(x),\ i = 1, \cdots, 4$, have with the horizontal line a (see Figure 7). Elaboration of this property yields the following result.

Theorem 4.7 *Consider the differential game (20–21), with* $\sigma_i = \frac{b_i^2 q_i}{r_i},\ i = 1, 2$. *Assume, without loss of generality, that* $\sigma_1 \geq \sigma_2$. *Moreover, let* $f_i(x),\ i = 1, \cdots, 4$, *be defined as in (25–28). Then, if*

1a. $\sigma_1 > 0$ *and* $\sigma_1 > \sigma_2$*, the game has*

- *one equilibrium if* $-\infty < a < \min f_3(x)$
- *two equilibria if* $a = \min f_3(x)$
- *three equilibria if* $a > \min f_3(x)$.

1b. $\sigma_1 = \sigma_2 > 0$ *the game has*

- *one equilibrium if* $a \leq \sqrt{\sigma_1}$
- *three equilibria if* $a > \sqrt{\sigma_1}$.

2a. $\sigma_1 < 0$ *and* $\sigma_1 > \sigma_2$*, the game has*

- *no equilibrium if* $\max f_1(x) < a \leq \sqrt{-\sigma_1} - \sqrt{-\sigma_2}$
- *one equilibrium if either*
  i) $a = \max f_1(x)$;
  ii) $a \leq -\sqrt{-\sigma_1} - \sqrt{-\sigma_2}$, *or*
  iii) $-\sqrt{-\sigma_1} - \sqrt{-\sigma_2} < a \leq \sqrt{-\sigma_1} + \sqrt{-\sigma_2}$
- *two equilibria if either*
  i) $-\sqrt{-\sigma_1} - \sqrt{-\sigma_2} < a < \max f_1(x)$, *or*
  ii) $-\sqrt{-\sigma_1} + \sqrt{-\sigma_2} < a \leq \sqrt{-\sigma_1} + \sqrt{-\sigma_2}$
- *three equilibria if* $a > \sqrt{-\sigma_1} + \sqrt{-\sigma_2}$.

2b. $\sigma_1 = \sigma_2 < 0$ *the game has*

- *no equilibrium if* $-\sqrt{-3\sigma} < a \leq 0$,
- *one equilibrium if* $a \leq -2\sqrt{-\sigma}$,
- *two equilibria if* $-2\sqrt{-\sigma} < a \leq -\sqrt{-3\sigma}$ *and* $0 < a \leq 2\sqrt{-\sigma}$,
- *three equilibria if* $a > 2\sqrt{-\sigma}$. □

Example 4.8
Take $a = 3$, $b_i = r_i = 1,\ i = 1, 2$, $q_1 = 9$ and $q_2 = 1$. Then, $\sigma_1 = 9 > 1 = \sigma_2$. Furthermore,

$$f_3(x) - 3 = x - 3 - \sqrt{x^2 - 9} + \sqrt{x^2 - 1} > 0, \text{ if } x \geq 3.$$

So, $\min_{x \geq 3} f_3(x) > 3$, and therefore

$$-\infty < a < \min f_3(x).$$

According Theorem 4.7, item 1a, the game has one feedback Nash equilibrium. In Figure 7 one can also see this graphically. □

Below we present a numerical algorithm to calculate all feedback Nash equilibria of the two-player scalar game (20–21). This algorithm can be generalized for the corresponding N-player scalar game (see [9]). The algorithm follows from the next result.

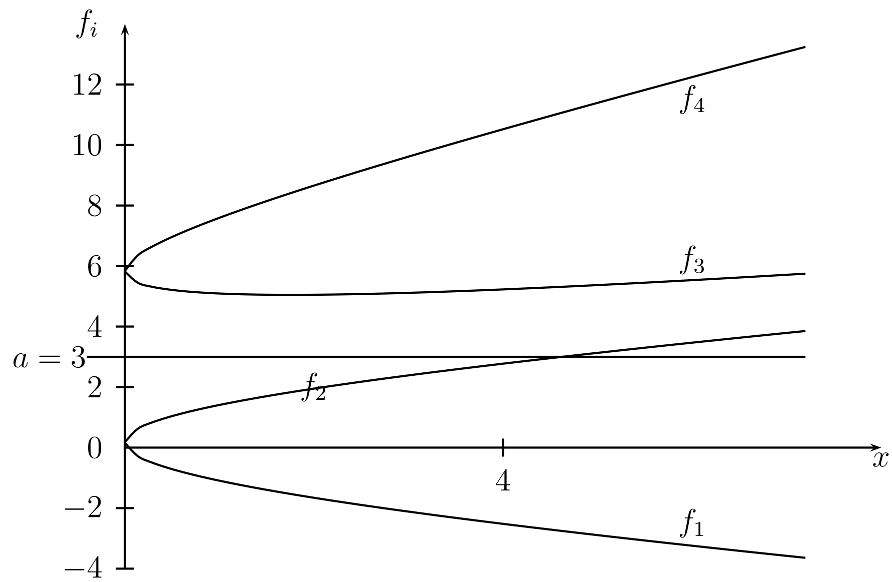

Figure 7: The curves f_i for $\sigma_1 = 9$; $\sigma_2 = 1$.

Theorem 4.9

1. *Assume that (k_1, k_2) is a feedback Nash equilibrium strategy. Then the negative of the corresponding closed-loop system parameter $\lambda := -a + \sum_{i=1}^{2} s_i k_i > 0$ is an eigenvalue of the matrix*

$$M := \begin{bmatrix} -a & s_1 & s_2 & 0 \\ q_1 & a & 0 & -s_2 \\ q_2 & 0 & a & -s_1 \\ 0 & \frac{1}{3}q_2 & \frac{1}{3}q_1 & \frac{1}{3}a \end{bmatrix}. \quad (29)$$

Furthermore, $[1, k_1, k_2, k_1 k_2]^T$ is a corresponding eigenvector and $\lambda^2 \geq \sigma_{max}$.

2. *Assume that $[1, k_1, k_2, k_3]^T$ is an eigenvector corresponding to a positive eigenvalue λ of M, satisfying $\lambda^2 \geq \sigma_{max}$, and that the eigenspace corresponding with λ has dimension one. Then, (k_1, k_2) is a feedback Nash equilibrium.* □

Algorithm 4.10 The following algorithm calculates all feedback Nash equilibria of the linear quadratic differential game (20,21).

Step 1: Calculate matrix M in (29) and $\sigma := \max_i \frac{b_i^2 q_i}{r_i}$.

Step 2: Calculate the eigenstructure (λ_i,m_i), $i = 1, \cdots, k$, of M, where λ_i are the eigenvalues and m_i the corresponding algebraic multiplicities.

Step 3: For $i = 1, \cdots, k$ repeat the following steps:

3.1) If i) $\lambda_i \in \mathbb{R}$; ii) $\lambda_i > 0$ and iii) $\lambda_i^2 \geq \sigma$ then proceed with Step 3.2 of the algorithm. Otherwise, return to Step 3.

3.2) If $m_i = 1$ then

3.2.1) calculate an eigenvector v corresponding with λ_i of M. Denote the entries of v by $[v_0, v_1, v_2]^T$. Calculate $k_j := \frac{v_j}{v_0}$ and $f_j := -\frac{b_j k_j}{r_j}$. Then, (f_1, f_2) is a feedback Nash equilibrium and $J_j = k_j x_0^2$, $j = 1, 2$. Return to Step 3.

If $m_i > 1$ then

3.2.2) Calculate $\sigma_i := \frac{b_i^2 q_i}{r_i}$.

3.2.3) For all 4 sequences (t_1, t_2), $t_k \in \{-1, 1\}$,

i) calculate

$$y_j := \lambda_i + t_j \sqrt{\lambda_i^2 - \sigma_j},\ j = 1, 2.$$

ii) If $\lambda_i = -a + \sum_{j=1}^{N} y_j$ then calculate $k_j := \frac{y_j r_j}{b_j^2}$ and $f_j := -\frac{b_j k_j}{r_j}$. Then, (f_1, f_2) is a feedback Nash equilibrium and $J_j = k_j x_0^2$, $j = 1, 2$.

Step 4: End of the algoritm. □

Example 4.11 Reconsider Example 3.9 where, for $a = -2; b_i = r_{ii} = q1 = 1$ and $q_2 = 4,\ i = 1, 2$, we calculated the open-loop Nash equilibrium for an infinite planning horizon. To calculate the feedback Nash equilibria for this game, according Algorithm 4.10, we first have to determine the eigenstructure of matrix

$$M := \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & -2 & 0 & -1 \\ 4 & 0 & -2 & -1 \\ 0 & \frac{4}{3} & \frac{1}{3} & \frac{-2}{3} \end{bmatrix}.$$

Using Matlab, we find the eigenvalues $\{-2.6959, -1.4626 \pm 0.7625i, 2.9543\}$. Since the square of 2.9543 is larger than $\sigma := 4$, we have to process Step 3 of the algorithm for this eigenvalue. Since its algebraic multiplicity is 1, we calculate an eigenvector v corresponding with this eigenvalue 2.9543. Choosing $v^T = [v_0,\ v_1,\ v_2\ v_3] := [0.7768,\ 0.1355,\ 0.6059,\ 0.1057]$ yields then

$$k_1 := \frac{v_1}{v_0} = 0.1744 \text{ and } k_2 := \frac{v_2}{v_0} = 0.7799$$

This gives the feedback Nash equilibrium actions

$$u_1(t) = -\frac{b_1 k_1}{r_1} x(t) = -0.1744 x(t) \text{ and } u_2(t) = -\frac{b_2 k_2}{r_2} x(t) = -0.7799 x(t).$$

The corresponding closed-loop system and cost are

$$\dot{x}(t) = -2.9543 x(t),\ x(0) = x_0;\ \text{and } J_1 = 0.1744 x_0^2, J_2 = 0.7799 x_0^2.$$

Note that these cost almost coincide with the open-loop case. □

5 The Uncertain Non-cooperative Game

Dynamic game theory brings together three features that are key to many situations in economy, ecology, and elsewhere: optimizing behavior, presence of multiple agents, and enduring consequences of decisions. In this section we add a fourth aspect, namely robustness with respect to variability in the environment. In our formulation of dynamic games, so-far, we specified a set of differential equations including input functions that are controlled by the players, and players are assumed to optimize a criterion over time. The dynamic model is supposed to be an exact representation of the environment in which the players act; optimization takes place with no regard of possible deviations. It can safely be assumed, however, that agents in reality follow a different strategy. If an accurate model can be formed at all, it will in general be complicated and difficult to handle. Moreover it may be unwise to optimize on the basis of a too detailed model, in view of possible changes in dynamics that may take place in the course of time and that may be hard to predict. It makes more sense for agents to work on the basis of a relatively simple model and to look for strategies that are robust with respect to deviations between the model and reality. In an economic context, the importance of incorporating aversion to specification uncertainty has been stressed for instance by Hansen *et al.* [18].

In control theory, an extensive theory of robust design is already in place; see e.g. Başar [6]. We use this background to arrive at suitable ways of describing aversion to model risk in a dynamic game context. We assume linear dynamics and quadratic cost functions. These assumptions are reasonable for situations of dynamic quasi-equilibrium, where no large excursions of the state vector are to be expected.

Following a pattern that has become standard in control theory two approaches can be considered. The first one is based on a stochastic approach. This approach assumes that the dynamics of the system are corrupted by a standard Wiener process (white-noise). Basic assumptions are that the players have access to the current value of the state of the system and that the positive definite covariance matrix does not depend on the state of the system. Basically it turns out that under these assumptions the feedback Nash equilibria also constitute an equilibrium in such an uncertain environment. For that reason we will not elaborate that case here any further. In the second approach, a malevolent disturbance input is introduced which is used in the modeling of aversion to specification uncertainty. That is, it is assumed that the dynamics of the system are corrupted by a deterministic noise component, and that each player has his own expectation about this noise. This is modeled by adapting for each player his cost function accordingly. The players cope with this uncertainty by considering a worst-case scenario. Consequently in this approach the equilibria of the game, in general, depend on the worst-case scenario expectations about the noise of the players.

In this section we restrict the analysis to the infinite-planning horizon case. We will first consider the open-loop one-player case.

Consider the problem to find

$$\inf_{u\in\mathcal{U}_s} \sup_{w\in L_2^q(0,\infty)} \int_0^\infty \{x^T(t)Qx(t) + u^T(t)Ru(t) - w^T(t)Vw(t)\}dt \tag{30}$$

subject to

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t),\ x(0) = x_0 \tag{31}$$

where $R > 0$, $V > 0$ and A is stable.

This problem formulation originates from the H_∞ disturbance attenuation control problem and is in literature known (see [4]) as the soft-constrained open-loop differential game. Since no definiteness assumptions were made in section 2 w.r.t. the matrices Q_i one can use the in that section obtained results directly to derive the next result.

Corollary 5.1 *Consider problem (30,31). Let $S := BR^{-1}B^T$ and $M := EV^{-1}E^T$. This problem has a solution for every initial state x_0 if*

1. *The coupled algebraic Riccati equations*

$$\begin{aligned} A^TP_1 + P_1A + Q - P_1SP_1 - P_1MP_2 &= 0, \\ A^TP_2 + P_2A - Q - P_2MP_2 - P_2SP_1 &= 0, \end{aligned}$$

   has a strongly stabilizing solution; and

2. *The two algebraic Riccati equations*

$$\begin{aligned} A^TK_1 + K_1A - K_1SK_1 + Q &= 0, \\ A^TK_2 + K_2A - K_2MK_2 - Q &= 0, \end{aligned}$$

   have a symmetric solution K_i such that $A - SK_1$ and $A - MK_2$ are stable.

Moreover, a worst-case control for the player is

$$u^*(t) = -R^{-1}B^TP_1x(t)$$

whereas the corresponding worst-case disturbance is

$$w^*(t) = -V^{-1}E^TP_2x(t).$$

Here $x(t)$ satisfies the differential equation

$$\dot{x}(t) = (A - SP_1 - MP_2)x(t);\ x(0) = x_0. \qquad \square$$

In case one additionalyy assumes that $Q \geq 0$, the above result simplifies even further and one gets the next result. More results (and, in particular, converse statements) on the soft-constrained open-loop differential game can be found in [4, Section 4.2.1 and Theorem 9.6] and in [21] some preliminary results for the multi-player case.

Corollary 5.2 *Consider problem (30,31). This problem has a solution for every initial state x_0 if the algebraic Riccati equations*

$$A^TP + PA + Q - P(S - M)P = 0$$

and

$$A^TK + KA + KMK + Q = 0.$$

have a solution $\bar{P}$ and $\bar{K}$, respectively, such that $A - (S - M)\bar{P}$ and $A + M\bar{K}$ are stable. Furthermore, a worst-case control for the player is

$$u^*(t) = -R^{-1}B^T\bar{P}x(t)$$

whereas the corresponding worst-case disturbance is

$$w^*(t) = V^{-1}E^T\bar{P}x(t).$$

Here $x(t)$ satisfies the differential equation

$$\dot{x}(t) = (A - (S - M)\bar{P})x(t);\ x(0) = x_0.$$

Moreover, $\bar{J} = x_0^T\bar{P}x_0$. □

From the above corollary we infer in particular that if $x_0 = 0$, the best open-loop worst-case controller is $u = 0$, whereas the worst-case signal in that case is $w = 0$. This independent of the choice of V, under the supposition that the Riccati equations have an appropriate solution. So if a stable system is in equilibrium (i.e. $x_0 = 0$) in this open-loop framework the best reaction to potential unknown disturbances is not to react.

Next we consider the corresponding problem within a feedback information framework. That is, consider

$$\dot{x} = (A + BF)x + Ew,\ x(0) = x_0, \tag{32}$$

with (A, B) stabilizable, $F \in \mathcal{F}$ and

$$J(F, w, x_0) = \int_0^\infty \{x^T(Q + F^TRF)x - w^TVw\}dt. \tag{33}$$

The matrices Q, R and V are symmetric, $R > 0$, and $V > 0$. The problem is to determine for each $x_0 \in \mathbb{R}^n$ the value

$$\inf_{F\in\mathcal{F}} \sup_{w\in L_2^q(0,\infty)} J(F, w, x_0). \tag{34}$$

Furthermore, if this infimum is finite, we like to know whether there is a feedback matrix $\bar{F} \in \mathcal{F}$ that achieves the infimum, and to determine all matrices that have this property. This soft-constrained differential game can also be interpreted as a model for a situation where the controller designer is minimizing the criterion (33) by choosing an appropriate $F \in \mathcal{F}$, while the uncertainty is maximizing the same criterion by choosing an appropriate $w \in L_2^q(0, \infty)$.

A necessary condition for the expression in (34) to be finite is that the supremum

$$\sup_{w\in L_2^q(0,\infty)} J(F, w, x_0)$$

is finite for at least one $F \in \mathcal{F}$. However, this condition is not sufficient. It may happen that the infimum in (34) becomes arbitrarily small. Below we present a sufficient condition under which the soft-constrained differential game has a saddle point.

Theorem 5.3 *Consider (32–33) and let S and M be as defined before. Assume that the algebraic Riccati equation*

$$Q + A^TX + XA - XSX + XMX = 0 \tag{35}$$

has a stabilizing solution X (i.e. $A - SX + MX$ stable) and that additionally $A - SX$ is stable. Furthermore, assume that there exists a real symmetric $n \times n$ symmetric matrix Y that satisfies the matrix inequality

$$Q + A^T Y + YA - YSY \geq 0. \tag{36}$$

Define $\bar{F} := -R^{-1}B^T X$ and $\bar{w}(t) := V^{-1}E^T X e^{(A-SX+MX)t}x_0$. Then the matrix $\bar{F}$ belongs to $\bar{\mathcal{F}}$, the function $\bar{w}$ is in $L_2^q(0,\infty)$, and for all $F \in \mathcal{F}$ and $w \in L_2^q(0,\infty)$

$$J(\bar{F}, w, x_0) \leq J(\bar{F}, \bar{w}, x_0) \leq J(F, \bar{w}, x_0).$$

Moreover, $J(\bar{F}, \bar{w}, x_0) = x_0^T X x_0$. □

Notice that if $Q \geq 0$, condition (36) is trivially satisfied by choosing $Y = 0$. Corollary 5.4 summarizes the consequences of Theorem 5.3 for the problem (34).

Corollary 5.4 *Let the assumptions of Theorem 5.3 hold and let X, $\bar{F}$, and $\bar{w}$ be as in that theorem. Then,*

$$\min_{F \in \mathcal{F}} \sup_{w \in L_2^q(0,\infty)} J(F, w, x_0) = \max_{w \in L_2^q(0,\infty)} J(\bar{F}, w, x_0) = x_0^T X x_0$$

and

$$\max_{w \in L_2^q(0,\infty)} \inf_{F \in \mathcal{F}} J(F, w, x_0) = \min_{F \in \mathcal{F}} J(F, \bar{w}, x_0) = x_0^T X x_0.$$ □

Next we consider the multi-player soft-constrained differential game. That is, we consider

$$\dot{x}(t) = (A + B_1 F_1 + B_2 F_2)x(t) + Ew(t), \;\; x(0) = x_0, \tag{37}$$

with $(A, [B_1 \;\; B_2])$ stabilizable, $(F_1, F_2) \in \mathcal{F}$ and

$$J_i(F_1, F_2, w, x_0) = \int_0^\infty \{x^T(t)(Q_i + F_1^T R_{i1} F_1 + F_2^T R_{i2} F_2)x(t) - w^T(t) V_i w(t)\} dt. \tag{38}$$

Here the matrices Q_i, R_{ij} and V_i are symmetric, $R_{ii} > 0$, $V_i > 0$, and

$$\mathcal{F} := \{(F_1, F_2) | A + B_1 F_1 + B_2 F_2 \text{ is stable}\}.$$

For this game we want to determine all soft-constrained Nash equilibria. That is, to find all $(\bar{F}_1, \bar{F}_2) \in \mathcal{F}$ such that

$$\sup_{w \in L_2^q(0,\infty)} J_1(\bar{F}_1, \bar{F}_2, w, x_0) \leq \sup_{w \in L_2^q(0,\infty)} J_1(F_1, \bar{F}_2, w, x_0), \text{ for all } (F_1, \bar{F}_2) \in \mathcal{F} \tag{39}$$

and

$$\sup_{w \in L_2^q(0,\infty)} J_2(\bar{F}_1, \bar{F}_2, w, x_0) \leq \sup_{w \in L_2^q(0,\infty)} J_2(\bar{F}_1, F_2, w, x_0), \text{ for all } (\bar{F}_1, F_2) \in \mathcal{F}, \tag{40}$$

for all $x_0 \in \mathbb{R}^m$.

Because the weighting matrix V_i occurs with a minus sign in (38), this matrix constrains the disturbance vector w in an indirect way so that it can be used to describe the aversion to model risk of player i. Specifically, if the quantity $w^T V_i w$ is large for a vector $w \in \mathbb{R}^q$, this means that player i does not expect large deviations of the nominal dynamics in the direction of Ew. Furthermore, the larger he chooses V_i, the closer the worst case signal he can be confronted with in this model will approach the zero input signal (that is: $w(.) = 0$).

From Corollary 5.4, a sufficient condition for the existence of a soft-constrained feedback Nash equilibrium follows in a straightforward way. Using the shorthand notation

$$S_i := B_i R_{ii}^{-1} B_i^T, S_{ij} := B_i R_{ii}^{-1} R_{ji} R_{ii}^{-1} B_i^T, \; i \neq j, \text{ and } M_i := E V_i^{-1} E^T,$$

we have the next result.

Theorem 5.5 *Consider the differential game defined by (37–40). Assume there exist real symmetric $n \times n$ matrices $X_i, \; i = 1, 2,$ and real symmetric $n \times n$ matrices $Y_i, \; i = 1, 2,$ such that*

$$-(A - S_2 X_2)^T X_1 - X_1(A - S_2 X_2) + X_1 S_1 X_1 - Q_1 - X_2 S_{21} X_2 - X_1 M_1 X_1 = 0, \quad (41)$$
$$-(A - S_1 X_1)^T X_2 - X_2(A - S_1 X_1) + X_2 S_2 X_2 - Q_2 - X_1 S_{12} X_1 - X_2 M_2 X_2 = 0, \quad (42)$$
$$A - S_1 X_1 - S_2 X_2 + M_1 X_1 \text{ and } A - S_1 X_1 - S_2 X_2 + M_2 X_2 \text{ are stable}, \quad (43)$$
$$A - S_1 X_1 - S_2 X_2 \text{ is stable} \quad (44)$$
$$-(A - S_2 X_2)^T Y_1 - Y_1(A - S_2 X_2) + Y_1 S_1 Y_1 - Q_1 - X_2 S_{21} X_2 \leq 0, \quad (45)$$
$$-(A - S_1 X_1)^T Y_2 - Y_2(A - S_1 X_1) + Y_2 S_2 Y_2 - Q_2 - X_1 S_{12} X_1 \leq 0. \quad (46)$$

Define $\overline{F} = (\overline{F}_1, \overline{F}_2)$ by

$$\overline{F}_i := -R_{ii}^{-1} B_i^T X_i, \; i = 1, 2.$$

Then $\overline{F} \in \mathcal{F}$, and $\overline{F}$ is a soft-constrained Nash equilibrium. Furthermore, the worst-case signal $\bar{w}_i$ from player i's perspective is

$$\bar{w}(t) = V_i^{-1} E^T X_i e^{(A - S_1 X_1 - S_2 X_2 + M_i X_i)t} x_0.$$

Moreover the cost for player i under his worst-case expectations are

$$\overline{J}_i^{SC}(\overline{F}_1, \overline{F}_2, x_0) = x_0^T X_i x_0, \; i = 1, 2.$$

Conversely, if $(\bar{F}_1, \bar{F}_2)$ is a soft-constrained Nash equilibrium, the equations (41–44) have a set of real symmetric solutions (X_1, X_2). □

Again, notice that if $Q_i \geq 0, \; i = 1, 2$ and $S_{ij} \geq 0, \; i, j = 1, 2$, the matrix inequalities (45–46) are trivially satisfied with $Y_i = 0, \; i = 1, 2$. So, under these conditions the differential game defined by (37–40) has a soft-constrained Nash equilibrium if and only if the equations (41–44) have a set of real symmetric $n \times n$ matrices $X_i, \; i = 1, 2$.

Theorem 5.5 shows that the equations (41–44) play a crucial role in the question whether the game (37–38) will have a soft-constrained Nash equilibrium. Every soft-constrained Nash equilibrium has to satisfy these equations. So, the question arises under which conditions (41–44) will have one or more solutions and, if possible, to calculate this (these) solution(s). This is a difficult open question. Similar remarks apply here as were made in Section 4 for solving the corresponding set of algebraic Riccati equations to determine the feedback Nash equilibria. But, again for the scalar case, one can devise an algorithm to calculate all soft-constrained Nash equilibria. This algorithm is in the spirit of Algorithm 4.10 and will be discussed now.

Like in Section 4 we will consider here just the 2-player case under the simplifying assumptions that $b_i \neq 0$ and players have no direct interest in eachothers control actions (i.e. $S_{ij} = 0,\ i \neq j$). For more details and the general N-player case we refer again to the literature ([9], [12]). Again lower case notation will be used to stress the fact that we are dealing with the scalar case. For notational convenience let Ω denote either the set $\{1\},\ \{2\}$ or $\{1, 2\}$. Furthermore, let

$$\tau_i := (s_i + m_i)q_i,\ \tau_{max} := \max_i \tau_i,\ \rho_i := \frac{s_i}{s_i + m_i},\ \gamma_i := -1 + 2\rho_i = \frac{s_i - m_i}{s_i + m_i}, \tag{47}$$

$$\text{and } \gamma_\Omega := -1 + 2\sum_{i\in\Omega} \rho_i.$$

With some small abuse of notation for a fixed index set Ω, γ_Ω will also be denoted without brackets and comma's. That is, if e.g. $\Omega = \{1, 2\}$, γ_Ω is also written as γ_{12}.

An analogous reasoning as in Theorem 4.9 gives the next Theorem.

Theorem 5.6

1. *Assume that (x_1, x_2) solves (41,42,44) and $\gamma_i \neq 0,\ i = 1, 2, 12$. Then $\lambda := -a + \sum_{i=1}^{2} s_i x_i > 0$ is an eigenvalue of the matrix*

$$M := \begin{bmatrix} -a & s_1 & s_2 & 0 \\ \frac{\rho_1 q_1}{\gamma_1} & \frac{a}{\gamma_1} & 0 & -\frac{s_2}{\gamma_1} \\ \frac{\rho_2 q_2}{\gamma_2} & 0 & \frac{a}{\gamma_2} & -\frac{s_1}{\gamma_2} \\ 0 & \frac{\rho_2 q_2}{\gamma_{12}} & \frac{\rho_1 q_1}{\gamma_{12}} & \frac{a}{\gamma_{12}} \end{bmatrix}. \tag{48}$$

   Furthermore, $[1, x_1, x_2, x_1x_2]^T$ is a corresponding eigenvector and $\lambda^2 \geq \tau_{max}$.

2. *Assume that $[1, x_1, x_2, x_3]^T$ is an eigenvector corresponding to a positive eigenvalue λ of M, satisfying $\lambda^2 \geq \tau_{max}$, and that the eigenspace corresponding with λ has dimension one. Then, (x_1, x_2) solves (41,42,44).* □

Using this result we can calculate soft-constrained feedback Nash equilibria by implementing the next numerical algorithm.

Algorithm 5.7 Let $s_i := \frac{b_i^2}{r_i}$ and $m_i := \frac{e^2}{v_i}$. Assume that for every index set Ω, $\gamma_\Omega \neq 0$. Then, the following algorithm calculates all solutions of (41,42,44).

Step 1: Calculate matrix M in (48) and $\tau := \max_i (s_i + m_i)q_i$.

Step 2: Calculate the eigenstructure (λ_i, n_i), $i = 1, \cdots, k$, of M, where λ_i are the eigenvalues and n_i the corresponding algebraic multiplicities.

Step 3: For $i = 1, \cdots, k$ repeat the following steps:

3.1) If i) $\lambda_i \in \mathbb{R}$; ii) $\lambda_i > 0$ and iii) $\lambda_i^2 \geq \tau$ then proceed with Step 3.2 of the algorithm. Otherwise, return to Step 3.

3.2) If $n_i = 1$ then

3.2.1) calculate an eigenvector z corresponding with λ_i of M. Denote the entries of z by $[z_0, z_1, z_2]^T$. Calculate $x_j := \frac{z_j}{z_0}$. Then, (x_1, x_2) solve (41,42,44). Return to Step 3.

If $n_i > 1$ then

3.2.2) Calculate $\tau_i := s_i q_i$.

3.2.3) For all 4 sequences (t_1, t_2), $t_k \in \{-1, 1\}$,

i) calculate

$$y_j := \lambda_i + t_j \frac{s_i}{s_i + m_i} \sqrt{\lambda_i^2 - \sigma_j},\ j = 1, 2.$$

ii) If $\lambda_i = -a + \sum_{j=1}^{2} y_j$ then calculate $x_j := \frac{y_j}{s_j + m_j}$. Then, (x_1, x_2) solves (41,42,44).

Step 4: End of the algorithm. □

Example 5.8 Reconsider Example 3.9. That is, consider the two-player scalar game with $a = -2$, $b_i = e = 1$, $r_{ii} = 1$, $v_i = \frac{1}{9}$, $i = 1, 2$, $q_1 = 1$ and $q_2 = 4$.
To calculate the soft-constrained Nash equilibria of this game, we first determine all solutions of (41,42,44). According Algorithm 5.7, we first have to determine the eigenstructure of the next matrix

$$M := \begin{bmatrix} 2 & 1 & 1 & 0 \\ -1/8 & 5/2 & 0 & 5/4 \\ -1/2 & 0 & 5/2 & 5/4 \\ 0 & -4/42 & -1/42 & 20/42 \end{bmatrix}.$$

Using Matlab, we find that M has the eigenvalues $\{2.4741, .5435, 2.2293 \pm 0.7671i\}$. Since none of the positive eigenvalues squared is larger than $\tau = 40$, the game has no equilibrium.
Next consider the case $v_i = 2$. In that case numerical computations show that M has the eigenvalues $\{-6.7017, -5.0291, -2.4819, 3.0126\}$. The only positive eigenvalue which is squared larger than $\tau = 6$ is $\lambda = 3.0126$. So we have to process Step 3 of the algorithm for this eigenvalue. Since this eigenvalue has a geometric multiplicity of one, there is one solution satisfying (41,42,44). From the corresponding eigenspace one obtains the solution tabulated below (with $a_{cl} = a - s_1 x1 - s_2 x_2 = -$eigenvalue):

eigenvalue	(x_1, x_2)	$a_{cl} + m_1 x_1$	$a_{cl} + m_2 x_2$
3.0126	(0.1735,0.8392)	-2.9259	-2.5931

From the last two columns of this table we see that the solution satisfies the additional conditions (43). Since $q_i > 0$, and thus (45,46) are satisfied with $y_i = 0$, it follows that this game has one soft-constrained Nash equilibrium. The with this equilibrium corresponding equilibrium actions are

$$u_1^*(t) = -0.1735x(t) \text{ and } u_2^*(t) = -0.8392x(t).$$

Assuming that the initial state of the system is x_0, the worst-case expected cost by the players are

$$J_1^* = 0.1735x_0^2 \text{ and } J_2^* = 0.8392x_0^2,$$

respectively.

Compared to the noise-free case we see that player one's cost decreases at least by .52% and player two incurs a cost increase that is maximal 7.60%. In particular this example shows that it is possible that due to the fact that player 2 takes noise into account in his decision making, player 1 gets better off compared to the noise free case. □

6 Concluding Remarks

In this paper we reviewed some main results in the area of linear quadratic differential games. For didactical reasons the results were presented for the two-player case. The exposition is based on [9] where one can find also additional results for the N-player case, proofs, references and a historical perspective for further reading. In particular this book also contains results on convergence properties of the finite planning horizon equilibria in case the planning horizon is extended to infinite. An issue that has not been addressed here.

For the cooperative game one can find an extension of results from section 2 to more general cost functions (like, e.g., indefinite Q_i matrices) in [14].

For the non-cooperative game results generalizing on the cost functions considered here can be found for the open-loop information case in more detail in [10]. For the feedback information case in [13] the existence of equilibria was considered if players can not observe the state of the system directly and use static output feedback to control the system.

A review on computational aspects involved with the calculation of the various equilibria can be found in [16], whereas [15] describes a numerical toolbox that is available on the web to calculate the unique open-loop Nash equilibrium for an infinite planning horizon.

For the infinite planning horizon case both in the open-loop and the feedback information case the number of equilibria the game may have can vary between zero and infinity. Theorem 3.7 presents both necessary and sufficient conditions under which the game has a unique equilibrium for the open-loop information case. An open problem is whether one can also find for the feedback information case conditions that are both necessary and sufficient for the existence of a unique equilibrium. Under such conditions the numerical calculation of this equilibrium is then possible using one of the algorithms proposed in the literature as described, e.g., in [16].

Several special cases of LQ differential games have been considered in literature. We like to mention here two cases where also recently new numerical results were reported.
First, the set of weakly coupled large-scale systems has been studied extensively by e.g. Mukaidani in a number of papers (see e.g. [23]). This are systems where each player controls a set of states which are only marginally affected by other players. So, the corresponding LQ game almost equals an ordinary optimal LQ control problem. It can be shown that under the assumption that the coupling between the various "subsystems" is marginal the LQ game will have a unique equilibrium.
In [3] the set of positive systems has recently been considered. That is, the case that both the state and used controls should be positive at any point in time. In this paper conditions are stated under which such a system has an equilibrium and some algorithms are devised to calculate an equilibrium.

We hope this survey convinced the reader that the last decennia progress has been made in the theory about linear quadratic differential games and that there are many challenges left in this area for research. Without being complete, important references that contributed to this research during the last decennium are Feucht [17], Weeren [29], Başar and Bernhard [4], Başar and Olsder [5], Kun [21], van den Broek [7] and Kremer [20].

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