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By Jeroen Suijs, Peter Borm, Herbert Hamers, Maurice Koster and Marieke Quant

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Communication and cooperation in public network situations

Jeroen Suijs[1]

Peter Borm[2]

Herbert Hamers[2]

Maurice Koster[3]

Marieke Quant[2]

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Abstract

This paper focuses on sharing the costs and revenues of maintaining a public network communication structure. Revenues are assumed to be bilateral and communication links are publicly available but costly. It is assumed that agents are located at the vertices of an undirected graph in which the edges represent all possible communication links. We take the approach from cooperative game theory and focus on the corresponding network game in coalitional form which relates any coalition of agents to its highest possible net bene…t, i.e., the net bene…t corresponding to an optimal operative network. Although …nding an optimal network in general is a di¢cult problem, it is shown that corresponding network games are (totally) balanced. In the proof of this result a speci…c relaxation, duality and techniques of linear production games with committee control play a role. Su¢cient conditions for convexity of network games are derived. Possible extensions of the model and its results are discussed.

Keywords: Public networks, cooperative games, total balancedness, convexity

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[1] CentER Accounting Research Group, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands.
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[2] CentER and Department of Econometrics and Operational Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands.

[3] Department of Quantitative Economics, University of Amsterdam, Roettersstraat 11, 1018WB Amsterdam, The Netherlands.

1 Introduction

This paper analyzes an allocation problem associated to maintaining a communication network between various economic agents. Communication links are widely observed in reality and our framework applies to many such situations like telecommunication, utilities, computer networks and information technology. The latter application is particularly interesting as …rms increasingly invest in information technology equipment to improve …rm-wide availability of divisional-speci…c (or lower-level) information. In principle the model assumes that all links within the underlying communication network are publicly available apart from possible exogenously determined restrictions. The use of a link however is assumed to be costly: a …xed cost is imposed on each link independent who exactly is using this particular link to establish communication. Next to these communication costs there are also revenues from communication. These revenues are assumed to be bilateral, i.e., the actual revenues of a group of agents is determined as the sum of the revenues of the pairs of those agents within this group who can directly or indirectly communicate via a sequence of communication links whose costs are accounted for by the group as a whole. If a group of agents chooses a particular subnetwork to be operative by paying the corresponding communication costs, this implicitly determines the total bene…ts from communication within this group. So the problem the agents face is to …nd an optimal operative network, i.e., an operative network with highest possible net bene…ts. Moreover, next to this optimization problem the agents also face an allocation problem: how to divide the net bene…ts of an optimal operative network among the agents?

Our setting constitutes a typical example in which the fundamental economic issue of cost and revenue allocation resulting from a cooperative endeavor takes place in the context of discrete optimization on networks (cf. Sharkey (1991)). The analysis will incorporate and intermingle techniques from optimization and cooperative game theory. Related literature with respect to restricted cooperation possibilities based on exogenous communication graphs was initiated by Myerson (1977), for a survey we refer to Slikker,van den Nouweland (2001). Closely related within this stream of literature is Slikker, van den Nouweland (2000) on network formation with costs for establishing links. There, however, the costs per link are assumed to be identical and the focus is not on a bilaterally based revenue structure. In our framework this means that the optimization problem with respect to …nding the optimal operative communication network is relatively easy to solve. In the same spirit as this paper on determining optimal operative networks and allocating the corresponding net bene…ts are e.g. Claus,Kleitman (1973) and Granot, Huberman (1981) on minimum cost spanning tree problems and games. In our setting, however, the focus is not solely on costs but to find in some sense an optimal compromise between maximizing joint revenues and minimizing joint costs.

The paper incorporates two main results. The first result is that the core of a network game, i.e. a cooperative game in coalitional form in which the value of a coalition equals the maximal net benefits of communication, is non-empty. This implies that core-allocation exists and that these allocations induce stable cooperation in the sense that no subgroup can improve their individual payoffs by establishing a communication network on their own. The proof of this result nicely combines the OR-techniques of relaxation and duality with a game theoretic technique of constructing core elements similar to the one used in Curiel, Derks, Tijs (1989) within the context of linear production situations (cf. Owen (1975)) with committee control.

The second result provides sufficient conditions on the network situation such that the corresponding network game is convex. The proof involves relations between optimal networks of various coalitions. The interest in convexity is motivated by the nice properties these games possess. For example, for convex games the core is equal to the convex hull of all marginal vectors (cf. Shapley (1971) and Ichiishi (1992)), and , as a consequence, the Shapley value is the bary centre of the core (Shapley (1971)). Moreover, the bargaining set and the core coincide, and the kernel coincides with the nucleolus (cf. Maschler, Peleg, Shapley (1972)). The proof is obtained by establishing relation between optimal networks of various coalitions.

The outline of the paper is as follows. Section 2 formalizes network situations and its associated cooperative games. Total balancedness of network games is shown in Section 3. Section 4 focuses on convexity. In general network games need not be convex. Sufficient conditions for convexity of the underlying situation are derived. Possible extensions of the model and its results, in particular with respect to directed graphs and the incorporation of public nodes are discussed in Section 5. An appendix contains the more technical proofs.

2 Network games

We will model the agents' decision problem regarding the use of a public communication network as a cooperative TU-game. A TU-game is a pair (N, v) with N representing the finite set of agents and $v : 2^N \to \mathbb{R}$ the characteristic function describing the gains of cooperation $v(S)$ for each coalition $S \subseteq N$. By assumption it holds that $v(\emptyset) = 0$. The core of a cooperative game (N, v) is the set of allocations of $v(N)$ for which no subcoalition S has an incentive to part company with the grand coalition N because it can do better on its own. Core allocations thus induce stable cooperation. The core $C(v)$ is defined as $C(v) = \{x \in \mathbb{R}^N \mid \forall_{S \subseteq N} : \sum_{i \in S} x_i \geq v(S); \sum_{i \in N} x_i = v(N)\}$. A game

$(N; v)$ is called balanced if the core is nonempty. In particular, $(N; v)$ is called totally balanced if the core of each subgame $(S; v|_S)$ is nonempty, where $v|_S(U) = v(U)$ for all $U \subseteq S$.

For defining network games, let N denote a finite set of agents. The revenues of communication between two agents i and j are denoted by $b_{\{i,j\}}$, with $b_{\{i,j\}} \geq 0$. Let $E \subseteq \{\{i,j\} | i, j \in N, i \neq j\}$ denote the set of available (communication) links. The cost of using the link $\{i,j\} \in E$ is denoted by $k_{\{i,j\}}$, with $k_{\{i,j\}} \geq 0$. Whenever we write $\{i,j\}$ it is implicitly assumed that $i \neq j$, an assumption that holds in the whole paper and which is adopted to avoid unnecessary notational inconveniences.

For communication, coalition $S \subseteq N$ has the links $E(S) \subseteq E$ at its disposal. Note that it may be possible that a coalition S is allowed to use links that involve agents outside S. So, agents i and j do not necessarily possess the ownership rights of the link $\{i,j\}$ in the sense that this link can only be used by other agents with the permission, i.e. cooperation, of agents i and j. We make the following natural assumptions:

(i) E connects all players in N,

(ii) if $S \subseteq T$, then $E(S) \subseteq E(T)$,

(iii) $E(N) = E$.

Two special networks can be viewed as two extreme situations. First, if each coalition can use all available links, i.e. $E(S) = E$ for all $S \in 2^N$, the network is called fully public. Second, if each coalition can only use links that connects players in that coalition, i.e.,
$E(S) = E \cap \{\{i,j\} \mid i, j \in S\}$ for all $S \in 2^N$, the network is called fully private.

A possible operative network for a coalition $S \subseteq N$ is represented by a subset $\mathcal{E} \subseteq E(S)$. A communication link $\{i,j\}$ is used and paid for if and only if $\{i,j\} \in \mathcal{E}$. Consequently, the total costs of the network $\mathcal{E}$ equal $\sum_{\{i,j\} \in \mathcal{E}} k_{\{i,j\}}$. Agents $i, j \in S$ can communicate with each other in the network $\mathcal{E}$ if there exists a path from agent i to agent j. By defining $C_{\{i,j\}}(\mathcal{E}) = 1$ if agents i and j can communicate with each other in $\mathcal{E}$ and $C_{\{i,j\}}(\mathcal{E}) = 0$ otherwise, the total revenues from communication in the network $\mathcal{E}$ equal $\sum_{\{i,j\} \subseteq S: C_{\{i,j\}}(\mathcal{E})=1} b_{\{i,j\}}$. Hence, the net (total) benefits equal $\sum_{\{i,j\} \subseteq S: C_{\{i,j\}}(\mathcal{E})=1} b_{\{i,j\}} - \sum_{\{i,j\} \in \mathcal{E}} k_{\{i,j\}}$. Because each coalition maximizes the benefits of cooperation, the corresponding cooperative network game $(N; v)$ is defined by

$$v(S) = \max_{\mathcal{E} \subseteq E(S)} \sum_{\{i,j\} \subseteq S: C_{\{i,j\}}(\mathcal{E})=1} b_{\{i,j\}} - \sum_{\{i,j\} \in \mathcal{E}} k_{\{i,j\}}, \qquad (1)$$

for all $S \subseteq N$.

Example 2.1 Consider the network given in Figure 2.1 with $N = \{1,2,3,4\}$ and $E = \{\{i,j\}|i,j \in N\}$. Each link $\{i,j\}$ comes with two numbers, the bold faced number represents the revenues $b_{\{i,j\}}$ of communication between agents i and j while the italic faced number represents the costs $k_{\{i,j\}}$ of the link $\{i,j\}$. So, for example, $b_{\{1,2\}} = 2$ and $k_{\{1,2\}} = 3$.

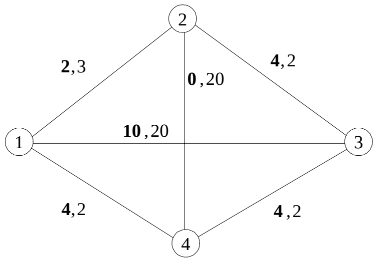

Figure 2.1: A network situation.

To illustrate the effect of the set $E(S)$ of available links on the corresponding game, define $E_1(S) = \{\{i,j\}|i,j \in S\}$ for all $S \subseteq N$. So, each coalition $S \subseteq N$ can only use links that connect agents in S, i.e., we have a fully private network. For coalition $\{1,3\}$ this means that $E_1(\{1,3\}) = \{\{1,3\}\}$. Since $b_{\{1,3\}} - k_{\{1,3\}} < 0$, coalition $\{1,3\}$ will not use the link $\{1,3\}$ so that in the corresponding game v_1 we have that $v_1(\{1,3\}) = 0$. Coalition $\{1,2,3\}$ has the links $E_1(\{1,2,3\}) = \{\{1,2\},\{1,3\},\{2,3\}\}$ at its disposal. Maximal benefits are obtained if they use the links $\{1,2\}$ and $\{2,3\}$, so $v_1(\{1,2,3\}) = b_{\{1,2\}} + b_{\{1,3\}} + b_{\{2,3\}} - k_{\{1,2\}} - k_{\{2,3\}} = 11$. In a similar way one obtains that $v_1(S) = 2$ if $S \in \{\{2,3\},\{3,4\},\{1,4\}\}$, $v_1(S) = 0$ if $S \in \{\{2,4\},\{1,2\}\}$, $v_1(\{1,2,4\}) = 2$, $v_1(\{1,3,4\}) = 14$, $v_1(\{2,3,4\} = 4$, and $v_1(\{1,2,3,4\}) = 18$. Note that in the optimal network for coalition $\{1,2,3,4\}$ the links $\{1,4\}$, $\{2,3\}$, and $\{3,4\}$ are used.

Next, define $E_2(S) = E$ for all $S \subseteq N$, i.e., a fully public network situation. Since coalition $\{1,3\}$ now can use the links $\{1,4\}$ and $\{3,4\}$, the maximal benefits that they can obtain equal $v_2(\{1,3\}) = b_{\{1,3\}} - k_{\{1,3\}} - k_{\{3,4\}} = 6$. Note that $v_2(\{1,3\}) > v_1(\{1,3\})$. In a similar way one obtains that $v_2(S) = v_1(S)$ for all $S \subseteq N \backslash \{1,3\}$.

3 Total balancedness

For Example 2.1 determining an optimal communication network is straightforward as the number of possible networks that need to be considered is relatively low. As the number of agents increases though, the number of possible networks grows exponentially, making the discrete optimization problem in (1) more complex. The following game (N, w) considers a relaxation of the optimization problem and coincides with the network game (N, v). For all $S \subseteq N$ define $w(S)$ by

$$w(S) = \max_{x_E, y_{\{i,j\}}} \sum_{\{i,j\} \subseteq N} y_{\{i,j\}} b_{\{i,j\}} - \sum_{E \subseteq \mathcal{E}} x_E \left(\sum_{\{i,j\} \in E} k_{\{i,j\}} \right) \quad (2)$$

$$\begin{aligned}
\text{s.t. } y_{\{i,j\}} - \sum_{E \subseteq \mathcal{E}} x_E C_{\{i,j\}}(E) &\leq 0 && \text{for all } \{i,j\} \subseteq N \\
y_{\{i,j\}} &\leq u_{\{i,j\}}(S) && \text{for all } \{i,j\} \subseteq N \\
x_E &\leq u_E(S) && \text{for all } E \subseteq \mathcal{E} \\
x_E &\geq 0 && \text{for all } E \subseteq \mathcal{E} \\
y_{\{i,j\}} &\geq 0 && \text{for all } \{i,j\} \subseteq N
\end{aligned}$$

where $(N, u_{\{i,j\}})$ is the unanimity game for coalition $\{i, j\}$, that is $u_{\{i,j\}}(S) = 1$ if $\{i, j\} \subseteq S$ and $u_{\{i,j\}}(S) = 0$ otherwise, and (N, u_E) is the game defined by $u_E(S) = 1$ if $E \in \mathcal{E}(S)$ and $u_E(S) = 0$ otherwise. Note that in an optimal solution it holds that $y_{\{i,j\}} = \min\{\sum_{E \subseteq \mathcal{E}(S)} x_E C_{\{i,j\}}(E); u_{\{i,j\}}(S)\}$ for all $\{i, j\} \subseteq N$. Hence, we can reduce (2) to the following nonlinear program

$$w(S) = \max \sum_{\{i,j\} \subseteq S} b_{\{i,j\}} \min\{\sum_{E \subseteq \mathcal{E}(S)} x_E C_{\{i,j\}}(E); 1\} - \sum_{E \subseteq \mathcal{E}(S)} x_E \left(\sum_{\{i,j\} \in E} k_{\{i,j\}} \right) \quad (3)$$

$$\text{s.t. } 0 \leq x_E \leq 1 \text{ for all } E \subseteq \mathcal{E}(S).$$

This game can be interpreted as a more dynamic version of the original game, in which the benefits do not only depend on whether or not communication takes place but also on the duration of the communication in an infinite horizon setting. For this, let $b_{\{i,j\}}$ denote the revenues of communication per time unit and let $k_{\{i,j\}}$ denote the operational cost per time unit of the link $\{i, j\}$. Suppose further that each network $E \subseteq \mathcal{E}$ can be maintained with a certain reliability $x_E \in [0, 1]$. The interpretation is that the network E is down $(1 - x_E)$ percent of the time due to repair. Repair is costless but takes some time during which the agents cannot communicate via the network E. To illustrate, consider the network $E = \{\{i, j\}\}$ that enables communication between agents i and j. Let x_E be the reliability of E. Then $(1 - x_E)$ percent of the time agents i and j cannot communicate because the network is down. As a result, the average revenue from communication per time period equals $x_{\{i,j\}} b_{\{i,j\}}$. Similarly, since the network is in operation for $x_{\{i,j\}}$ percent of time, the average operational cost per time period equals $x_{\{i,j\}} k_{\{i,j\}}$. More general, suppose that agents $i, j \in S$ are connected in the networks $E_1, E_2 \subseteq \mathcal{E}(S)$, which are maintained with reliability x_{E_1} and x_{E_2}, respectively. Since $(1 - x_{E_1})$ percent of the time the network E_1 is down, agents i and j can communicate with each other through the network E_1 for x_{E_1} percent of the time. Hence, communication via the network E_1 yields agents i and j an average revenue per time period of $x_{E_1} b_{\{i,j\}}$. Similarly, communication via the network E_2 yields an average revenue per time period of $x_{E_2} b_{\{i,j\}}$. Since agents i and j can not communicate more than 100 percent of the time and we consider an infinite horizon, the average total revenue per time period can be set to $\min\{x_{E_1} + x_{E_2}; 1\} b_{\{i,j\}}$. Similarly, the average operational costs per time period equal $x_{E_1} \sum_{\{s,t\} \in E_1} k_{\{s,t\}} + x_{E_2} \sum_{\{s,t\} \in E_2} k_{\{s,t\}}$. Summarizing, (3) expresses that each coalition $S \subseteq N$ wants to maximize the net average benefits (per time unit) over the reliabilities of the networks $E \subseteq \mathcal{E}(S)$ that they can use. Obviously, the maximal average benefits (per time unit) equals at least the maximal benefits a coalition can obtain in the static case, i.e. $v(S)$, because they can always choose the network that maximizes (1) with reliability 1 and all other networks with reliability 0. The following proposition states the converse is also true. The proof of this proposition can be found in the Appendix.

Proposition 3.1 For each $S \subseteq N$ it holds that $v(S) = w(S)$.

The game $(N; w)$ as defined in (2) closely resembles the formulation of linear production games with committee control as considered by Curiel, Derks, Tijs (1989). Linear production games were introduced in Owen (1975) and describe the benefits of cooperation when agents combine their individual resource bundles to produce and subsequently sell commodities. Curiel, Derks, Tijs (1989) extended this model to linear production situations with committee control, where resource bundles may be controlled by coalitions instead of individuals. They showed that linear production games with committee control have a nonempty core if the cooperative games describing the resources are simple games with nonempty cores. To illustrate the similarity, consider, for instance, the variable $y_{\{i,j\}}$. The 'resources' for $y_{\{i,j\}}$ are described by the unanimity game $(N; u_{\{i,j\}})$, that is coalition S has an amount 1 of the resource $y_{\{i,j\}}$ if $\{i,j\} \subseteq S$ and an amount zero otherwise. This means that for coalition S it holds true that $y_{\{i,j\}} \leq u_{\{i,j\}}(S)$. Note that the game $(N; u_{\{i,j\}})$ has a nonempty core. Similarly, we have that $x_E \leq u_E(S)$. So, for the reliability of the network $E \in \mathcal{E}$, the 'resources' are described by the game $(N; u_E)$ with $u_E(S) = 1$ if and only if $E \in \mathcal{E}(S)$. This game, however, is not balanced if, for example, $\mathcal{E}(S) = \mathcal{E}$ for all $S \subseteq N$. So, the game defined in (2) does not meet the balancedness conditions of Curiel, Derks, Tijs (1989). Nevertheless, the same type of techniques as in Curiel, Derks, Tijs (1989) can be used to show that the game $(N; w)$, and hence $(N; v)$, has a nonempty core.

Theorem 3.2 The network game $(N; v)$ is totally balanced.

The proof of Theorem 3.2 is given in the Appendix. It considers an optimal solution $(\lambda^*_{\{i,j\}})_{\{i,j\}\subseteq N}, (\mu^*_{\{i,j\}})_{\{i,j\}\subseteq N}, (\nu^*_E)_{E\subseteq \mathcal{E}}$ of the dual program with respect to $w(N)$:

$$
\begin{array}{rclll}
w(N) & = & \min\limits_{\lambda_{\{i,j\}},\mu_{\{i,j\}},\nu_E} & \sum\limits_{\{i,j\}\subseteq N} \lambda_{\{i,j\}} + \sum\limits_{E\subseteq \mathcal{E}} \nu_E u_E(N) & \\
& & \text{s.t.:} \quad \lambda_{\{i,j\}} + \mu_{\{i,j\}} & \geq b_{\{i,j\}}, & \text{for all } \{i,j\}\subseteq N \\
& & \sum\limits_{\{i,j\}\subseteq N} \mu_{\{i,j\}} C_{\{i,j\}}(E) & \leq \nu_E + \sum_{\{i,j\}\in E} k_{\{i,j\}} & \text{for all } E\subseteq \mathcal{E} \\
& & \lambda_{\{i,j\}} & \geq 0 & \text{for all } \{i,j\}\subseteq N \\
& & \mu_{\{i,j\}} & \geq 0 & \text{for all } \{i,j\}\subseteq N \\
& & \nu_E & \geq 0 & \text{for all } E\subseteq \mathcal{E}.
\end{array}
\qquad (4)
$$

Moreover, it shows that the allocation $x_i = \frac{1}{2}\sum_{j\in N\setminus\{i\}} \lambda^*_{\{i,j\}}$, $i \in N$, is a core-allocation for the game (N, v).

In its present formulation the dual program (4) consists of $2n(n-1) + 2^{|\mathcal{E}|}$ variables and $n(n-1) + 2^{|\mathcal{E}|}$ restrictions. It includes a variable and a restriction for each network $E \in \mathcal{E}$. Since the number of possible networks can be very large, this program is not (very) practical to solve. We can reduce the number of variables and restrictions to $\frac{1}{2}n(n-1)$ and 2^n, respectively.

A coalition $S \subseteq N$ is called connected if S is connected in the graph $(N, \mathcal{E}(S))$. For a connected coalition S, $T^*(S)$ denotes the set of edges of a minimum cost spanning tree for S in the graph $(N, \mathcal{E}(S))$.

Proposition 3.3

$$
\begin{array}{rcll}
w(N) & = & \sum\limits_{\{i,j\}\subseteq N} b_{\{i,j\}} - \max\limits_{\mu_{\{i,j\}}} \sum\limits_{\{i,j\}\subseteq N} \mu_{\{i,j\}} & \\
& \text{s.t.:} & \sum\limits_{\{i,j\}\subseteq S} \mu_{\{i,j\}} \leq \kappa(S) & \text{for all connected } S\subseteq N \\
& & \mu_{\{i,j\}} \geq 0 & \text{for all } \{i,j\}\subseteq N
\end{array}
\qquad (5)
$$

where for each connected $S \subseteq N$, $\kappa(S) := \min\{\sum_{\{i,j\}\subseteq S} b_{\{i,j\}}, \sum_{\{i,j\}\in T^*(S)} k_{\{i,j\}}\}$.

Given an optimal solution $(\mu^*_{\{i,j\}})_{\{i,j\}\subseteq N}$ of the optimization problem (5), a core-allocation now can be defined by $x_i = \frac{1}{2}\sum_{j\in N\setminus\{i\}}(b_{\{i,j\}} - \mu^*_{\{i,j\}})$ for all $i \in N$. Furthermore, notice that an equal distribution of $b_{\{i,j\}} - \mu^*_{\{i,j\}}$ is not necessary to obtain a core-allocation, any nonnegative distribution suffices.

Each optimal solution of the dual program (5) results in core-allocations for the corresponding network game by varying the nonnegative distribution of the pairwise net benefits $b_{\{i,j\}} - \mu^*_{\{i,j\}}$. However, not every core-allocation can be obtained in this way. The following example shows that the core can be much larger than the allocations that arise from optimal dual solutions.

Example 3.4 Consider the network in Figure 3.1. We assume that $E(S) = \{\{i,j\} | i,j \in S\}$ for all $S \subseteq N$, .i.e., the fully private case.

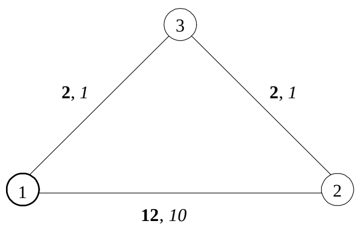

Figure 3.1: A 3-person network.

The resulting network game $(N;v)$ is given by $v(\{i\}) = 0$ for all $i \in N$, $v(\{1,2\}) = 2$, $v(\{1,3\}) = v(\{2,3\}) = 1$, and $v(\{1,2,3\}) = 14$. The core of this game is depicted in Figure 3.2. The set of optimal solutions $\{\mu^*_{\{1,2\}}, \mu^*_{\{1,3\}}, \mu^*_{\{2,3\}}\}$ of the dual program (5).

$$
\begin{aligned}
v(N) = b_{\{1,2\}} + b_{\{1,3\}} + b_{\{2,3\}} - \max_{\mu_{\{i,j\}}} \quad & \mu_{\{1,2\}} + \mu_{\{1,3\}} + \mu_{\{2,3\}} \\
\text{s.t.:} \quad & \mu_{\{1,2\}} \leq 10 \\
& \mu_{\{1,3\}} \leq 1 \\
& \mu_{\{2,3\}} \leq 1 \\
& \mu_{\{1,2\}} + \mu_{\{1,3\}} + \mu_{\{2,3\}} \leq 2 \\
& \mu_{\{i,j\}} \geq 0 \quad \text{for all } \{i,j\} \subseteq N;
\end{aligned}
$$

is given by $\mathrm{Conv}(\{(2,0,0),(1,1,0),(0,1,1)\})$. All core-allocations corresponding to optimal dual solutions are depicted in Figure 3.2.

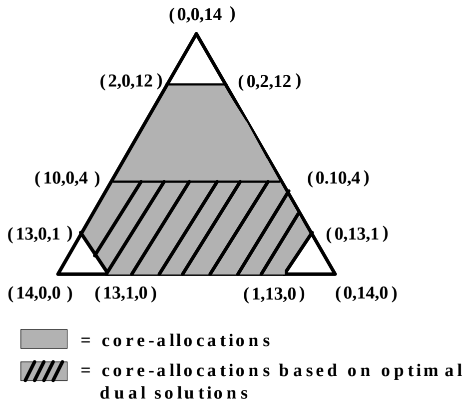

Figure 3.2: The core of a network game.

As Figure 3.2 illustrates, not all core-allocations are supported by optimal dual solutions. This 'deficiency' is caused by the fact that core-allocations based on optimal dual solutions give each pair of connected agents i and j their benefits of communication $b_{\{i,j\}}$ minus some part of the total costs of the optimal network that they have to pay. Since in our example, the optimal network costs 2, the pair $\{1,2\}$ receives at least $b_{\{1,2\}} - 2 = 10$, which is much more than they can obtain on their own, i.e. $v(\{1,2\}) = 2$. To make a cheap connection, the pair of agents 1 and 2 need the cooperation of agent 3. Agent 3, however, does not profit from the additional benefits that the pair $\{1,2\}$ makes in this way, if the allocation is based on an optimal dual solution.

4 Convexity

This section considers two special classes of network situations that yield convex network games. Both focus on network situations in which the underlying graph is a tree. As before, a public network situation is called fully public if $E(S) = E$ for all $S \subseteq N$. It is called fully private if $E(S) = E \cap \{\{i,j\} \mid i,j \in S\}$.

Before we present the convexity result, we recall the definition of a convex game. A game (N,v) is called convex if for $k \in N$ and any $S \subseteq T \subseteq N \setminus \{k\}$ it holds

$$v(T \cup \{k\}) - v(T) \geq v(S \cup \{k\}) - v(S).$$

The following example illustrates that both fully public and private network games need not be convex.

Example 4.1 Consider the network presented in Example 2.1. The fully public network game $(N;v_1)$ is not convex since

$$v_1(\{1;3;4\}) - v_1(\{1;3\}) = 14 - 6 > 18 - 11 = v_1(\{1;2;3;4\}) - v_1(\{1;2;3\}).$$

Similarly, the fully private network game $(N;v_2)$ is not convex since

$$v_2(\{1;3;4\}) - v_2(\{1;3\}) = 14 - 0 > 18 - 11 = v_2(\{1;2;3;4\}) - v_2(\{1;2;3\}).$$

For network situations in which the available communication links form a tree, the fully public and fully private case leads to a convex network games.

Theorem 4.2 For any fully public or private network situation in which E is a tree, the corresponding network game is convex.

For the proof of this theorem we refer to the Appendix. The driving lemma is the following.

Lemma 4.3 Consider a network situation where E is a tree. Let $S \subseteq T \subseteq N$. If D is an optimal network for S and F is and optimal network for T, then $D \cup F$ is also optimal for T.

In particular, Lemma 4.3 implies that within a network situation where E is a tree, any optimal network for a specific coalition S can be extended to an optimal operational network for a larger coalition containing S. Note that with respect to the network of Example 4.1 this is not the case for e.g. $S = \{1;2;3\}$ in both the fully public and private setting.

5 Concluding remarks

In network games, the structure of the revenues of communication have a bilateral additive structure; there are no synergies or positive/negative external effects of communication. Similar to Myerson (1977), one could describe the revenues of communication by a cooperative TU-game $(N;b)$ with the interpretation that $b(S)$ equals the revenues of communication for coalition $S \subseteq N$. The corresponding spanning network game $(N;v_b)$ is then defined by

$$v_b(S) = \max_{E \subseteq E(S)} \sum_{U \in C(E)} b(U \cap S) - \sum_{\{i;j\} \in E} k_{\{i;j\}} \tag{6}$$

for all $S \subseteq N$, where $C(E)$ denotes the connected components of N in the network E. The core of such a game, however, can be empty, even if the game $(N;b)$ has a nonempty core, as the following example shows.

Example 5.1 Let $N = \{1,2,3\}$, $E = \{\{i,j\} \mid i,j \in N\}$ and let $b(\{i\}) = 0$ for all $i \in N$, $b(\{1,2\}) = b(\{1,3\}) = b(\{2,3\}) = 4$, and $b(\{1,2,3\}) = 6$. Note that the core of the game (N,b) equals $\{(2,2,2)\}$. Next, let the maintenance costs of the links be equal to one, that is $k_{\{i,j\}} = 1$ for all $\{i,j\} \subseteq N$ and take $E(S) = \{\{i,j\} | i,j \in S\}$ for all $S \subseteq N$. Then the corresponding network game (N, v_b) equals $v_b(\{i\}) = 0$ for all $i \in N$, $v_b(\{1,2\}) = v_b(\{1,3\}) = v_b(\{2,3\}) = 3$, and $v_b(\{1,2,3\}) = 4$. The core of this game is empty.

Note that Example 5.1 features negative external effects of communication since $b(\{1,2,3\}) < b(\{1,2\}) + b(\{1,3\}) + b(\{2,3\})$. The following example shows that also in the positive externality case with $C(b) \neq \emptyset$, the corresponding network game may not be balanced.

Example 5.2 Let $N = \{1,2,3,4\}$, $E = \{\{i,j\} \mid i,j \in N\}$ and let $b(\{i\}) = 0$ for all $i \in N$, $b(S) = 2$ if $|S| = 2$, $b(S) = 9$ if $|S| = 3$, and $b(\{1,2,3,4\}) = 13$. Note that the game (N,b) satisfies $b(S) \geq \sum_{\{i,j\} \subseteq S} b(\{i,j\})$ for all $S \subseteq N$. Next, let the maintenance costs of each link be $k_{\{i,j\}} = 4$, and take $E(S) = \{\{i,j\} | i,j \in S\}$ for all $S \subseteq N$. Then the corresponding network game (N, v_b) equals $v_b(S) = 0$ if $|S| \leq 2$, $v_b(S) = 1$ if $|S| \geq 3$. The core of this game is empty while the core of the game (N,b) is nonempty.

From the examples above it follows that the structure of the revenues $b(S), S \subseteq N$, requires more than just balancedness to induce stable cooperation in network games. A sufficient condition is additivity, that is $b(S) = \sum_{\{i,j\} \subseteq S} b(\{i,j\})$.

A second extension concerns the characteristics of communication links. In the present model communication links are undirected. Dependent on the underlying situation, directed links may be more appropriate to consider, for instance when the links in the network represent railroad or motor traffic. Our results on network games extend straightforwardly to directed networks.

The final extension introduces public nodes. To illustrate, consider the network presented in Figure 5.1 with three agents and one public node. In the absence of the public node, the minimum cost spanning tree for the agents 1, 2, and 3 costs $2\sqrt{2}$. If, however, they can also use the links that connect to the public node, the minimum cost spanning tree is less expensive at $2\frac{2}{3}$. Public nodes have a practical meaning in network games as they can represent, for instance, switchboards in a telephone network or switches in railroads.

For the inclusion of public nodes in our model, let M with $N \cap M = \emptyset$ denote the finite set of public nodes and define $E \subseteq \{\{i,j\} | i,j \in N \cup M\}$ as the set of available links. In particular, let $E(S) \subseteq E$ be the available links for coalition $S \subseteq N$ and make the same basic assumptions as in Section 2. Notice that we do not assume that each coalition can use all existing links with public

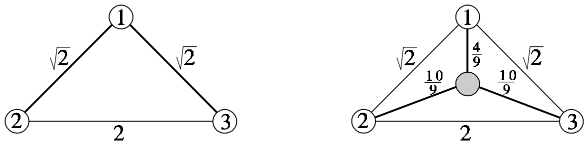

Figure 5.1: Public nodes in a network.

nodes. In that sense, the term public node may be somewhat misleading. The extended network game $(N; v_p)$ is now defined by by

$$v_p(S) = \max_{E \subseteq E(S)} \sum_{\{i,j\} \subseteq S : C_{\{i,j\}}(E)=1} b_{\{i,j\}} - \sum_{\{i,j\} \in E} k_{\{i,j\}} \tag{7}$$

for all $S \subseteq N$. Note that the agents do not obtain any benefits from connections with public nodes. Only connections with other agents might be profitable. It can be shown that the corresponding network game is totally balanced.

6 Appendix

Proof of Proposition 3.1: The proof consists of three steps. In the first step, we reformulate the optimization problem (2) based on some properties of the optimal solution. In the second step, we show that there exists an optimal solution of (2) in which the reliability x_{E^*} of a given optimal network $E^* \subseteq E(S)$ with respect to (1) is equal to one. Hence, we may assume that $x_{E^*} = 1$.In the third step we then show that there exists an optimal solution of (2) in which $x_E = 0$ for all other networks $E \subseteq E(S)$ with $E \neq E^*$, so that $w(S) = v(S)$.

Let $S \subseteq N$. Consider the linear program as formulated in (2). Since $x_E = 0$ for all $E \not\subseteq E(S)$ we can restrict our attention to $E \subseteq E(S)$. For ease of notation, let $E(S) = \{E_1; E_2; \ldots; E_r\}$ such that $\sum_{\{i,j\} \subseteq S : C_{\{i,j\}}(E_p)=1} b_{\{i,j\}} - \sum_{\{i,j\} \in E_p} k_{\{i,j\}} \geq \sum_{\{i,j\} \subseteq S : C_{\{i,j\}}(E_q)=1} b_{\{i,j\}} - \sum_{\{i,j\} \in E_q} k_{\{i,j\}}$ if $p < q$. So, E_1 is an optimal network with respect to (1). Further, define $K_p = \sum_{\{i,j\} \in E_p} k_{\{i,j\}}$ for each $p \in \{1; 2; \ldots; r\}$.

First, note that $y_{\{i,j\}} \leq \min \left\{ \sum_{p=1}^{r} x_p C_{\{i,j\}}(E_p); u_{\{i,j\}}(S) \right\}$ for all $\{i; j\} \subseteq N$, where x_p is a short notation for x_{E_p}. Furthermore, since each $y_{\{i,j\}}$ has a nonnegative contribution to the objective function, it follows that in an optimal solution $y_{\{i,j\}} = \min \left\{ \sum_{p=1}^{r} x_p C_{\{i,j\}}(E_p); u_{\{i,j\}}(S) \right\}$ for all

$\{i,j\} \subseteq N$. Hence,

$$
\begin{aligned}
w(S) \;=\; & \max_{x_p; y_{\{i,j\}}} \sum_{\{i,j\}\subseteq N} y_{\{i,j\}} b_{\{i,j\}} - \sum_{p=1}^{r} x_p K_p \qquad (8)\\
& \text{s.t.} \quad y_{\{i,j\}} - \sum_{p=1}^{r} x_p C_{\{i,j\}}(E_p) \le 0 \quad \text{for all } \{i,j\} \subseteq N\\
& \qquad\qquad y_{\{i,j\}} \le u_{\{i,j\}}(S); \quad \text{for all } \{i,j\} \subseteq N\\
& \qquad\qquad x_p \le 1 \quad \text{for all } p \in \{1,2,\dots,r\}\\
& \qquad\qquad x_p \ge 0 \quad \text{for all } p \in \{1,2,\dots,r\}\\
& \qquad\qquad y_{\{i,j\}} \ge 0 \quad \text{for all } \{i,j\} \subseteq N\\
=\; & \max_{0 \le x_p \le 1} \sum_{\{i,j\}\subseteq N} \min\left\{ \sum_{p=1}^{r} x_p C_{\{i,j\}}(E_p); u_{\{i,j\}}(S) \right\} b_{\{i,j\}} - \sum_{p=1}^{r} x_p K_p\\
=\; & \max_{0 \le x_p \le 1} \sum_{\{i,j\}\subseteq S} \min\left\{ \sum_{p=1}^{r} x_p C_{\{i,j\}}(E_p); 1 \right\} b_{\{i,j\}} - \sum_{p=1}^{r} x_p K_p
\end{aligned}
$$

Define the function $R : \mathbb{R}^r_+ \to \mathbb{R}$ by

$$
R(x) = \sum_{\{i,j\}\subseteq S} \min\left\{ \sum_{p=1}^{r} x_p C_{\{i,j\}}(E_p); 1 \right\} b_{\{i,j\}} - \sum_{p=1}^{r} x_p K_p; \qquad (9)
$$

for each $x \in \mathbb{R}^r_+$, so that $w(S) = \max\{R(x) | 0 \le x_p \le 1; \forall_{p \in \{1,2,\dots,r\}}\}$. Furthermore, notice that $R(e^p)$, where $e^p \in \mathbb{R}^r_+$ is defined by $e^p_q = 1$ if $q = p$ and $e^p_q = 0$ otherwise, equals the net total benefits of the network E_p, i.e. revenues minus costs. So, by definition we have that $R(e^1) \ge R(e^2) \ge \dots \ge R(e^r)$ and $v(S) = R(e^1)$. Hence, it is sufficient to show that $w(S) = R(e^1)$.

Next, we show that it is optimal to take $x_1 = 1$. Let $x \in \mathbb{R}^r_+$ be such that $x_1 < 1$. We distinguish two cases: $\sum_{p=1}^{r} x_p < 1$ and $\sum_{p=1}^{r} x_p \ge 1$.

If $\sum_{p=1}^{r} x_p < 1$, then

$$
\begin{aligned}
R(x) &= \sum_{\{i,j\}\subseteq S} \min\left\{ \sum_{p=1}^{r} x_p C_{\{i,j\}}(E_p); 1 \right\} b_{\{i,j\}} - \sum_{p=1}^{r} x_p K_p\\
&= \sum_{\{i,j\}\subseteq S} \left(\sum_{p=1}^{r} x_p C_{\{i,j\}}(E_p) \right) b_{\{i,j\}} - \sum_{p=1}^{r} x_p K_p\\
&= \sum_{p=1}^{r} \sum_{\{i,j\}\subseteq S : C_{\{i,j\}}(E_p) = 1} x_p b_{\{i,j\}} - \sum_{p=1}^{r} x_p K_p\\
&= \sum_{p=1}^{r} x_p \left(\sum_{\{i,j\}\subseteq S : C_{\{i,j\}}(E_p) = 1} x_p b_{\{i,j\}} - K_p \right)\\
&= \sum_{p=1}^{r} x_p R(e^p)\\
&\le \sum_{p=1}^{r} R(e^1) \le R(e^1):
\end{aligned}
$$

Hence, each vector x with $\sum_{p=1}^r x_p < 1$ yields lower benefits than the vector e^1.
If $\sum_{p=1}^r \geq 1$ and $x_1 < 1$, then take $y \in \mathbb{R}^r_+$ such that $y_1 = 1$ and $y_p \in [0, x_p]$ such that $\sum_{p=2}^r (x_p - y_p) = 1 - x_1$. So, the increase from x_1 to $y_1 = 1$ is compensated by decreasing x_p to y_p for all $p > 1$. Since

$$R(y) = \sum_{\{i,j\}\subseteq S} \min\left\{\sum_{p=1}^r y_p C_{\{i,j\}}(E_p), 1\right\} b_{\{i,j\}} - \sum_{p=1}^r y_p K_p$$

and

$$R(x) = \sum_{\{i,j\}\subseteq S} \min\left\{\sum_{p=1}^r x_p C_{\{i,j\}}(E_p), 1\right\} b_{\{i,j\}} - \sum_{p=1}^r x_p K_p$$

we have that

$$\begin{aligned}
R(y) - R(x) = {} & \\
= {} & \sum_{\{i,j\}\subseteq S} \left(\min\{\sum_{p=1}^r y_p C_{\{i,j\}}(E_p), 1\} - \min\{\sum_{p=1}^r x_p C_{\{i,j\}}(E_p), 1\}\right) b_{\{i,j\}} \\
& - \sum_{p=1}^r (y_p - x_p) K_p \\
= {} & \sum_{\{i,j\}\subseteq S : C_{\{i,j\}}(E_1)=1} \left(\min\{\sum_{p=1}^r y_p C_{\{i,j\}}(E_p), 1\} - \min\{\sum_{p=1}^r x_p C_{\{i,j\}}(E_p), 1\}\right) b_{\{i,j\}} \\
& + \sum_{\{i,j\}\subseteq S : C_{\{i,j\}}(E_1)=0} \left(\min\{\sum_{p=1}^r y_p C_{\{i,j\}}(E_p), 1\} - \min\{\sum_{p=1}^r x_p C_{\{i,j\}}(E_p), 1\}\right) b_{\{i,j\}} \\
& - (y_1 - x_1) K_1 - \sum_{p=2}^r (y_p - x_p) K_p \\
= {} & \sum_{\{i,j\}\subseteq S : C_{\{i,j\}}(E_1)=1} \left(1 - \min\{\sum_{p=1}^r x_p C_{\{i,j\}}(E_p), 1\}\right) b_{\{i,j\}} \\
& + \sum_{\{i,j\}\subseteq S : C_{\{i,j\}}(E_1)=0} \left(\min\{\sum_{p=2}^r y_p C_{\{i,j\}}(E_p), 1\} - \min\{\sum_{p=2}^r x_p C_{\{i,j\}}(E_p), 1\}\right) b_{\{i,j\}} \\
& - (1 - x_1) K_1 - \sum_{p=2}^r (y_p - x_p) K_p \\
= {} & \sum_{\{i,j\}\subseteq S : C_{\{i,j\}}(E_1)=1} (1 - x_1) b_{\{i,j\}} + \sum_{\{i,j\}\subseteq S : C_{\{i,j\}}(E_1)=1} \left(x_1 - \min\{\sum_{p=1}^r x_p C_{\{i,j\}}(E_p), 1\}\right) b_{\{i,j\}} \\
& + \sum_{\{i,j\}\subseteq S : C_{\{i,j\}}(E_1)=0} \left(\min\{\sum_{p=2}^r y_p C_{\{i,j\}}(E_p), 1\} - \min\{\sum_{p=2}^r x_p C_{\{i,j\}}(E_p), 1\}\right) b_{\{i,j\}} \\
& - (1 - x_1) K_1 - \sum_{p=2}^r (y_p - x_p) K_p
\end{aligned}$$

$$
\begin{aligned}
&= (1 - x_1)R(e^1) + \sum_{\{i,j\}\subseteq S : C_{\{i,j\}}(E_1)=1} \left(x_1 - \min\left\{ \sum_{p=1}^{r} x_p C_{\{i,j\}}(E_p); 1 \right\} \right) b_{\{i,j\}} \\
&\quad + \sum_{\{i,j\}\subseteq S : C_{\{i,j\}}(E_1)=0} \left(\min\left\{ \sum_{p=2}^{r} y_p C_{\{i,j\}}(E_p); 1 \right\} - \min\left\{ \sum_{p=2}^{r} x_p C_{\{i,j\}}(E_p); 1 \right\} \right) b_{\{i,j\}} \\
&\quad - \sum_{p=2}^{r} (y_p - x_p) K_p \\
&\geq (1 - x_1)R(e^1) + \sum_{\{i,j\}\subseteq S : C_{\{i,j\}}(E_1)=1} (x_1 - 1) b_{\{i,j\}} \\
&\quad + \sum_{\{i,j\}\subseteq S : C_{\{i,j\}}(E_1)=0} \left(\sum_{p=2}^{r} y_p C_{\{i,j\}}(E_p) - \sum_{p=2}^{r} x_p C_{\{i,j\}}(E_p) \right) b_{\{i,j\}} \\
&\quad - \sum_{p=2}^{r} (y_p - x_p) K_p \\
&= (1 - x_1)R(e^1) + \sum_{\{i,j\}\subseteq S : C_{\{i,j\}}(E_1)=1} (x_1 - 1) b_{\{i,j\}} \\
&\quad + \sum_{\{i,j\}\subseteq S : C_{\{i,j\}}(E_1)=0} \left(\sum_{p=2}^{r} (y_p - x_p) C_{\{i,j\}}(E_p) \right) b_{\{i,j\}} - \sum_{p=2}^{r} (y_p - x_p) K_p \\
&= (1 - x_1)R(e^1) + \sum_{\{i,j\}\subseteq S : C_{\{i,j\}}(E_1)=1} \sum_{p=2}^{r} (y_p - x_p) b_{\{i,j\}} \\
&\quad + \sum_{\{i,j\}\subseteq S : C_{\{i,j\}}(E_1)=0} \left(\sum_{p=2}^{r} (y_p - x_p) C_{\{i,j\}}(E_p) \right) b_{\{i,j\}} - \sum_{p=2}^{r} (y_p - x_p) K_p \\
&= (1 - x_1)R(e^1) + \sum_{\{i,j\}\subseteq S} \left(\sum_{p=2}^{r} (y_p - x_p) C_{\{i,j\}}(E_p) \right) b_{\{i,j\}} - \sum_{p=2}^{r} (y_p - x_p) K_p \\
&= (1 - x_1)R(e^1) + \sum_{p=2}^{r} \sum_{\{i,j\}\subseteq S : C_{\{i,j\}}(E_p)=1} (y_p - x_p) b_{\{i,j\}} - \sum_{p=2}^{r} (y_p - x_p) K_p \\
&= (1 - x_1)R(e^1) + \sum_{p=2}^{r} (y_p - x_p) \left(\sum_{\{i,j\}\subseteq S : C_{\{i,j\}}(E_p)=1} b_{\{i,j\}} - K_p \right) \\
&= (1 - x_1)R(e^1) + \sum_{p=2}^{r} (y_p - x_p) R(e^p) \\
&\geq (1 - x_1)R(e^1) + \sum_{p=2}^{r} (y_p - x_p) R(e^1) \\
&= 0;
\end{aligned}
$$

where the first inequality follows from $a - \min\{b; 1\} \geq a - 1$ for all $a; b \in \mathbb{R}$ and $\min\{a; 1\} - \min\{b; 1\} \geq a - b$ if $a \leq b$; the subsequent equality follows from $\sum_{p=2}^{r} y_p - x_p = x_1 - 1$ and the last inequality follows from $y_p - x_p \leq 0$ and $R(e^1) \geq R(e^p)$ for all $p \in \{2; 3; \ldots; r\}$.

So, there exists an optimal solution for which $x_1 = 1$. What remains to show is that $R(e^1) \geq$

$R(x)$ for all $x \in \mathbb{R}^r_+$ with $x_1 = 1$. Therefore, let x be such a vector. Then

$$\begin{aligned}
R(x) &= \sum_{\{i,j\}\subseteq S} \min\Big\{\sum_{p=1}^{r} x_p C_{\{i,j\}}(E_p), 1\Big\} b_{\{i,j\}} - \sum_{p=1}^{r} x_p K_p \\
&= \sum_{\{i,j\}\subseteq S: C_{\{i,j\}}(E_1)=1} \min\Big\{\sum_{p=1}^{r} x_p C_{\{i,j\}}(E_p), 1\Big\} b_{\{i,j\}} - x_1 K_1 \\
&\quad + \sum_{\{i,j\}\subseteq S: C_{\{i,j\}}(E_1)=0} \min\Big\{\sum_{p=1}^{r} x_p C_{\{i,j\}}(E_p), 1\Big\} b_{\{i,j\}} - \sum_{p=2}^{r} x_p K_p \\
&= \sum_{\{i,j\}\subseteq S: C_{\{i,j\}}(E_1)=1} b_{\{i,j\}} - K_1 \\
&\quad + \sum_{\{i,j\}\subseteq S: C_{\{i,j\}}(E_1)=0} \min\Big\{\sum_{p=1}^{r} x_p C_{\{i,j\}}(E_p), 1\Big\} b_{\{i,j\}} - \sum_{p=2}^{r} x_p K_p \\
&= R(e^1) + \sum_{\{i,j\}\subseteq S: C_{\{i,j\}}(E_1)=0} \min\Big\{\sum_{p=1}^{r} x_p C_{\{i,j\}}(E_p), 1\Big\} b_{\{i,j\}} - \sum_{p=2}^{r} x_p K_p
\end{aligned}$$

so that

$$\begin{aligned}
R(e^1) - R(x) &= -\sum_{\{i,j\}\subseteq S: C_{\{i,j\}}(E_1)=0} \min\Big\{\sum_{p=1}^{r} x_p C_{\{i,j\}}(E_p), 1\Big\} b_{\{i,j\}} + \sum_{p=2}^{r} x_p K_p \\
&\geq -\sum_{\{i,j\}\subseteq S: C_{\{i,j\}}(E_1)=0} \sum_{p=1}^{r} x_p C_{\{i,j\}}(E_p) b_{\{i,j\}} + \sum_{p=2}^{r} x_p K_p \\
&= -\sum_{p=1}^{r} \sum_{\{i,j\}\subseteq S: C_{\{i,j\}}(E_1)=0 \text{ and } C_{\{i,j\}}(E_p)=1} x_p b_{\{i,j\}} + \sum_{p=2}^{r} x_p K_p \\
&= -\sum_{p=2}^{r} x_p \left(\sum_{\{i,j\}\subseteq S: C_{\{i,j\}}(E_1)=0 \text{ and } C_{\{i,j\}}(E_p)=1} b_{\{i,j\}} - K_p \right) \\
&\geq 0,
\end{aligned}$$

where the first inequality follows from the fact that $\min\{a, b\} \leq a$ and the last equality follows from $\{\{i,j\} \subseteq S : C_{\{i,j\}}(E_1) = 0 \text{ and } C_{\{i,j\}}(E_p) = 1\} = \emptyset$ for $p = 1$. Regarding the last inequality, suppose that $\sum_{\{i,j\}\subseteq S: C_{\{i,j\}}(E_1)=0 \text{ and } C_{\{i,j\}}(E_p)=1} b_{\{i,j\}} - K_p > 0$ for some $p \in \{2, 3, \ldots, r\}$. Consider the network $E = E_1 \cup E_p$. Then

$$\begin{aligned}
&\sum_{\{i,j\}\subseteq S: C_{\{i,j\}}(E)=1} b_{\{i,j\}} - \sum_{\{i,j\}\in E} k_{\{i,j\}} = \\
&\quad \geq \sum_{\{i,j\}\subseteq S: C_{\{i,j\}}(E_1)=1 \text{ or } C_{\{i,j\}}(E_p)=1} b_{\{i,j\}} - \sum_{\{i,j\}\in E} k_{\{i,j\}} \\
&\quad = \sum_{\{i,j\}\subseteq S: C_{\{i,j\}}(E_1)=1} b_{\{i,j\}} + \sum_{\{i,j\}\subseteq S: C_{\{i,j\}}(E_1)=0 \text{ and } C_{\{i,j\}}(E_p)=1} b_{\{i,j\}} - \sum_{\{i,j\}\in E} k_{\{i,j\}} \\
&\quad = \sum_{\{i,j\}\subseteq S: C_{\{i,j\}}(E_1)=1} b_{\{i,j\}} - \sum_{\{i,j\}\in E_1} k_{\{i,j\}}
\end{aligned}$$

$$
\begin{aligned}
& + \sum_{\{i,j\}\subseteq S:\, C_{\{i,j\}}(E_1)=0 \text{ and } C_{\{i,j\}}(E_p)=1} b_{\{i,j\}} - \sum_{\{i,j\}\in E_p} k_{\{i,j\}} \\
= \;& R(e^1) + \sum_{\{i,j\}\subseteq S:\, C_{\{i,j\}}(E_1)=0 \text{ and } C_{\{i,j\}}(E_p)=1} b_{\{i,j\}} - \sum_{\{i,j\}\in E_p} k_{\{i,j\}} \\
> \;& R(e^1);
\end{aligned}
$$

which contradicts the optimality of E_1. Hence, $\sum_{\{i,j\}\subseteq S:\, C_{\{i,j\}}(E_1)=0 \text{ and } C_{\{i,j\}}(E_p)=1} b_{\{i,j\}} - K_p \leq 0$. □

Proof of Theorem 3.2: From Proposition 3.1 we know that it is sufficient to prove that the game (N, w) is balanced. Recall that

$$
\begin{aligned}
w(N) = \;& \max_{x_E, y_{\{i,j\}}} \sum_{\{i,j\}\subseteq N} y_{\{i,j\}} b_{\{i,j\}} - \sum_{E\subseteq \mathcal{E}} x_E \left(\sum_{\{i,j\}\in E} k_{\{i,j\}} \right) \\
\text{s.t.: } & y_{\{i,j\}} \leq u_{\{i,j\}}(N); && \text{for all } \{i,j\}\subseteq N \\
& y_{\{i,j\}} - \sum_{E\subseteq \mathcal{E}} x_E C_{\{i,j\}}(E) \leq 0 && \text{for all } \{i,j\}\subseteq N \\
& x_E \leq u_E(N) && \text{for all } E\subseteq \mathcal{E} \\
& x_E \geq 0 && \text{for all } E\subseteq \mathcal{E} \\
& y_{\{i,j\}} \geq 0 && \text{for all } \{i,j\}\subseteq N
\end{aligned}
$$

From duality theory we know that

$$
\begin{aligned}
w(N) = \;& \min_{\lambda_{\{i,j\}}, \mu_{\{i,j\}}, \nu_E} \sum_{\{i,j\}\subseteq N} \lambda_{\{i,j\}} + \sum_{E\subseteq \mathcal{E}} \nu_E u_E(N) \\
\text{s.t.: } & \lambda_{\{i,j\}} + \mu_{\{i,j\}} \geq b_{\{i,j\}}; && \text{for all } \{i,j\}\subseteq N \\
& \sum_{\{i,j\}\subseteq N} \mu_{\{i,j\}} C_{\{i,j\}}(E) \leq \nu_E + \sum_{\{i,j\}\in E} k_{\{i,j\}} && \text{for all } E\subseteq \mathcal{E} \\
& \lambda_{\{i,j\}} \geq 0 && \text{for all } \{i,j\}\subseteq N \\
& \mu_{\{i,j\}} \geq 0 && \text{for all } \{i,j\}\subseteq N \\
& \nu_E \geq 0 && \text{for all } E\subseteq \mathcal{E}.
\end{aligned}
\tag{10}
$$

Let $\lambda^*_{\{i,j\}}, \mu^*_{\{i,j\}}, \nu^*_E$ be an optimal solution of (10) such that $\nu^*_{\bar{E}} > 0$ for some $\bar{E} \subseteq \mathcal{E}(N)$. We will show that there exists an optimal solution for which $\nu_{\bar{E}} = 0$. The optimality of $\lambda^*_{\{i,j\}}, \mu^*_{\{i,j\}}, \nu^*_E$ implies that $\sum_{\{i,j\}\subseteq N} \mu^*_{\{i,j\}} C_{\{i,j\}}(\bar{E}) = \nu^*_{\bar{E}} + \sum_{\{i,j\}\in \bar{E}} k_{\{i,j\}}$. Since $k_{\{i,j\}} \geq 0$ for all $\{i,j\}\subseteq N$, we can take $\Delta\mu_{\{i,j\}} \in [0, \mu^*_{\{i,j\}}]$ such that $\sum_{\{i,j\}\subseteq N: C_{\{i,j\}}(\bar{E})=1} \Delta\mu_{\{i,j\}} = \nu^*_{\bar{E}}$ and $\Delta\mu_{\{i,j\}} = 0$ for all $\{i,j\}\subseteq N$ with $C_{\{i,j\}}(\bar{E}) = 0$. For each $\{i,j\}\subseteq N$, define $\mu_{\{i,j\}} = \mu^*_{\{i,j\}} - \Delta\mu_{\{i,j\}}$ and $\lambda_{\{i,j\}} = \max\{b_{\{i,j\}} - \mu_{\{i,j\}}, 0\}$, and for each $E \subseteq \mathcal{E}$, define $\nu_{\bar{E}} = 0$ and $\nu_E = \nu^*_E$ for all $E\subseteq \mathcal{E}$ such that $E \neq \bar{E}$. Note that $\lambda_{\{i,j\}}, \mu_{\{i,j\}}, \nu_E$ is a feasible solution for the dual program because $\mu_{\{i,j\}} \leq \mu^*_{\{i,j\}}$ for all $\{i,j\}\subseteq N$. Furthermore, note that $\Delta\lambda_{\{i,j\}} = \lambda^*_{\{i,j\}} - \lambda_{\{i,j\}}$

$= \max\{b_{\{i,j\}} - \mu^*_{\{i,j\}}, 0\} - \max\{b_{\{i,j\}} - \mu_{\{i,j\}}, 0\} \le \Delta\mu_{\{i,j\}}$ for all $\{i,j\} \subseteq N$. Now, it holds that

$$\begin{aligned}
&\sum_{\{i,j\}\subseteq N} \lambda_{\{i,j\}} + \sum_{E\subseteq\mathcal{E}} \nu_E u_E(N) \\
&= \sum_{\{i,j\}\subseteq N} \lambda^*_{\{i,j\}} + \sum_{\{i,j\}\subseteq N} \Delta\lambda_{\{i,j\}} + \sum_{E\subseteq\mathcal{E}: E\neq\tilde{E}} \nu_E u_E(N) \\
&\le \sum_{\{i,j\}\subseteq N} \lambda^*_{\{i,j\}} + \sum_{\{i,j\}\subseteq N} \Delta\mu_{\{i,j\}} + \sum_{E\subseteq\mathcal{E}: E\neq\tilde{E}} \nu^*_E u_E(N) \\
&= \sum_{\{i,j\}\subseteq N} \lambda^*_{\{i,j\}} + \nu^*_{\tilde{E}} + \sum_{E\subseteq\mathcal{E}: E\neq\tilde{E}} \nu^*_E u_E(N) \\
&= \sum_{\{i,j\}\subseteq N} \lambda^*_{\{i,j\}} + \sum_{E\subseteq\mathcal{E}} \nu^*_E u_E(N);
\end{aligned}$$

so that $\lambda_{\{i,j\}}, \mu_{\{i,j\}}, \nu_{\{i,j\}}$ is also an optimal solution. Hence, we may assume that $\nu^*_E = 0$ for all $E \subseteq E(N)$.

Let $\lambda^*_{\{i,j\}}, \mu^*_{\{i,j\}}, \nu^*_E$ be an optimal solution with $\nu^*_E = 0$ for all $E \subseteq E(N)$. Define the allocation $z \in \mathbb{R}^N$ by $z_i = \sum_{j\in N} \frac{1}{2}\lambda^*_{\{i,j\}}$ for all $i \in N$. We will show that z is a core-allocation for the game $(N; w)$. Take $S \subseteq N$. Duality theory implies that

$$\begin{aligned}
w(S) \;=\; \min_{\lambda_{\{i,j\}},\mu_{\{i,j\}},\nu_E} \quad & \sum_{\{i,j\}\subseteq N} \lambda_{\{i,j\}} u_{\{i,j\}}(S) + \sum_{E\subseteq\mathcal{E}} \nu_E u_E(S) \\
\text{s.t.:} \quad & \lambda_{\{i,j\}} + \mu_{\{i,j\}} \ge b_{\{i,j\}}, && \text{for all } \{i,j\}\subseteq N \\
& \sum_{\{i,j\}\subseteq N} \mu_{\{i,j\}} C_{\{i,j\}}(E) \le \nu_E + \sum_{\{i,j\}\in E} k_{\{i,j\}} && \text{for all } E \subseteq \mathcal{E} \\
& \lambda_{\{i,j\}} \ge 0 && \text{for all } \{i,j\}\subseteq N \\
& \mu_{\{i,j\}} \ge 0 && \text{for all } \{i,j\}\subseteq N \\
& \nu_E \ge 0 && \text{for all } E \subseteq \mathcal{E}.
\end{aligned} \tag{11}$$

Since $\lambda^*_{\{i,j\}}, \mu^*_{\{i,j\}}, \nu^*_E$ is a feasible solution of this optimization program, we have that

$$\begin{aligned}
w(S) &\le \sum_{\{i,j\}\subseteq N} \lambda^*_{\{i,j\}} u_{\{i,j\}}(S) + \sum_{E\subseteq\mathcal{E}} \nu^*_E u_E(S) \\
&= \sum_{\{i,j\}\subseteq S} \lambda^*_{\{i,j\}} \\
&= \sum_{i\in S}\sum_{j\in S} \frac{1}{2}\lambda^*_{\{i,j\}} \\
&\le \sum_{i\in S}\sum_{j\in N} \frac{1}{2}\lambda^*_{\{i,j\}} \\
&= \sum_{i\in S} z_i;
\end{aligned}$$

where the first equality follows from $u_E(S) = 0$ if $E \not\subseteq E(S)$ and $\nu^*_E = 0$ if $E \subseteq E(S) \subseteq E(N)$. Hence, z is a core-allocation of the game $(N; w)$. By using the same argument (and it can be used by the monotonicity condition (ii) in Section 2 one can show that the game is totally balanced if

$\mathcal{E}(U) \subseteq \mathcal{E}(S)$ for all $U, S \subseteq N$ with $U \subseteq S$. $\Box$

Proof of Proposition 3.3: Consider the dual of $w(N)$ as formulated in (4). From the proof of Theorem 3.2 we know that $\gamma^*_E = 0$ for all $E \in \mathcal{E}(N)$. Hence, it holds true that

$$
\begin{aligned}
w(N) \quad = \quad & \min_{\lambda_{\{i,j\}}, \mu_{\{i,j\}}} \sum_{\{i,j\} \subseteq N} \lambda_{\{i,j\}} \\
\text{s.t.:} \quad & \lambda_{\{i,j\}} + \mu_{\{i,j\}} \geq b_{\{i,j\}}, && \text{for all } \{i,j\} \subseteq N \\
& \sum_{\{i,j\} \subseteq N} \mu_{\{i,j\}} C_{\{i,j\}}(E) \leq \textstyle\sum_{\{i,j\} \in E} k_{\{i,j\}} && \text{for all } E \subseteq \mathcal{E}(N) \\
& \lambda_{\{i,j\}} \geq 0 && \text{for all } \{i,j\} \subseteq N \\
& \mu_{\{i,j\}} \geq 0 && \text{for all } \{i,j\} \subseteq N.
\end{aligned}
$$

Next, we show that

$$
\begin{aligned}
\{\{\mu_{\{i,j\}}\}_{\{i,j\} \subseteq N} \mid \forall_{E \subseteq \mathcal{E}(N)} : \sum_{\{i,j\} \subseteq N} \mu_{\{i,j\}} C_{\{i,j\}}(E) \leq \sum_{\{i,j\} \in E} k_{\{i,j\}}\} = \qquad (12) \\
\{\{\mu_{\{i,j\}}\}_{\{i,j\} \subseteq N} \mid \forall_{S \subseteq N; S \text{ connected}} : \sum_{\{i,j\} \subseteq S} \mu_{\{i,j\}} \leq \sum_{\{i,j\} \in T^*(S)} k_{\{i,j\}}\}.
\end{aligned}
$$

The $\subseteq$-part in (12) follows from the fact that $T^*(S) \in \mathcal{E}(N)$ for all $S \subseteq N$. To see the $\supseteq$-part, take $E \subseteq \mathcal{E}(N)$ and let $\mathcal{C}(E) = \{U_1, U_2, \ldots, U_m\}$ be the maximally connected components of N with respect to the network E. This means that the agents in U_p are connected to each other while agents in U_p are not connected to agents in U_q, $p \neq q$. Then

$$
\begin{aligned}
\sum_{\{i,j\} \subseteq N} \mu_{\{i,j\}} C_{\{i,j\}}(E) &= \sum_{p=1}^{m} \sum_{\{i,j\} \subseteq U_p} \mu_{\{i,j\}} C_{\{i,j\}}(E) = \sum_{p=1}^{m} \sum_{\{i,j\} \subseteq U_p} \mu_{\{i,j\}} C_{\{i,j\}}(E|_{U_p}) \\
&= \sum_{p=1}^{m} \sum_{\{i,j\} \subseteq U_p} \mu_{\{i,j\}} \leq \sum_{p=1}^{m} \sum_{\{i,j\} \in T^*(U_p)} k_{\{i,j\}} \\
&\leq \sum_{p=1}^{m} \sum_{\{i,j\} \in E|_{U_p}} k_{\{i,j\}} = \sum_{\{i,j\} \in E} k_{\{i,j\}},
\end{aligned}
$$

where the second inequality follows from the fact that $T^*(U_p)$ is a minimum cost spanning tree for U_p.

Having (12), the dual can now be reduced to

$$
\begin{aligned}
w(N) \quad = \quad & \min_{\lambda_{\{i,j\}}, \mu_{\{i,j\}}} \sum_{\{i,j\} \subseteq N} \lambda_{\{i,j\}} \\
\text{s.t.:} \quad & \lambda_{\{i,j\}} + \mu_{\{i,j\}} \geq b_{\{i,j\}}, && \text{for all } \{i,j\} \subseteq N \\
& \sum_{\{i,j\} \subseteq S} \mu_{\{i,j\}} \leq \textstyle\sum_{\{i,j\} \in T^*(S)} k_{\{i,j\}} && \text{for all } S \subseteq N; S \text{ connected} \\
& \lambda_{\{i,j\}} \geq 0 && \text{for all } \{i,j\} \subseteq N \\
& \mu_{\{i,j\}} \geq 0 && \text{for all } \{i,j\} \subseteq N.
\end{aligned}
$$

Clearly, $\mu_{\{i,j\}} \le b_{\{i,j\}}$ and $\lambda_{\{i,j\}} = 0$ if $\mu_{\{i,j\}} \ge b_{\{i,j\}}$. Consequently, it holds for an optimal dual solution that $\lambda_{\{i,j\}} = b_{\{i,j\}} - \mu_{\{i,j\}}$ for all $\{i,j\} \subseteq N$. Using these observations in the dual program we obtain

$$
\begin{aligned}
w(N) = \sum_{\{i,j\}\subseteq N} b_{\{i,j\}} - \max_{\mu_{\{i,j\}}} & \sum_{\{i,j\}\subseteq N} \mu_{\{i,j\}} \\
\text{s.t.:} \quad & \mu_{\{i,j\}} \le b_{\{i,j\}}; && \text{for all } \{i,j\} \subseteq N \\
& \sum_{\{i,j\}\subseteq S} \mu_{\{i,j\}} \le \sum_{\{i,j\}\in T^*(S)} k_{\{i,j\}} && \text{for all } S \subseteq N; S \text{ connected} \\
& \mu_{\{i,j\}} \ge 0 && \text{for all } \{i,j\} \subseteq N.
\end{aligned}
$$

The restrictions $\mu_{\{i,j\}} \le b_{\{i,j\}}$ imply that $\sum_{\{i,j\}\subseteq S} \mu_{\{i,j\}} \le \sum_{\{i,j\}\subseteq S} b_{\{i,j\}}$ for all $S \subseteq N$. Define $\kappa(S) = \min\{\sum_{\{i,j\}\subseteq S} b_{\{i,j\}}; \sum_{\{i,j\}\in T^*(S)} k_{\{i,j\}}\}$ for all $S \subseteq N$. Then

$$
\begin{aligned}
w(N) = \sum_{\{i,j\}\subseteq N} b_{\{i,j\}} - \max_{\mu_{\{i,j\}}} & \sum_{\{i,j\}\subseteq N} \mu_{\{i,j\}} \\
\text{s.t.:} \quad & \sum_{\{i,j\}\subseteq S} \mu_{\{i,j\}} \le \kappa(S) && \text{for all } S \subseteq N; S \text{ connected} \\
& \mu_{\{i,j\}} \ge 0 && \text{for all } \{i,j\} \subseteq N;
\end{aligned}
$$

which proves the result. $\Box$

Proof of Lemma 4.3: Consider a network situation where E is a tree and denote the corresponding network game by $(N;v)$. For convenience we denote the set of optimal operative networks for coalition U with respect to (1) by $E^*(U)$. Let $S \subseteq T \subseteq N$ and take $D \in E^*(S)$ and $F \in E^*(T)$. Then D can be partitioned in D_1 and D_2 such that $D_1 \subseteq F$ and $D_2 \subseteq E \setminus F$. We show that $D \cup F \in E^*(T)$, or equivalently, that $D_2 \cup F \in E^*(T)$. It follows that

$$
\begin{aligned}
& \sum_{\{i,j\}\subseteq T: C_{\{i,j\}}(D_2\cup F)=1} b_{\{i,j\}} - \sum_{\{i,j\}\in D_2\cup F} k_{\{i,j\}} \\
= & \sum_{\{i,j\}\subseteq T: C_{\{i,j\}}(D_2)=1} b_{\{i,j\}} + \sum_{\{i,j\}\subseteq T: C_{\{i,j\}}(F)=1} b_{\{i,j\}} \\
+ & \sum_{\substack{\{i,j\}\subseteq T:\; C_{i,j}(D_2\cup F)=1; \\ C_{\{i,j\}}(F)=0; \\ C_{\{i,j\}}(D_2)=0}} b_{\{i,j\}} - \sum_{\{i,j\}\in D_2\cup F} k_{\{i,j\}} \\
= & \; v(T) + \sum_{\{i,j\}\subseteq T: C_{\{i,j\}}(D_2)=1} b_{\{i,j\}} + \sum_{\substack{\{i,j\}\subseteq T:\; C_{\{i,j\}}(D_2\cup F)=1; \\ C_{\{i,j\}}(F)=0; \\ C_{\{i,j\}}(D_2)=0}} b_{\{i,j\}} - \sum_{\{i,j\}\in D_2} k_{\{i,j\}}
\end{aligned}
$$

$$
\begin{aligned}
&\geq v(T) + \sum_{\{i,j\}\subseteq S: C_{\{i,j\}}(D_2)=1} b_{\{i,j\}} + \sum_{\substack{\{i,j\}\subseteq S:\ C_{\{i,j\}}(D_1 \cup D_2)=1;\\ C_{\{i,j\}}(D_1)=0;\\ C_{\{i,j\}}(D_2)=0}} b_{\{i,j\}} - \sum_{\{i,j\}\in D_2} k_{\{i,j\}} \\
&= v(T) + \sum_{\substack{\{i,j\}\subseteq S:\ C_{\{i,j\}}(D_1 \cup D_2)=1;\\ C_{\{i,j\}}(D_1)=0}} b_{\{i,j\}} - \sum_{\{i,j\}\in D_2} k_{\{i,j\}} \\
&= v(T) + v(S) - \Big(\sum_{\{i,j\}\subseteq S: C_{\{i,j\}}(D_1)=1} b_{\{i,j\}} - \sum_{\{i,j\}\in D_1} k_{\{i,j\}}\Big) \\
&\geq v(T);
\end{aligned}
$$

where the first equality holds since $D_2 \cup F$ is a forest, the first inequality holds since $D_1 \subseteq F$ and $S \subseteq T$ and the last inequality from the fact that $D_1 \cup D_2 \in E^*(S)$. $\square$

Proof of Theorem 4.2: Let $(N; v)$ be a network game corresponding to a fully public or fully private network situation where E is a tree. Take $S \subseteq T \subseteq N\backslash\{k\}$. We will show that $v(S \cup \{k\}) - v(S) \leq v(T \cup \{k\}) - v(T)$. According to Lemma 4.3 we can take optimal operative networks $E^*(S); E^*(S \cup \{k\}); E^*(T); E^*(T \cup \{k\})$ corresponding to coalition $S; S \cup \{k\}; T; T \cup \{k\}$, respectively, such that $E^*(S) \subseteq E^*(S \cup \{k\}) \subseteq E^*(T \cup \{k\})$ and $E^*(S) \subseteq E^*(T) \subseteq E^*(T \cup \{k\})$.

Define the set A_1 as the set of edges that is contained in $E^*(S \cup \{k\})$ and $E^*(T)$, but not in $E^*(S)$, i.e.,

$$A_1 = \{\{i,j\} \in E \mid \{i,j\} \in E^*(S \cup \{k\})\backslash E^*(S); \{i,j\} \in E^*(T)\}.$$

Note that $A_1 \subseteq E(S)$ is satisfied directly in fully public but also in fully private networks.

The set A_2 is the set of edges that is contained in $E^*(S \cup \{k\})$, but not in $E^*(S)$ and not in $E^*(T)$, i.e.,

$$A_2 = \{\{i,j\} \in E \mid \{i,j\} \in E^*(S \cup \{k\})\backslash E^*(S); \{i,j\} \notin E^*(T)\}.$$

Observe that the definitions of A_1 and A_2 imply that $E^*(S \cup \{k\})$ can be written as the union of three disjoint sets, i.e.,

$$E^*(S \cup \{k\}) = E^*(S) \cup A_1 \cup A_2. \tag{13}$$

Next we introduce the following notation. With $A; B \subseteq E$ and $\{i,j\} \subseteq N$ we define $C^A_{\{i,j\}}(B) = 1$ if the unique path between i and j in E is contained in B and this path contains at least one link of A; otherwise we set $C^A_{\{i,j\}}(B) = 0$.

Now we can make the following observation

$$\sum_{\substack{\{i,j\}\subseteq S:\ C^{A_1}_{\{i,j\}}(E^*(S\cup\{k\})=1\\ C^{A_2}_{\{i,j\}}(E^*(S\cup\{k\})=0}} b_{ij} - \sum_{\{i,j\}\in A_1} k_{ij} \leq 0. \qquad (14)$$

To prove (14) note that the revenues to S of the network $E^*(S)\cup A_1 \subseteq E(S)$ is smaller than or equal to the revenues to S of $E^*(S)$, since $E^*(S)$ is an optimal network. Hence the extra revenues to S by adding A_1 to $E^*(S)$ are smaller than or equal to zero, i.e.,

$$\sum_{\{i,j\}\subseteq S:C^{A_1}_{\{i,j\}}(E^*(S)\cup A_1)=1} b_{ij} - \sum_{\{i,j\}\in A_1} k_{ij} \leq 0 \qquad (15)$$

From equation (13) it follows that

$$\{\{i,j\}\subseteq S : C^{A_1}_{\{i,j\}}(E^*(S)\cup A_1)=1\} =$$

$$\{\{i,j\}\subseteq S : C^{A_1}_{\{i,j\}}(E^*(S\cup\{k\})=1; C^{A_2}_{\{i,j\}}(E^*(S\cup\{k\})=0\};$$

which proves (14).

Now, we introduce the set A_3, which consists of all edges in $E^*(T\cup\{k\})$ that are not in $E^*(S\cup\{k\})$ and not in $E^*(T)$, i.e.,

$$A_3 = \{\{i,j\}\in E \mid \{i,j\}\in E^*(T\cup\{k\})\setminus E^*(T); \{i,j\}\notin E^*(S\cup\{k\})\}.$$

Observe that the definitions of A_2 and A_3 imply that $E^*(T\cup\{k\}$ can be written as the union of three disjoint sets, i.e.,

$$E^*(T\cup\{k\}) = E^*(T)\cup A_2\cup A_3. \qquad (16)$$

Now, we have

$$\begin{aligned}
& v(S\cup\{k\}) - v(S) \\
= & \sum_{i\in S:C_{\{i,k\}}(E^*(S\cup\{k\}))=1} b_{\{i,k\}} + \sum_{\substack{\{i,j\}\subseteq S:\ C_{\{i,j\}}(E^*(S\cup\{k\}))=1\\ C_{\{i,j\}}(E^*(S))=0}} b_{\{i,j\}} \\
& - \sum_{\{i,j\}\in E^*(S\cup\{k\})\setminus E^*(S)} k_{\{i,j\}} \\
= & \sum_{i\in S:C_{\{i,k\}}(E^*(S\cup\{k\})=1} b_{\{i,k\}} + \sum_{\substack{\{i,j\}\subseteq S:\ C^{A_1}_{\{i,j\}}(E^*(S\cup\{k\})=1\\ C^{A_2}_{\{i,j\}}(E^*(S\cup\{k\})=0}} b_{\{i,j\}} - \sum_{\{i,j\}\in A_1} k_{\{i,j\}} \\
& + \sum_{\{i,j\}\subseteq S:C^{A_2}_{\{i,j\}}(E^*(S\cup\{k\})=1} b_{\{i,j\}} - \sum_{\{i,j\}\in A_2} k_{\{i,j\}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i\in S:C_{\{i,k\}}(E^*(S\cup\{k\})=1} b_{\{i,k\}} + \sum_{\{i,j\}\subseteq S:C^{A_2}_{\{i,j\}}(E^*(S\cup\{k\})=1} b_{\{i,j\}} - \sum_{\{i,j\}\in A_2} k_{\{i,j\}} \\
&\leq \sum_{\substack{i\in T:\ C_{\{i,k\}}(E^*(T\cup\{k\})=1\\ C^{A_3}_{\{i,k\}}(E^*(T\cup\{k\})=0}} b_{\{i,k\}} + \sum_{\substack{\{i,j\}\subseteq T:\ C^{A_2}_{\{i,j\}}(E^*(T\cup\{k\})=1\\ C^{A_3}_{\{i,j\}}(E^*(T\cup\{k\})=0}} b_{\{i,j\}} - \sum_{\{i,j\}\in A_2} k_{\{i,j\}} \\
&\leq \sum_{\substack{i\in T:\ C_{\{i,k\}}(E^*(T\cup\{k\})=1\\ C^{A_3}_{\{i,k\}}(E^*(T\cup\{k\})=0}} b_{\{i,k\}} + \sum_{\substack{\{i,j\}\subseteq T:\ C^{A_2}_{\{i,j\}}(E^*(T\cup\{k\})=1\\ C^{A_3}_{\{i,j\}}(E^*(T\cup\{k\})=0}} b_{ij} - \sum_{\{i,j\}\in A_2} k_{\{i,j\}} \\
&\quad + \sum_{\{i,j\}\subseteq T\cup\{k\}:C^{A_3}_{\{i,j\}}(E^*(T\cup\{k\})=1} b_{\{i,j\}} - \sum_{\{i,j\}\in A_3} k_{\{i,j\}} \\
&= \sum_{i\in T:C_{\{i,k\}}(E^*(T\cup\{k\})=1} b_{\{i,k\}} + \sum_{\substack{\{i,j\}\subseteq T:\ C_{\{i,j\}}(E^*(T\cup\{k\})=1\\ C_{\{i,j\}}(E^*(T))=0}} b_{\{i,j\}} \\
&\quad - \sum_{\{i,j\}\in E^*(T\cup\{k\})\setminus E^*(T)} k_{\{i,j\}} \\
&= v(T\cup\{k\}) - v(T)
\end{aligned}
$$

where the first equality holds by the definition of a network game, the second equality follows from (13),the first inequality holds by (14), the second inequality is a consequence of $E^*(S\cup\{k\}\subseteq E^*(T\cup\{k\}$ and the definition of A_3, the third inequality holds by the optimality of $E^*(T\cup\{k\})$, thee third equality follows from (16) and the last equality holds by the definition of a network game. $\square$

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