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Abstract

By introducing the notions of projects and shares, this paper studies a class of economic environments, the so-called project-allocation situations, in which society may profit from cooperation, i.e., by reallocating the initial shares of projects among agents. This paper mainly focuses on the associated issues of compensation of losses and surplus sharing arising from the reallocation of projects. For this purpose, we construct and analyze an associated project-allocation game and a related system of games that explicitly models the underlying cooperative process. Specific solution concepts are proposed.

JEL classification codes: C71, H70
Keywords: projects-allocation situations/games; loss compensation; surplus sharing.

1 Introduction

This paper has two aims. Firstly, it develops a general framework for studying a class of economic environments in which coalitions of agents are optimally reassigned to some bundles of projects: project-allocation situations. Secondly, since this reassignment causes some agents losing jobs or positions, we analyze the problems associated with valuating such reshuffling, such as compensation for losses and sharing surpluses arising from the enhanced efficiency.

In an economy characterized by changing capabilities and preferences of agents and changing technology embodied in projects, people need to continuously adapt their positions to obtain efficiency. That is how our societies have evolved into prosperity. Every change in the production structure requires a reshuffling of responsibilities, which is hard or impossible to implement if possible "losers" are not sufficiently compensated to cooperate. Only when all parties gain from the reassignment, is it a win-win situation. Examples are abundant. Similarly, when extra profit is generated simply by cooperation after reshuffling, a surplus sharing problem occurs.

Obviously, solving the problems of compensation and surplus sharing is essential for creating and maintaining flexibility and creating efficiency in an economy. However, generally speaking, the two concepts are not well distinguished in theoretical research so that the corresponding practical problems can not be treated adequately. In a strict sense, compensation refers to a financial remuneration to an agent for the loss caused by her being removed from some project. On the other hand, surplus sharing deals with the extra benefits created by cooperation among agents assigned to some combination of projects, which benefits are in excess of the sum of individual payoffs. Hence, if compensation is not properly distinguished from benefit/surplus sharing, some individuals may lose on the whole after a reshuffling. Those individuals will strongly oppose and obstruct such a reshuffling. If there exists an authority who can impose reassignments from above, without minding too much about individual sacrifices, the overall approach is sufficient. But even then there are several value concepts available that have a characteristic influence on the outcome. That will be our point of departure.

One may observe that trade unions have forced firms to adopt generic rules for laborers that include some compensation for lay-offs in a firm, as well as labor laws and other safety nets on the macro-level. Our paper focuses on the micro-level. We assume that gains and losses for every particular situation can be endogenously specified and may serve as a basis for the issues of compensation of losses and surplus sharing.

Consider, for example, a restaurant and a boat company, working independently, both situated on the shore of the same beautiful lake. The restaurant, project A, is operated by agent 1 who can be understood as a group of managers, waiters and kitchen staff. Agent 2, a group of people as well, manages project B, the boat company. They are considering collaboration and have two proposals. The first one is simply setting up a joint lunch-sightseeing program, $\{1_A, 2_B\}$ that will benefit both parties. The second proposal is more involved and induces a reshuffling of the two projects, i.e. the restaurant and the boat company. Since agent 2 has excellent expertise in both travelling and restaurant management, much more profit will be generated if the restaurant is also managed by her. The technical possibilities in this situation are represented in the following diagram:

$\{1_A\}$	$\{2_B\}$	$\{1_A, 2_B\}$	$\{1, 2_{A,B}\}$
5	10	18	26

Note that the two types of cooperation, $\{1_A, 2_B\}$ and $\{1, 2_{A,B}\}$ are different in nature: the former corresponds to the first proposal where those two agents have their own projects and coordinate with each other; the latter can be understood as that agent 1 renders A to agent 2 and then works with 2 (just on his human capital). Whereas the first proposal only entails a surplus sharing problem, the second one is further complicated by the problem of compensating agent 1 for giving up his access or user rights of the restaurant project.

This paper analyzes both the loss compensation and surplus sharing problem as illustrated by the second proposal in the above example from a cooperative game theoretic point of view. In our framework, the value of some coalition of agents crucially depends on the involvement of the agents in this coalition in a well-defined set of projects. The involvement is measured by the notion of an agent's shares in projects. That defines a so-called *project-allocation situation* (in short, P-A situation) and an associated *project-allocation game* (in short, P-A game). The value function of the project-allocation game is derived from the underlying profit functions for every coalition given a specific share profile of the projects. So in particular, the value function of this game can be viewed as a generalization of the neoclassical profit function, with labor and capital as inputs and with prices given.

Naturally, any specific solution concept for a cooperative TU (transferable utility) game may of course be applied to solve project-allocation games, and implicitly solve the *combined* loss compensation and surplus sharing problems. (The combined compensation and surplus sharing problem corresponding to the second proposal in the above example can be modelled as a TU game in which $v(\{1, 2\})$ equals 26, the joint value generated by the cooperation between agent 1 and agent 2 after agent 1 transferred the restaurant project to agent 2.) We restrict our attention to two additive one-point solution concepts: the Shapley value (Shapley (1953)) and the *consensus value* (cf. Ju, Borm and Ruys (2004)). Arguments for the suitability of these rules in this specific context are provided.

However, since P-A games are just a partial abstraction of P-A situations, this traditional approach is incapable of disentangling all necessary details to adequately model the basic mechanisms concerning the physical reallocation of projects (loss compensation) and cooperation in joint projects (surplus sharing). In fact, the process to realize the maximal gain of a coalition is a blackbox. Therefore, in order to make the framework operational for solving practical problems, one further step has to be made. By explicitly incorporating an underlying cooperative structure in terms of project reallocation and cooperation afterwards, we devise two different stages in such a way that the loss compensation due to project reallocation and the sharing of extra surplus from cooperation can be clearly and logically distinguished. Hence, this two stage approach makes voluntary acceptance of a reshuffling and a bottom up approach possible, and is even more compelling if reassignment means that agents are laid off and have no chance to participate in the benefit sharing. For each of the two stages, a game is constructed.[1] Consistently, the same solution concept is applied to each of the stage games. Thus, following a general stage approach, also a solution concept for the combined problem is obtained.

Although there exists some fundamental work that is helpful for our research, it seems that the problem of compensating losses has largely been ignored in economic research. The analysis of cost sharing situations (cf. Moulin (1987), Tijs and Branzei (2002)) and linear production situations (Owen (1975)) is in the same spirit but does not explicitly discriminate between the problems of surplus sharing and loss compensation. An exception in a somewhat different context is the work on sequencing games (Curiel et al (1989), Hamers (1995), Klijn (2000)). In this framework, time slots could be considered as projects. Agents change the initial order (shares or rights on time slots) into an optimal one so that the individual payoffs are changed and compensation is needed. Moreover, since joint total costs decrease as well, also the issue of surplus sharing becomes prominent.

In addition to this section introducing the problem and reviewing the literature briefly, the remaining part of the paper is structured as follows. In the next section, we present the main analytical framework by formally introducing project-allocation situations and define project-allocation games. Section 3 addresses the possible solution concepts. Section 4 distinguishes stages to explicitly solve the problems of compensation and surplus sharing in project-allocation situations separately. The final section provides an example of public-private partnerships, which indicates an interesting application of the framework into real economic issues.

[1] The combined problem in the above example can be decomposed into two stages. The first stage considers project reallocation such that agent 1 only renders his A project to agent 2 but does not make further cooperation. The second stage may correspond to a new situation: imagine that agent 1 does not have any project while agent 2 has both A and B; now agent 2 would like agent 1 to work for her.

2 Project-allocation situations and P-A games

Consider a situation in which there exists a finite set N of agents/players who can operate a finite set M of *projects*. We use the word "project" in this paper in a very general sense. A project is a specific entity that can be exploited or operated for some purpose(s) (and mostly for value-creation). It can be a machine, a research project, a firm, or a public utility, etc. Generally, a project can either be *divisible* or *indivisible*. A project is divisible if it is capable of being separated into parts and can be partially operated or owned by some party, without loss of its original function. For instance, a tree farm as a project can be perfectly divided among agents. Indivisibility means that for the purpose of value-creation, a project can only be completely owned or exploited as a whole. A truck is then an indivisible project because it will lose the basic function if divided into parts. Since divisibility is a context-dependent concept which may imply physical divisibility, operational divisibility or ownership divisibility, we have to point out that this paper focuses on the operational divisibility.

The basic idea of a project-allocation situation is that individual agents from the set N have received user rights to operate individual projects from the set M. Each such agent-project combination results in an outcome, called a payoff. Agents may also cooperate and form a coalition that operates a bundle of projects. Since both the agents and the projects are specialized, some agent-project combinations may generate a higher payoff than other combinations. So, for a given assignment of user rights over agents, each coalition of agents operates a specific bundle of projects, which generates a payoff. When, however, some coalition would perform better when it is assigned another bundle of projects, then a feasible reshuffling of coalitions of agents may increase the efficiency of the situation.

This problem of project-allocation is formalized as follows. Each agent receives a *share* in each project, which is a real number $\rho_{i,k}$ between 0 and 1, indicating[2] the fraction of project k that agent i may use or operate. If the agent has an exclusive right, respectively no right on the project, the share equals 1, respectively 0. If the agent has to share rights with other agents in N, and the project is divisible, the fraction corresponds with the distribution of the project over the agents, satisfying the feasibility restriction $\sum_{i\in N} \rho_{i,k} \leq 1$. For instance, one agent may own half of a project, the share is then 0.5. If the project is not divisible and assigned to a coalition S of agents, then the project is - fictitiously - equally distributed among the agents in S. So in that case the individual share $\rho_{i,k}$ equals $1/|S|$,

[2]However, we do not restrict the implications of *shares*, or in another sentence, we do not give a definite economic interpretation but only care about how the shares (in a general sense) in projects that agents have will affect cooperation or even determine compensation and surplus sharing. It may have different meanings in different contexts. For example, it can also represent the ownership/property rights or managerial rights.

for all $i \in S$. This allows for describing each agent's share in any indivisible project and for solving the problems of loss compensation and surplus sharing in such cases mathematically. That is, despite that an indivisible project can not be divided in itself, the value generated from it can be shared among the agents who jointly own it in some way.

The assignment of individual user rights for operating individual projects to the agents in N is thus specified by an $N \times M$ matrix[3] ρ, called a *share profile*. The set of share profiles, R, is defined by

$$R := \left\{ \rho \in [0,1]^{N \times M} | \sum_{i=1}^{n} \rho_{i,k} \leq 1, \forall k \in M \right\} \tag{1}$$

A share profile determines *feasible* agent-project combinations. Feasibility is integrated with technical performance by the following function. The map $f^\rho : 2^N \longrightarrow \mathbb{R}$, assigning for any share profile ρ in R, to any coalition S in N, a real number - the payoff - is called the *payoff function under share profile* ρ. So $f^\rho(S)$ is the payoff of coalition S under share profile ρ. An empty coalition has a zero payoff. Specialization implies that an arbitrary bundle of agent-project combinations may neither be optimal: other combinations may perform better; nor feasible: according to the given share profile it may not have access to projects required for a performance, in which case the payoff equals 0.

Thus, feasible reshuffling of a share profile is required to obtain optimality or efficiency. For that purpose we define the concept of a feasible allocation. Let an initial share profile ρ^0 in R be given. A reallocation of shares within some coalition S in N is called *feasible* for S, if the sum of initially allocated shares in each project to the agents in S equals the sum of the reallocated shares in each corresponding project to the agents in S, while the other agents in N keep their initial shares. So, the set of feasible allocations or feasible share profiles for coalition S with respect to ρ^0, $F(S, \rho^0)$, is defined by

$$F(S, \rho^0) := \left\{ \rho \in R | \sum_{i \in S} \rho_{i,k} = \sum_{i \in S} \rho^0_{i,k}, \forall k \in M \text{ and } \rho_{N \setminus S} = \rho^0_{N \setminus S} \right\} \tag{2}$$

For notational simplicity, we use $F(S)$ if there is no confusion about ρ^0. Thus, feasibility means that agents can re-arrange their shares in projects subject to the capacity determined by the initial share profile within the coalition they participate in, without affecting the allocations outside the coalition.

Based on the above description, we are able to define a project-allocation situation.

[3] Given agent set N of size n and project set M of size m, ρ is in fact an $n \times m$ matrix. We use $N \times M$ to emphasize that ρ is a matrix associated to agent set and project set. The same explanation applies in cases of other matrices, for instance, when we say that ρ_S is an $S \times M$ matrix.

Definition 2.1 *A* project-allocation situation $P(\rho^0)$ *is a tuple* $(N, M, R, \rho^0, \{f^\rho\}_{\rho\in R})$*, where* N *is the set of agents,* M *is the set of projects,* R *is the set of share profiles,* ρ^0 *is the initial share profile, and* $f^\rho : 2^N \longrightarrow \mathbb{R}$ *is the payoff function under a share profile* $\rho \in R$*.*

For analytical convenience, we use the following assumptions at different stages.

Assumption 1 (continuity)*:*
for any $S \in 2^N$*,* $f^\rho(S)$ *is continuous with respect to the share profile* $\rho \in R$*.*

So a small change in the share profile has only a small effect on the value distribution.

Assumption 2 (no externality among coalitions)*:*
$f^{\rho^1}(S) = f^{\rho^2}(S)$,
for all $S \in 2^N$*, whenever* $\rho^1_S = \rho^2_S$*.*

Here ρ_S is the $S \times M$ submatrix of ρ. It follows that the distribution of values within a coalition is independent from the share profile outside that coalition.

Assumption 3 (gains from cooperation)*:*
$f^\rho(S \cup T) \geq f^\rho(S) + f^\rho(T)$,
for all $\rho \in R$ *and for all* $S, T \in 2^N$ *with* $S \cap T = \emptyset$*.*

Assumption 4 (ordinary cooperation)*:*
$f^{\rho^1}(S) \geq f^{\rho^2}(S) \Rightarrow \sum_{i\in S} f^{\rho^1}(\{i\}) \geq \sum_{i\in S} f^{\rho^2}(\{i\})$,
for all $\rho^1, \rho^2 \in R$ *and for all* $S \in 2^N$*.*

The last assumption means that if a share profile is preferable for a coalition of players in cooperation, then the corresponding stand-alone situation is also preferable in terms of the sum of their individual payoffs. It can be understood as a type of consistency between cooperation and its stand-alone basis.

The class of all project-allocation situations with player set N and project set M and the payoff functions satisfying the above assumptions is denoted by $PAS^{N,M}$.

The project-allocation situation $P(\rho^0)$ provides room for reshuffling and optimizing the initial share profile ρ^0. This reallocation process can be described as a TU game, in which the value of a coalition is defined as the maximal payoff that this coalition can achieve by means of feasible share profile.

Given a project-allocation situation $P(\rho^0) = (N, M, R, \rho^0, \{f^\rho\}_{\rho\in R}) \in PAS^{N,M}$, the cooperative game with transferable utility (N, v) defined by

$$v(S) := \max_{\rho\in F(S)} f^\rho(S) \tag{3}$$

for all coalitions S in N with $v(\emptyset) = 0$, is called a **project-allocation (P-A) game**.

A share profile $\rho \in F(S)$ with $f^{\rho}(S) = v(S)$ is called an *optimal share profile* for coalition S, denoted by $\rho^*(S)$.

We want to note that despite the fact that there may exist multiple optimal share profiles for a coalition, the corresponding value for the coalition is uniquely determined.

The P-A games in this paper are endowed with the following property.

Proposition 2.2 *Project-allocation games are superadditive.*

Proof. Let $P(\rho^0) = (N, M, R, \rho^0, \{f^{\rho}\}_{\rho \in R})$ be a project-allocation situation and let the corresponding P-A game be given by (N, v). We need to show $v(S \cup T) \geq v(S) + v(T)$ for all S, T in N with $S \cap T = \emptyset$.

Consider optimal share profiles $\rho^*(S \cup T)$, $\rho^*(S)$ and $\rho^*(T)$ for coalitions $S \cup T$, S, T, respectively. Since $S \cap T = \emptyset$, we can construct a new share profile $\tilde{\rho} \in F(S \cup T)$ such that $\tilde{\rho_S} = \rho^*_S(S)$ and $\tilde{\rho_T} = \rho^*_T(T)$. Then, by definition and Assumption 2 and 3, we have

$$\begin{aligned}
& v(S \cup T) \\
= \; & f^{\rho^*(S \cup T)}(S \cup T) \\
\geq \; & f^{\tilde{\rho}}(S \cup T) \\
\geq \; & f^{\tilde{\rho}}(S) + f^{\tilde{\rho}}(T) \\
= \; & f^{\rho^*(S)}(S) + f^{\rho^*(T)}(T) \\
= \; & v(S) + v(T)
\end{aligned}$$

■

3 Solution concepts for project-allocation games

In this section, we consider two related solution concepts: the well known Shapley value Φ and a newly introduced solution concept, called the *consensus value* γ.

Let TU^N denote the class of all TU games with player set N. Recall that the **Shapley value** is defined by

$$\Phi(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m^{\sigma}(v)$$

for all $v \in TU^N$. Here $\Pi(N)$ denotes the set of all bijections $\sigma : \{1, 2, ..., |N|\} \longrightarrow N$ of N and the marginal vector $m^{\sigma}(v) \in \mathbb{R}^N$, for $\sigma \in \Pi(N)$, is defined by

$$m^{\sigma}_{\sigma(k)} := v(\{\sigma(1), ..., \sigma(k)\}) - v(\{\sigma(1), ..., \sigma(k-1)\})$$

for all $k \in \{1, ..., |N|\}$.

When we go over the definition and the properties of the Shapley value, we can find that it may not be entirely adequate to analyze the project-allocation situations mainly by the following two reasons.

Firstly, the Shapley value relies on the basic notion of marginal vectors. Here, given some ordering of players entering a game, the payoffs are determined by the marginal contributions, which is not satisfying in a constructive or bargaining type of physical setting since a later entrant gets the whole surplus. In a superadditive game, the incumbents will not accept such an arrangement as their contributions are not reflected. While if a game is subadditive, the entrant will not accept such a contract. Apparently, in the practice of a project-allocation situation, a marginal vector is even harder to implement as it may involve reshuffling of projects by current incumbents.

Secondly, the dummy property does not seem too imperative in P-A situations. Rather, this purely utilitarian requirement assigning nothing more than the individual value to a dummy player may hinder the possible collaborations in a P-A situation. Payoffs can only be verified after actual project reallocations. Each agent can be a dummy player, while no one would like to pay effort for nothing. Furthermore, the balance in tradeoff between utilitarianism and egalitarianism is also critical in real life situations.

We propose an alternative solution concept for TU games: the consensus value. This rule follows from a natural and simple idea to share coalition values. For more information including an axiomatic characterization of this solution concept, we refer to Ju, Borm and Ruys (2004).

Consider the following 3-player example. We first have two players: 1 and 2. They cooperate with each other and form a coalition $\{1, 2\}$. The coalition value $v(\{1, 2\})$ is generated. Suppose now player 3 enters the scene, who would like to cooperate with player 1 and 2. But because the coalition $\{1, 2\}$ has been already formed before she enters the game, player 3 will actually cooperate with the existing coalition $\{1, 2\}$ instead of simply cooperating with 1 and 2 individually (Consensus is needed here). If $\{1, 2\}$ agrees as well, the coalition value $v(\{1,2,3\})$ will be generated. How to share it between $\{3\}$ and $\{1, 2\}$? Generally, no rule is better than splitting the joint surplus $v(\{1, 2, 3\}) - v(\{1, 2\}) - v(\{3\})$ equally and assigning half to each of the two parties in addition to their own values (Consensus is obtained once again). Then, what remains (the so-called *remainder*) for $\{1, 2\}$ is $\frac{1}{2}\left(v(\{1, 2, 3\}) + v(\{1, 2\}) - v(\{3\})\right)$. Apparently, 1 and 2 will share this remainder in a similar way: besides their individual values, each of them gets $\frac{1}{2}\left(\frac{1}{2}\left(v(\{1, 2, 3\}) + v(\{1, 2\}) - v(\{3\})\right) - v(\{1\}) - v(\{2\})\right)$. Extending this argument[4]

[4]Indeed, this argument is based on a backward process. Alternatively, we can construct a forward

to an n-player case, we then have a general method, which can be understood as a *standardized remainder rule* as we take the 2-player game standard solution as a base and apply it to solving games by taking the existing coalition as one player. Furthermore, since no order is pre-determined for a TU game, we take all possible ordering of players into account and average the corresponding outcomes, which serves as the final payoff for players.

Formally, this rule is defined as follows. For a given $\sigma \in \Pi(N)$ and $k \in \{1, 2, ..., |N|\}$ we define $S_k^\sigma = \{\sigma(1), \sigma(2), ..., \sigma(k)\}$ and $S_0^\sigma = \emptyset$. Recursively, we define

$$r(S_k^\sigma) = \begin{cases} v(N) & \text{if } k = |N| \\ v(S_k^\sigma) + \frac{1}{2}\left(r(S_{k+1}^\sigma) - v(S_k^\sigma) - v(\{\sigma(k+1)\})\right) & \text{if } k \in \{1, 2, ..., |N| - 1\} \end{cases}$$

where $r(S_k^\sigma)$ is the *standardized remainder* for coalition S_k^σ: the value left for S_k^σ after allocating surplus to later entrants $N \backslash S_k^\sigma$.

We construct the *individual standardized remainder vector* $s^\sigma(v)$, which corresponds to the situation where the players enter the game one by one in the order $\sigma(1), \sigma(2), ..., \sigma(|N|)$ and assign each player $\sigma(k)$, besides her individual payoff $v(\{\sigma(k)\})$, half of the net surplus from the standardized remainder obtained by (the cooperation between) her and the group of players already present. Formally, it is the vector in $\mathbb{R}^N$ defined by

$$s_{\sigma(k)}^\sigma = \begin{cases} v(\{\sigma(k)\}) + \frac{1}{2}\left(r(S_k^\sigma) - v(S_{k-1}^\sigma) - v(\{\sigma(k)\})\right) & \text{if } k \in \{2, ..., |N|\} \\ r(S_1^\sigma) & \text{if } k = 1 \end{cases}$$

The **consensus value** is defined to be the average of the individual standardized remainder vectors, i.e.

$$\gamma(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} s^\sigma(v).$$

A more descriptive name for the consensus value could be *the average serial standardized remainder value*. In the same spirit, an alternative name for the Shapley value could be *the average serial marginal contribution value*.

By Ju, Borm and Ruys (2004), surprisingly, the consensus value is in fact the average of the Shapley value and the *equal surplus solution*:

$$\gamma_i(v) = \frac{1}{2}\Phi_i(v) + \frac{1}{2}\left(\frac{v(N) - \sum_{j \in N} v(\{j\})}{|N|} + v(\{i\})\right)$$

for all $i \in N$, where $\Phi_i(v)$ is the Shapley value of the game.

process to model the idea, which yields the same result. The consistency is provided in Ju, Borm and Ruys (2004).

Example 3.1 *Consider a P-A situation* $P(\rho^0) = (N, M, R, \rho^0, \{f^\rho\}_{\rho \in R}) \in PAS^{N,M}$ *where* $N := \{1, 2, 3\}$*,* $M := \{A, B\}$*, and*

$$\rho^0 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Furthermore,

$$\begin{aligned}
f^\rho(\{1\}) &= 10\rho_{1,A} + 1\rho_{1,B} + \rho_{1,A}\rho_{1,B} \\
f^\rho(\{2\}) &= 8\rho_{2,A} + 3\rho_{2,B} + \rho_{2,A}\rho_{2,B} \\
f^\rho(\{3\}) &= 6\rho_{3,A} + 5\rho_{3,B} + \rho_{3,A}\rho_{3,B} \\
f^\rho(\{1,2\}) &= \sum_{i \in \{1,2\}} f^\rho(\{i\}) + 3 \left(\sum_{i \in \{1,2\}} \sum_{k \in M} \rho_{i,k} \right) \\
f^\rho(\{1,3\}) &= \sum_{i \in \{1,3\}} f^\rho(\{i\}) + 4 \left(\sum_{i \in \{1,3\}} \sum_{k \in M} \rho_{i,k} \right) \\
f^\rho(\{2,3\}) &= \sum_{i \in \{2,3\}} f^\rho(\{i\}) + 2 \left(\sum_{i \in \{2,3\}} \sum_{k \in M} \rho_{i,k} \right) \\
f^\rho(\{1,2,3\}) &= \sum_{i \in \{1,2,3\}} f^\rho(\{i\}) + 6 \left(\sum_{i \in \{1,2,3\}} \sum_{k \in M} \rho_{i,k} \right)
\end{aligned}$$

It is easy to see that these payoff functions satisfy Assumption 1-4. The corresponding P-A game is given by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{12\}$	$\{13\}$	$\{23\}$	$\{123\}$
$v(S)$	0	12	0	19	0	17	27

with $\gamma(v) = \left(4\frac{3}{4}, 18, 4\frac{1}{4}\right)$ *and* $\Phi(v) = \left(4\frac{1}{2}, 19, 3\frac{1}{2}\right)$.

It is our opinion that the consensus value fits quite well in the reshuffling process and the admission structure of the P-A situations. Consider an existing coalition S and a new entrant i. Ex ante, S is a well formed coalition: they had reallocated projects with each other and now cooperate well; they also share the joint surplus in some way. Now, player i would join this coalition. What happens? Obviously, i could not work with any sub-coalition of S but only with S as a whole since S has already been formed over there, comparable to the case that two players cooperate. The immediate (standard) and also a practical solution is then to share the extra revenues from the cooperation equally between S and i.

However, the approach to model the whole P-A situation as one cooperative game is not completely satisfying: P-A games do not take all practical features of P-A situations into account. The underlying process of realizing and allocating the maximal gain of the grand coalition $f^{\rho^*(N)}(N)$ starting from the individual payoffs is still a blackbox.

4 The two stage approach: compensation and surplus sharing

We now focus on an underlying process of obtaining and redistributing the maximal payoff of the grand coalition $f^{\rho^*(N)}(N)$ in a project-allocation situation $P(\rho^0)$.

The initial share distribution ρ^0 yields an *stand-alone value distribution before reallocation*:

$$\beta_i^0 := f^{\rho^0}(\{i\}), \text{ for } i = 1, ..., n.$$

Reallocation of shares not only changes this individual value distribution based on stand-alone activities, but also the payoff generated by new combinations of coalitions: the surplus generated by cooperation. For determining the boundaries of individual compensation, we focus on the stand-alone situations and compare the initial stand-alone value with the stand-alone value generated by the optimal share distribution[5] $\rho^*(N)$. So the optimal share distribution $\rho^*(N)$ yields an *stand-alone value distribution after reallocation*:

$$\beta_i^* := f^{\rho^*(N)}(\{i\}), \text{ for } i = 1, ..., n.$$

If the difference $(\beta_i^* - \beta_i^0)$ is positive, it indicates the stand-alone gain from reallocation, which is also the maximal compensation agent i is willing to pay to other agents. If it is negative, it gives the stand-alone loss from reallocation, which is the minimal compensation agent i is asking from other agents for agreeing with the reallocation of shares.

Taking β^* as a watershed, we can distinguish two stages in the reshuffling process. The first stage considers the compensation issue while the second one focuses on surplus sharing.

- **Stage 1: The compensation game** $(N, \bar{w})$

[5] As noted in Section 2, there may exist multiple optimal share profiles for N. For simplicity, in this paper, we focus the analysis on the cases with unique optimal share profile. For coalitions we do not have to impose such a condition because by Assumption 4 possible multiplicity does not play a role in the procedure.

This stage consists of project reallocation towards the optimal share profile $\rho^*(N)$ for the grand coalition N and finally yields β^*. To determine agents' true value in this reallocation stage, we construct a *stand-alone game*, in which not only the stand-alone values for the grand coalition are taken into account, but also the stand-alone values generated by other coalitions. Given a P-A situation $P(\rho^0)$, the stand-alone game (N, w) is defined by $w(S) = \sum_{i\in S} f^{\rho^*(S)}(\{i\})$, where $\rho^*(S) = \arg\max_{\rho\in F(S,\rho^0)} f^\rho(S)$. So, in particular, $w(\{i\}) = \beta_i^0$ and $w(N) = \sum_{i\in N} \beta_i^*$.

Proposition 4.1 *Stand-alone games are superadditive.*

Proof. Let $P(\rho^0) = (N, M, R, \rho^0, \{f^\rho\}_{\rho\in R})$ be a project-allocation situation and let the corresponding stand-alone game be given by (N, w). We need to show $w(S\cup T) \geq w(S) + w(T)$ for all S, T in N with $S \cap T = \emptyset$.

Let $\rho^*(S\cup T)$, $\rho^*(S)$ and $\rho^*(T)$ be the optimal share profiles for coalitions $S \cup T$, S, T, respectively. Let $\tilde{\rho} \in F(S \cup T)$ be such that $\tilde{\rho_S} = \rho^*_S(S)$ and $\tilde{\rho_T} = \rho^*_T(T)$. Then, we have

$$\begin{aligned}
& w(S\cup T) \\
= & \sum_{i\in S\cup T} f^{\rho^*(S\cup T)}(\{i\}) \\
\geq & \sum_{i\in S\cup T} f^{\tilde{\rho}}(\{i\}) \\
= & \sum_{i\in S} f^{\tilde{\rho}}(\{i\}) + \sum_{i\in T} f^{\tilde{\rho}}(\{i\}) \\
= & \sum_{i\in S} f^{\rho^*(S)}(\{i\}) + \sum_{i\in T} f^{\rho^*(T)}(\{i\}) \\
= & w(S) + w(T)
\end{aligned}$$

Here, the inequality follows from the fact $f^{\rho^*(S\cup T)}(S\cup T) \geq f^{\tilde{\rho}}(S \cup T)$ and Assumption 4. ■

In general, $\beta^* \neq \beta^0$; and apparently, the agents incurred losses due to project reallocation need to be compensated. To explicitly determine compensations, we will consider solutions of the associated *compensation game* $(N, \bar{w})$ defined by

$$\bar{w}(S) = w(S) - \sum_{i\in S} \beta_i^*.$$

Note that $\bar{w}(N) = 0$. The specific values of compensation depend on the solution concept to be chosen, such as the Shapley value $\Phi(\bar{w})$ or the consensus value $\gamma(\bar{w})$.

- **Stage 2: The surplus sharing game** $(N, \bar{\omega})$

This stage considers cooperation after the reallocation in the first stage, i.e. co-working on projects based on the optimal share profile $\rho^*(N)$. This type of cooperation yields a *co-working* game (N, ω) defined as the project-allocation game corresponding to a project-allocation situation $(N, M, R, \rho^*(N), \{f^\rho\}_{\rho \in R})$.

Proposition 4.2 *Co-working games are superadditive.*

Proof. Apparently, co-working games are superadditive as they are project-allocation games. ■

Indeed, this game still takes project reallocation into consideration so that agents are allowed to reallocate shares before joint production. However, the initial share profile itself in this situation is the optimal share profile and $\omega(N) = v(N) = f^{\rho^*(N)}(N)$, players do not reallocate projects in the grand coalition any more (although it may happen in theory within sub-coalitions) but directly work with each other with their current shares. So no compensation is needed. What entails is only surplus sharing. For this aspect, we consider solutions of the associated *surplus sharing* game $(N, \bar{\omega})$ given by $\bar{\omega}(S) = \omega(S) - \sum_{i \in S} \beta_i^*$.

It is obvious that the two stage approach decomposes the maximal payoff of the grand coalition into three elements: $v(N) = \sum_{i \in N} \beta_i^* + \bar{w}(N) + \bar{\omega}(N)$.

The above description on the two stages in a project-allocation situation implies a natural and reasonable way to share the maximal payoff $f^{\rho^*(N)}(N)$. Firstly, an agent i has her stand-alone value after reallocation, β_i^*, due to optimal project reallocation. In addition, to determine the compensations, one solves the compensation game $\bar{w}$; and to solve the surplus sharing problem, one solves the surplus sharing game $\bar{\omega}$. A player's final payoff is the sum of these three parts.

Solving both games with the same one-point solution concept yields a (stage-based) consistent one-point solution concept for P-A situations:

$$\psi_i^*(P(\rho^0)) := \beta_i^* + \psi_i(\bar{w}) + \psi_i(\bar{\omega})$$

where $\psi : TU^N \longrightarrow \mathbb{R}^N$ is a one-point solution concept for TU games.

Generally, it will be the case that the immediate application of ψ to the P-A game v will yield a different solution, i.e., $\psi_i^*(P(\rho^0)) \neq \psi_i(v)$. One may wonder under which conditions the equality holds and both the one stage and the two stage approach give the same result. We require two weak conditions on ψ, i.e., $\psi(0) = 0$ and translation invariance $\psi(v + b) = \psi(v) + b$ for all $v \in TU^N$ and $b \in \mathbb{R}^{|N|}$ (b is an additive game), and strengthen Assumption 3 in the following way.

Assumption 3' *f is additive with respect to coalitions, i.e. $f^\rho(S \cup T) = f^\rho(S) + f^\rho(T)$ for all $\rho \in R$ and for all S, T in N with $S \cap T = \emptyset$.*

Now we can show

Proposition 4.3 *With Assumption 3', if ψ satisfies translation invariance and $\psi(0) = 0$, then $\psi_i^*(P(\rho^0)) = \psi_i(v)$, for all $i \in N$, where ψ, $P(\rho^0)$ and v are defined as above.*

Proof. Clearly, Assumption 3' implies that $f^{\rho^*(N)}(N) = \sum_{i \in N} f^{\rho^*(N)}(\{i\})$. Consequently, $w(S) = v(S)$ for all S in N and $\bar{\omega}(S) = 0$ for all S in N. What remains is obvious: $\psi^*(P(\rho^0)) = \psi(v)$. ■

In particular, the Shapley value $\Phi^*(P(\rho^0))$ of a project-allocation situation $P(\rho^0)$ is given by

$$\Phi_i^*(P(\rho^0)) = \beta_i^* + \Phi_i(\bar{w}) + \Phi_i(\bar{\omega}) \text{ for all } i \in N$$

Similarly, the consensus value $\gamma^*(P(\rho^0))$ of a project-allocation situation $P(\rho^0)$ is given by

$$\gamma_i^*(P(\rho^0)) = \beta_i^* + \gamma_i(\bar{w}) + \gamma_i(\bar{\omega}) \text{ for all } i \in N$$

One may also, in principle, choose a solution concept for the compensation game that is different from the solution concept for the surplus sharing game.

5 An example: disintegration in the water sector

Let us look at an example considering the reform of disintegration and reallocation in the water sector. In this setting, we have three players $N := \{1, 2, 3\}$: player 1 is a provincial government, 2 is a local government, and 3 is a company; two projects: water business (A) and related business (B) such as a golf club or recreation park built on the water source land. So, $M := \{A, B\}$. Initially, both projects are owned by the local government. Consequently, the initial share profile is

$$\rho^0 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Unlike the for-profit project B, water business is usually seen as a public utility. So, the payoff of running the water project can be interpreted as the social welfare/value instead of individual profit. Moreover, we assume that the company has speciality in operating a commercial business while the provincial government may create higher social value if she controls the water project. However, they do have some relative weaknesses. For example, the private company is not good at running public utilities. This type of situation is modelled by the corresponding payoff functions, which are provided in Example 3.1.

Without cooperation, players' individual payoffs come from two parts: the stand-alone payoffs generated from project A or B and the payoff due to the cross-subsidy effect between two projects. With cooperation, in addition to players' individual payoffs, there are some extra gains from cooperation, which, for instance, is expressed by $2\left(\sum_{i\in\{2,3\}}\sum_{k\in M}\rho_{i,k}\right)$ in the payoff function of coalition $\{2,3\}$.

The optimal reform plans for the various coalitions will be: $\rho^*(\{1\}) = \rho^0$;

$$\rho^*(\{1,2\}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}; \qquad \rho^*(\{1,3\}) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}; \qquad \rho^*(\{2,3\}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and for grand coalition $\{1,2,3\}$,

$$\rho^*(\{1,2,3\}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

One readily checks that $\beta^0 := (0,12,0)$, and $\beta^* = (10,0,5)$.

Moreover, beside the project-allocation game (N,v) for the whole situation in this example, we have a compensation game and a surplus sharing game:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{12\}$	$\{13\}$	$\{23\}$	$\{123\}$
$v(S)$	0	12	0	19	0	17	27
$\bar{w}(S)$	-10	12	-5	3	-15	8	0
$\bar{\omega}(S)$	0	0	0	3	8	2	12

The solutions based on the Shapley value or the consensus value can be found in the following two tables.

$\Phi(v)$	$\left(4\frac{1}{2}, 19, 3\frac{1}{2}\right)$
β^*	$(10, 0, 5)$
$\Phi(\bar{w})$	$\left(-9\frac{1}{6}, 13\frac{1}{3}, -4\frac{1}{6}\right)$
$\Phi(\bar{\omega})$	$\left(5\frac{1}{6}, 2\frac{1}{6}, 4\frac{2}{3}\right)$
$\Phi^*(P(\rho^0))$	$\left(6, 15\frac{1}{2}, 5\frac{1}{2}\right)$

$\gamma(v)$	$\left(4\frac{3}{4}, 18, 4\frac{1}{4}\right)$
β^*	$(10, 0, 5)$
$\gamma(\bar{w})$	$\left(-9\frac{1}{12}, 13\frac{1}{6}, -4\frac{1}{12}\right)$
$\gamma(\bar{\omega})$	$\left(4\frac{7}{12}, 3\frac{1}{12}, 4\frac{1}{3}\right)$
$\gamma^*(P(\rho^0))$	$\left(5\frac{1}{2}, 16\frac{1}{4}, 5\frac{1}{4}\right)$

Hence, according to the consensus value, the local government is compensated by the provincial government and the company with a total amount $13\frac{1}{6}$ due to project reallocation, and obtains $3\frac{1}{12}$ from the joint surplus generated by joint production based on the new share profile.

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