Inventory modelling for a manufacturer of sweets: an evaluation of an adjusted compound renewal approach for B-items with a relative short production lead time

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Abstract

In this paper we are especially interested how to optimize the production/inventory control for a manufacturer of sweets, under the following circumstances: short production lead times in combination with an intermittent demand pattern for the so-called B-taste items. As for A-taste items a compound renewal approach appeared appropriate to control inventory/production, we formulated and tested an adjusted compound renewal approach for B-taste items, because a certain condition was not satisfied for those items. For several experiments where the condition was not satisfied and the adjusted approach was compared with the unadjusted one, it then appears that the difference in performance was small. So the first two moments with the compound renewal model appear to be more robust than the restriction given in the literature.

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1 Introduction and problem formulation

Let's introduce the problem by first describing the production process of a manufacturer of sweets in the Netherlands, namely Van Melle Inc. (see Stoop [6]). Van Melle in Breda, The Netherlands, produces the following brands: Mentos, Dummy and Meller. The production process consists of several phases: preparation of different kinds of dough, gumming, drying, attaching a protective layer, sorting and packing.

During the production process, the mint-drops are sorted by thickness. After this sorting process has taken place, the semi-finished products are kept in stock until they can be packed. In order to have an optimal production/inventory control procedure, taking into account the uncertainty in demand for different types of mint-drops, the company wanted to have a trade-off between inventory costs and the direct availability of products for customers. Specifying a given customer service level, one would like to know at which inventory level of the semi-finished products a new production run should be started and how large it should be. However, as the so-called B-tastes have a less smooth demand pattern than A-tastes, an adjusted procedure for demand description is needed. (See for more details section 4).

In this research we are primarily interested how to handle B-tastes, as the standard renewal approach cannot be safely used in this situation.

2 The demand process

Since the inventory level of semi-finished products has to be optimized, the packing department's demand for the various types of Mentos is relevant. In this research only Mentos products were considered, as for other brands the same idea could be used. For every type of Mentos, the demand is approximated by a probability distribution and it is investigated whether the chosen distribution fits the data.

2.1 ABC-analysis

An ABC-analysis was performed to classify the different types of Mentos into three groups. Therefore all tastes are sorted by their average demand.

- The tastes with the highest average demand that are cumulative responsible for 80% of the total demand are classified as A-tastes;
- The tastes with the lowest average demand that cumulatively cause about 5% of the total demand are classified as C-tastes;
- All other tastes are classified as B-tastes.

The results of the ABC-analysis are given in appendix A.

For the three categories, different control policies are needed. Since C-tastes hardly contribute to company profit, the control of these products should be as simple as possible. A make-to-order strategy would be appropriate. For A- and B-tastes the (R, s, Q)

inventory model could be used, as the production quantities are fixed at level Q in this model. An (s, Q) model could also have been used, but since the (s, Q) model is a special case of the (R, s, Q) model, it suffices to analyse the latter model.

2.2 Renewal processes for demand description

Tijms [7] defines a renewal process as a stochastic process counting the number of events that take place as a function of time, where the interoccurrence times are independent and identically distributed (iid). Formally, let $X_1, X_2, \ldots$ be iid interoccurrence times with X_n the time between the $(n-1)$-th and n-th event. Define $S_0 = 0$ and

$$S_n = \sum_{i=1}^{n} X_i, n = 1, 2, \ldots.$$

Then S_n is the time at which the n-th event occurs. Let $N(t)$ be the largest non-negative integer n for which $S_n \leqslant t$. $N(t)$ represents the number of events up to time t. The counting process $\{N(t), t \geqslant 0\}$ is called the renewal process generated by the interoccurrence times $X_1, X_2, \ldots$.

A compound renewal process is a stochastic process $\{X(t), t \geqslant 0\}$ with

$$X(t) = \sum_{i=1}^{N(t)} D_i,$$

where $\{N(t), t \geqslant 0\}$ is a renewal process and $D_1, D_2, \ldots$ are iid and independent of $\{N(t)\}$. It is easily seen that the ordinary renewal process is a special case of the compound renewal process, by taking $D_i = 1$ for all $i = 1, 2, \ldots$. In a compound renewal process, customers arrive according to a renewal process and every customer asks a specific amount of the product. This amount can be larger than 1 unit.

At Van Melle, the demand for A-tastes of Mentos can be modelled as a compound renewal process. The reason that a compound renewal process is chosen is that there are days without any demand for Mentos. Renewal processes are appropriate if there are periods with zero demand. Other models often assume that every period demand is positive. The compound renewal process is a good model for A-tastes because the average interarrival time of the demand epochs is smaller than the production lead time of a few days. Therefore the formulas provide good approximations. See section 3.7 for a formal condition under which the approximations are good.

For example, if one is interested in the variance of demand during a relatively short time interval, the formula that estimates this variance will provide a good approximation only if enough events occurred during the considered time interval. For B-tastes the average interarrival time of the demand epochs is probably larger than the production lead time. In that case, the demand for B-tastes cannot be modelled by a compound renewal process. However, Janssen [1] suggests an adjustment to the compound renewal process that can be used to model the demand for B-tastes. This adjusted model will be discussed in section 4.

3 The (R, s, Q) inventory model with compound renewal demand

3.1 Introduction

In the (R, s, Q) inventory model the lot size is fixed. Note that the inventory position is defined as the on-hand inventory plus outstanding orders minus backorders. Every R periods the inventory position is checked and if this level has dropped to s or below, a production order of size Q is placed. The following assumptions are made in this model:

- The expected demand is constant over time
- The undershoot is strictly positive
- No crossing of orders
- The first two moments of the probability distributions of the demand order sizes, interarrival times and lead times are known
- Shortages are backordered, so there are no lost sales
- Shortages at the beginning of a delivery cycle are allowed

Next, the parameters of the (R, s, Q) model have to be estimated. Because of the limited production capacity, it should be checked whether the solution of the model is feasible. This can be done using a simulation program, which also provides information about the realized average inventory levels. The value of s depends on the chosen cost criterion or service criterion. In this case, a P_2-service criterion is used, where P_2 denotes the average fraction of demand that can be satisfied during a delivery cycle. The reason why a service criterion is used is the fact that few information is available about all kinds of costs. The P_2-criterion is chosen because it was considered to be suitable for this firm. Since there is a lack of data concerning all sorts of costs, the production order sizes are determined intuitively instead of using the economic order quantity.

The demand for A-tastes during the production lead time is modelled by a compound renewal process. This process can be described by interarrival times between demand epochs and the demand quantities at these epochs. A χ^2-goodness-of-fit test shows that the gamma distribution is a good approximation of the distribution of the demand quantities. An advantage of the gamma distribution is the fact that gamma distributed random variables only take non-negative values. For example, if the demand is assumed to be normally distributed, there is a positive probability of a negative demand, which is nonsense. Another advantage of the gamma distribution is the fact that it can be quite good approximated by mixed Erlang distributions, which are numerically attractive. Furthermore, the gamma distribution is skewed to the right, which often occurs in practical data. By changing the two positive parameters λ and α of the gamma distribution, a lot of different shapes can be created.

3.2 Notation

To analyse the (R, s, Q) inventory model, the following notation is used (see also Janssen, Heuts, de Kok [2], [3]).

R	$:=$	length of the review interval
s	$:=$	reorder point
Q	$:=$	order quantity
D_i	$:=$	demand quantity of i-th order
D	$:=$	an arbitrary demand order size
A_i	$:=$	interarrival time between $(i-1)$-th and i-th order
A	$:=$	an arbitrary interarrival time
L_m	$:=$	lead time of the m-th production order
$Z(n)$	$:=$	$\sum_{i=1}^{n} D_i$, total demand during n periods
X_t	$:=$	inventory position at time t
T_k	$:=$	the k-th time epoch that X_t decreases beyond s
U_k	$:=$	$s - X(T_k)$, the k-th undershoot
U	$:=$	an arbitrary undershoot
τ_k	$:=$	the first review epoch after T_k
W_k	$:=$	$\tau_k - T_k$, waiting time until the next review epoch
L'_k	$:=$	$L_k + W_k$. In this paper, L'_k is called 'pseudo lead time'
L'	$:=$	pseudo lead time of an arbitrary production order
$U_{R,k}$	$:=$	$s - X(\tau_k)$, number of units below the reorder level at the review epoch
V_k	$:=$	$Z(L'_k)$, demand during the pseudo lead time
V	$:=$	demand during an arbitrary pseudo lead time
Z_k	$:=$	$V_k + U_k = Z(L'_k) + U_k$, undershoot plus demand during pseudo lead time
Z	$:=$	demand during an arbitrary pseudo lead time plus undershoot

The term 'pseudo lead time' is used because of the following. At a time epoch at which the inventory level reaches the reorder point, it is not possible to place an order immediately. One has to wait for the next review moment. Therefore, the time span between the epoch at which the inventory level reaches the reorder point and the time at which the order arrives, consists of the waiting time until the next review moment plus the actual production lead time. Here the sum of these two terms is called the pseudo lead time.

3.3 Assumptions

Besides the assumptions of the (R, s, Q) model stated in section 3.1, some additional assumptions are made:

- The demand process is a renewal process
- $D_1, D_2, \ldots$ iid with expectation μ_D, standard deviation σ_D and coefficient of variation $c_D := \sigma_D/\mu_D$, where $D_i > 0$ for all $i = 1, 2, \ldots$
- $A_1, A_2, \ldots$ iid with expectation μ_A, standard deviation σ_A and coefficient of variation $c_A := \sigma_A/\mu_A$, where $A_i > 0$ for all $i = 1, 2, \ldots$
- A_i and D_i independent
- $L_1, L_2, \ldots$ iid with expectation μ_L and standard deviation σ_L, where $L_m > 0$ for all $m = 1, 2, \ldots$
- The demand order sizes are gamma distributed: $D_i \sim \Gamma(\alpha, \lambda)$, where α is the shape parameter and λ is the scale parameter. These parameters can be estimated using $E(D)$ and $E(D^2)$, which are assumed to be known. See the assumptions of the (R, s, Q) inventory model. The estimation of the parameters is discussed in the next section.
- $Z_k = Z(L'_k) + U_k \sim \Gamma(\alpha, \lambda)$. Not only the demand quantity per order is gamma distributed, but also the undershoot plus the aggregated demand during the pseudo lead time. It should be noted however, that the parameters α and λ need not be the same as for the D_i process.

Not all of these assumptions hold in practice. For example, the interarrival time and the demand order size will probably be interrelated. However, these assumptions are a reasonable approximation of reality.

3.4 The method of moments for the gamma parameters

Tijms [7, page 355] shows how to estimate the parameters of the gamma distribution, using the first two moments of the empirical distribution of D. Let $\alpha_1 = E(D)$ and

$\alpha_2 = E(D^2)$. The mean of the $\Gamma(\alpha, \lambda)$ distribution is α/λ and the variance is α/λ^2. It follows that the second moment equals

$$E(D^2) = [E(D)]^2 + Var(D) = \left(\frac{\alpha}{\lambda}\right)^2 + \frac{\alpha}{\lambda^2} = \frac{\alpha(1+\alpha)}{\lambda^2} \tag{1}$$

To express α and λ as functions of α_1 and α_2, solve the following equalities for α and λ.

$$\alpha_1 = \frac{\alpha}{\lambda} \tag{2}$$

$$\alpha_2 = \frac{\alpha(1+\alpha)}{\lambda^2} \tag{3}$$

This system of equations has a unique solution if and only if $\alpha_2 \neq \alpha_1^2$, which is given by

$$\lambda = \frac{\alpha_1}{\alpha_2 - \alpha_1^2} \tag{4}$$

$$\alpha = \frac{\alpha_1^2}{\alpha_2 - \alpha_1^2} \tag{5}$$

If $\alpha_2 = \alpha_1^2$, then it is not possible to fit a gamma distribution. In practice this will not be a problem. It holds that $Var(D) = E(D^2) - [E(D)]^2 = \alpha_2 - \alpha_1^2$, so $\alpha_2 = \alpha_1^2$ implies that the variance is 0, i.e. that demand is deterministic. In that case, nothing has to be estimated at all.

3.5 The P_2-service equation to determine the reorder point

For all A-tastes the reorder point s is determined using the P_2-service criterion. Here P_2 is the fraction of demand that can be satisfied during a delivery cycle. This fraction will be denoted by β, which is a function of R, s and Q. The delivery cycle is the time span from the epoch at which an order has just arrived until the epoch just before the next order arrives. Taking into account the possibility of a shortage at the beginning of a delivery cycle, the expected demand that cannot be satisfied during a delivery cycle can be expressed as the expected shortage at the end of a delivery cycle minus the expected shortage at the beginning of a delivery cycle.

Using the notation introduced above, $\tau_2 + L_2$ represents the delivery epoch of the order placed at the second review epoch. The expected net stock just before this order arrives is equal to $E\{X_{\tau_2+L_2}\}_- = s - E\{Z_2\}$ and the expected net stock just after this order arrives is equal to $E\{X_{\tau_2+L_2}\}_+ = s + Q - E\{Z_1\}$. Let f be the density of X and denote $\max\{0, x-s\}$ by $(x-s)^+$, such that

$$E\{X - s\}^+ := \int_s^\infty (x-s) f(x)\,\mathrm{d}x \tag{6}$$

Let f_Z be the density of the undershoot plus the demand during the pseudo lead time. Then the expected shortage at the end of a delivery cycle is

$$E\{Z_2 - s\}^+ = \int_s^\infty (x-s) f_Z(x)\,\mathrm{d}x \tag{7}$$

The expected shortage at the beginning of a delivery cycle is

$$E\{Z_1 - (s+Q)\}^+ = \int_{s+Q}^{\infty} (x - (s+Q))\, f_Z(x)\, \mathrm{d}x \tag{8}$$

Hence, the expected demand that cannot be satisfied per delivery cycle is

$$\int_{s}^{\infty} (x - s) f_Z(x)\, \mathrm{d}x - \int_{s+Q}^{\infty} (x - (s+Q))\, f_Z(x)\, \mathrm{d}x \tag{9}$$

The expected total demand per delivery cycle is Q, because shortages are backordered. The service equation is then given by

$$\beta(R, s, Q) = 1 - \frac{1}{Q} \left[\int_{s}^{\infty} (x - s) f_Z(x)\, \mathrm{d}x - \int_{s+Q}^{\infty} (x - (s+Q))\, f_Z(x)\, \mathrm{d}x \right] \tag{10}$$

From this service equation, the optimal value of s can be solved. The distribution of the demand during the pseudo lead time plus undershoot is needed, in order to solve for s. In appendix C some information is given about how to evaluate the integrals in (10) easily.

3.6 The first two moments of the undershoot

The first two moments of the distribution of the undershoot U can be approximated using a result from renewal theory, which states that if Q is large enough and D_i is gamma distributed for all $i = 1, \ldots, n$ with scale parameter λ and shape parameter α, then

$$\mu_U \approx \frac{\alpha + 1}{2\lambda} \text{ and } \sigma_U^2 \approx \frac{(\alpha+1)(\alpha+5)}{12\lambda^2} \tag{11}$$

where $\mu_U := E(U)$ and $\sigma_U^2 := Var(U)$. Note that α and λ are the parameters of the D_i process in this case. Using a method of moments, the distribution of the undershoot can now be approximated, analogous to section 3.4.

3.7 The first two moments of the demand during the pseudo lead time

The expectation and the variance of the demand during the pseudo lead time (V) can also be approximated using results from renewal theory, see Tijms [7]. However, these results only hold if the production lead time is at least as large as some value t_0, where

$$t_0 = \begin{cases} \frac{3}{2} c_A^2 \mu_A, & \text{if } c_A^2 > 1 \\ \mu_A, & \text{if } 0.2 < c_A^2 \leqslant 1 \\ \frac{1}{2c_A} \mu_A, & \text{if } 0 < c_A^2 \leqslant 0.2 \end{cases}$$

Recall that c_A denotes the coefficient of variation of the interarrival times of demand moments.

If the condition is satisfied, the following results can safely be used, see De Kok [4]:

$$\mu_V \approx \frac{\mu_L}{\mu_A}\mu_D + \frac{(c_A^2 - 1)}{2}\mu_D \tag{12}$$

$$\sigma_V^2 \approx \frac{\mu_L}{\mu_A}\sigma_D^2 + \frac{\mu_L}{\mu_A}c_A^2\mu_D^2 + \frac{\mu_D^2}{\mu_A^2}\sigma_L^2 + \frac{(c_A^2 - 1)}{2}\sigma_D^2 + \frac{(1 - c_A^4)}{12}\mu_D^2 \tag{13}$$

where $\mu_V := E(V)$ and $\sigma_V^2 := Var(V)$.

For A-tastes the condition is satisfied, so the results can safely be used for these items (see appendix C.2 for an example). However, for B-tastes the condition is probably not satisfied and so (12) and (13) cannot be used. To resolve this we shall consider a so-called adjusted compound renewal process for B-tastes in section 4.

3.8 Distribution of the demand during the pseudo lead time plus undershoot and an expression for the average physical inventory level

Consider the distribution of the demand during the pseudo lead time plus undershoot (Z). The expectation μ_Z and the variance σ_Z^2 of Z can be approximated using the relationship $Z_k = U_k + V_k$, i.e. by considering the undershoot and the demand during the pseudo lead time separately. Since the variables U and V are independent, the following results can be used:

$$\mu_Z = \mu_U + \mu_V \tag{14}$$

$$\sigma_Z^2 = \sigma_U^2 + \sigma_V^2 \tag{15}$$

The aforementioned method of moments can be used to approximate the distribution of Z. Finally, the optimal values of s can be solved from equation (10).

Let X be the physical inventory level, then once s is known, the average physical inventory level for an (s, Q) inventory model can be computed with the following formula (see e.g. de Kok [4]):

$$E(X) \approx \frac{\int_0^{s+Q}(s + Q - x)^2 f_V(x)\,\mathrm{d}x - \int_0^{s}(s - x)^2 f_V(x)\,\mathrm{d}x}{2Q}, \tag{16}$$

where f_V is the density of the demand during the pseudo lead time. Using simulation, one can compare this value with the actual physical inventory level. In appendix D it is shown how to evaluate the integrals in (16) easily.

4 The (R, s, Q) inventory model with adjusted moments for the compound renewal demand

Janssen [1] suggests an adjusted procedure to determine the first two moments of the compound renewal process,which can be used for cases in which the condition of section 3.7 is not satisfied. This concept could be used to model the demand process for B-tastes of Mentos during a short period of time.

4.1 Notation

In the adjusted model the same notation is used as in section 3.1, together with some extra variables. Let $N(0,t)$ indicate the number of customer arrivals during $(0,t]$ and let $N(T)$ denote the number of arrivals during a time interval of length T. For example, $N(L')$ denotes the number of customer arrivals during the pseudo lead time of a production order. Let $D(0,t)$ be the total demand during the time interval $(0,t]$. Then $V := D(L')$ denotes the demand during the pseudo lead time of a production order. Furthermore, recall that for a renewal process the variable S_k denotes the time of the k-th renewal.

4.2 Quantities of interest

The goal of this model is to approximate the demand for B-tastes during the pseudo lead time of a production order. According to Janssen [1], the following relationships hold:

$$E\{D(0,t)\} = E\{N(0,t)\} \cdot E(D) \tag{17}$$

$$E\{D(0,t)^2\} = E\{N(0,t)\} \cdot Var(D) + E\{N(0,t)^2\} \cdot (E(D))^2 \tag{18}$$

From these relationships it follows that the first two moments of $N(0,t)$ need to be computed, in order to determine the first two moments of the demand during a certain period $(0,t]$. For example, the first two moments of $N(L')$ are needed in order to determine the first two moments of $D(L')$, the demand during the pseudo lead time of a production order.

4.3 The algorithm

The following algorithm can be used to determine the first two moments of $N(L')$.

- Calculate the first two moments of L' and $S_1, S_2, \ldots$. Recall that L' is the pseudo lead time of a production order and that S_k is the sum of k successive interarrival times. This means that in the time interval of length S_k exactly k customers have arrived.

- Fit a mixed Erlang distribution on L' and on $S_1, S_2, \ldots$. Mixed Erlang distributions are discussed in appendix B.

- Compute $P(S_k \leqslant L')$. Note that a closed form solution exists in case of mixed Erlang distributions (see appendix E).

- Using the property
$$\{S_k \leqslant L'\} = \{N(L') \geqslant k\}, \tag{19}$$
compute $P\{N(L') = 0\} = 1 - P\{N(L') \geqslant 1\} = 1 - P(S_1 \leqslant L')$. Furthermore, compute $P\{N(L') = k\} = P(S_k \leqslant L') - P(S_{k+1} \leqslant L')$ for $k = 1, 2, \ldots, k_{\max}$, where $k_{\max}$ is chosen such that $\sum_{j=0}^{k_{\max}} P\{N(L') = j\} \geqslant 0.99999$.

- The first two moments of $N(L')$ can be computed as follows:
$$E\{N(L')\} = \sum_{j=1}^{k_{\max}} j \cdot P\{N(L') = j\} \tag{20}$$
$$E\{N(L')^2\} = \sum_{j=1}^{k_{\max}} j^2 \cdot P\{N(L') = j\} \tag{21}$$

$E(V)$ and $E(V^2)$ can now be computed using equations (17) and (18). This gives

$$E(V) \ : \ = E\{D(L')\} = E\{N(L')\} \cdot E(D) \tag{22}$$
$$E(V^2) \ : \ = E\{D(L')^2\} = E\{N(L')\} \cdot Var(D) + E\{N(L')^2\} \cdot (E(D))^2 \tag{23}$$

4.4 The first two moments of Z for B-taste items

Using μ_U and σ_U, see section 3.6, $\mu_V = E(V)$ and $\sigma_V^2 = E(V^2) - [E(V)]^2$, the mean and variance of Z can be determined, just like in section 3.8:

$$\mu_Z = \mu_U + \mu_V \tag{24}$$
$$\sigma_Z^2 = \sigma_U^2 + \sigma_V^2 \tag{25}$$

4.5 Order policy for B-taste items

The P_2-service criterion is also used for B-tastes. It follows that the optimal values of the reorder points (s) can again be solved from the service equation (10). In addition, the average physical inventory level can again be approximated by (16).

5 Differences in performance for a B-taste item when the adjusted moments procedure is compared with the unadjusted one

To investigate whether the (s, Q) inventory model with the adjusted moments approach performs better than with the unadjusted one, we considered the following example.

The review interval is 1 day, so it is always possible to place an order immediately. This implies that $L' = L$. The average lead time is $\mu_L = 2.008$ days, with standard deviation $\sigma_L = 0.4$. The squared coefficient of variation is $c_L^2 = 0.03968$. The probability that demand on a day is positive is 0.23. Suppose the mean μ_S, standard deviation σ_S and squared coefficient of variation c_S^2 of $S_k,\ k = 1, \ldots, 10$ are as given in table 1.

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
μ_S	4.22	8.40	12.59	16.80	21.01	25.25	29.46	33.67	37.89	42.14
σ_S	3.36	4.45	5.39	6.29	6.93	7.82	8.69	9.51	10.33	11.09
c_S^2	0.634	0.281	0.183	0.140	0.109	0.096	0.087	0.080	0.074	0.069

Table 1: Information about S_k.

The coefficient of variation of interarrival times is $c_A^2 = 0.634$, so the condition in section 3.7 states that the ordinary model could be used if the production lead time is at least as large as the average interarrival time. Since $\mu_L = 2.008$ and $\mu_A = 4.22$, the condition is not satisfied. Therefore, we will perform the adjusted approach.

The first step is to fit a mixed Erlang distribution on $S_1, S_2, \ldots, S_{10}$. All coefficients of variation but one are smaller than 0.5. Since the coefficient of variation of S_1 is larger than 0.5, the following hyperexponential density with parameters p, φ_1 and φ_2 will be used for S_1:

$$f(x) = \sum_{j=1}^{2} p_j \varphi_j^{k_j} \frac{x^{k_j - 1}}{(k_j - 1)!} e^{-\varphi_j x},\ x \geqslant 0, \tag{26}$$

with

$$\begin{aligned}
k_1 &= 1;\ k_2 = 1 \\
p &= \frac{\varphi_1(\varphi_2 \mu_S - 1)}{\varphi_2 - \varphi_1} = -0.9603 \\
p_1 &= p = -0.9603;\ p_2 = 1 - p = 1.9603 \\
\varphi_1 &= \frac{2}{\mu_S}\left(1 + \sqrt{\frac{c_S^2 - \frac{1}{2}}{c_S^2 + 1}}\right) = 0.6102 \\
\varphi_2 &= \frac{4}{\mu_S} - \varphi_1 = 0.3386.
\end{aligned}$$

The following density will be used for $S_2, \ldots, S_{10}$:

$$f(x) = \sum_{j=1}^{2} p_j \varphi^{k_j} \frac{x^{k_j - 1}}{(k_j - 1)!} e^{-\varphi x}, \; x \geqslant 0, \tag{27}$$

with

$$\begin{aligned} k &= \left\lfloor \frac{1}{c_S^2} + 1 \right\rfloor \\ k_1 &= k - 1; \; k_2 = k \\ p &= \frac{1}{1 + c_S^2} \left(k c_S^2 - \sqrt{k(1 + c_S^2) - k^2 c_S^2} \right) \\ p_1 &= p; \; p_2 = 1 - p \\ \varphi &= \frac{k - p}{\mu_S}. \end{aligned}$$

The parameter values are given in table 2.

	S_2	S_3	S_4	S_5
k	4	6	8	10
p	0.2584	0.3283	0.6360	0.5640
φ	0.4457	0.4505	0.4384	0.4491

	S_6	S_7	S_8	S_9	S_{10}
k	11	12	13	14	15
p	0.3566	0.2939	0.2768	0.3274	0.3367
φ	0.4216	0.3974	0.3779	0.3608	0.3480

Table 2: Parameter values in mixed Erlang densities.

The same mixed Erlang distribution as for $S_2, \ldots, S_{10}$ will be used for L, but with other symbols. Let $g(s)$ be the mixed Erlang density of L.

$$g(s) = \sum_{i=1}^{2} q_i \rho^{l_i} \frac{s^{l_i - 1}}{(l_i - 1)!} e^{-\rho s}, \; s \geqslant 0, \tag{28}$$

with

$$\begin{aligned} l &= \left\lfloor \frac{1}{c_L^2} + 1 \right\rfloor = 26 \\ l_1 &= l - 1 = 25; \; l_2 = l = 26 \\ q &= \frac{1}{1 + c_L^2} \left(l c_L^2 - \sqrt{l(1 + c_L^2) - l^2 c_L^2} \right) = 0.5550 \\ q_1 &= q = 0.5550; \; q_2 = 1 - q = 0.4450 \\ \rho &= \frac{l - q}{\mu_L} = 12.6718. \end{aligned}$$

The next step is to compute $P(S_k \leqslant L)$ for $k = 1, \ldots, 10$. The derivation of a closed form solution can be found in appendix E. From this derivation, it follows that

$$\begin{aligned} P(S_k \leqslant L) &= \sum_{j=1}^{2}\sum_{i=1}^{2} p_j q_i \left(1 - \sum_{t=0}^{k_j-1} \binom{t+l_i-1}{t} \frac{\varphi^t \rho^{l_i}}{(\varphi+\rho)^{t+l_i}}\right) \qquad (29) \\ &= pq\left(1 - \sum_{t=0}^{k-2} \binom{t+l-2}{t} \frac{\varphi^t \rho^{l-1}}{(\varphi+\rho)^{t+l-1}}\right) \\ &\quad + p\,(1-q)\left(1 - \sum_{t=0}^{k-2} \binom{t+l-1}{t} \frac{\varphi^t \rho^{l}}{(\varphi+\rho)^{t+l}}\right) \\ &\quad + (1-p)\,q\left(1 - \sum_{t=0}^{k-1} \binom{t+l-2}{t} \frac{\varphi^t \rho^{l-1}}{(\varphi+\rho)^{t+l-1}}\right) \\ &\quad + (1-p)\,(1-q)\left(1 - \sum_{t=0}^{k-1} \binom{t+l-1}{t} \frac{\varphi^t \rho^{l}}{(\varphi+\rho)^{t+l}}\right). \end{aligned}$$

For $k = 1$ this reduces to:

$$P(S_k \leqslant L) = \sum_{j=1}^{2}\sum_{i=1}^{2} p_j q_i \left(1 - \left(\frac{\rho}{\varphi_j+\rho}\right)^{l_i}\right) \qquad (30)$$

For $k = 2, \ldots, 10$ formula (29) is used. The results are given in table 3.

$P(S_1 \leqslant L) = 0.28803091711052$	$P(S_6 \leqslant L) = 3.784919 \times 10^{-8}$
$P(S_2 \leqslant L) = 0.02797809545450$	$P(S_7 \leqslant L) = 1.84248 \times 10^{-9}$
$P(S_3 \leqslant L) = 0.00135190636514$	$P(S_8 \leqslant L) = 9.1706 \times 10^{-11}$
$P(S_4 \leqslant L) = 0.00004592232223$	$P(S_9 \leqslant L) = 4.9229 \times 10^{-12}$
$P(S_5 \leqslant L) = 0.00000082585922$	$P(S_{10} \leqslant L) = 2.3624 \times 10^{-13}$

Table 3: $P(S_k \,\square\, L)$ for $k = 1, \ldots, 10$.

Now it is possible to compute $P\{N(L) = k\}$. The results are summarized in table 4. $k_{\max} = 4$, because $\sum_{j=0}^{4} P\{N(L) = j\} \geqslant 0.99999$.

From (20) and (21) it follows that $E\{N(L')\} = 0.3174$ and $E\{N(L')^2\} = 0.3790$. Now $E(V)$ and $E(V^2)$ can be computed using (22) and (23). The average daily demand on days with a positive demand is $E(D) = 12.638$ tons with standard deviation 10.543. The variance is $Var(D) = 111.15$. Hence,

$$\begin{aligned} E(V) &= 4.0113 \\ E(V^2) &= 95.82 \\ c_V^2 &= 4.9549. \end{aligned}$$

$P\{N(L)=0\} = 0.71196908$	Cumulative:
$P\{N(L)=1\} = 0.26005282$	$\sum_{j=0}^{1} P\{N(L)=j\} = 0.972022$
$P\{N(L)=2\} = 0.02662619$	$\sum_{j=0}^{2} P\{N(L)=j\} = 0.998648$
$P\{N(L)=3\} = 0.00130598$	$\sum_{j=0}^{3} P\{N(L)=j\} = 0.999954$
$P\{N(L)=4\} = 0.00004510$	$\sum_{j=0}^{4} P\{N(L)=j\} = 0.999999$

Table 4: $P\{N(L)=k\}$ for $k = 0, 1, \ldots, k_{\max}$.

Next the reorder point will be determined, given the service level, using the procedure in appendix C. The order quantity is $Q = 30$ tons. The other parameter values are $\alpha_1 = 12.638$, $\alpha_2 = 270.87$, $\lambda = 0.1137$, $\alpha = 1.4369$, $\mu_U = 10.7166$, $\sigma_U^2 = 101.12$, $\mu_Z = 14.7280$, $\sigma_Z^2 = 180.85$, $c_Z^2 = 0.8337$. The parameters of the mixed Erlang distribution of Z are $k_1 = 1$, $k_2 = 1$, $\mu_1 = 0.1937$, $\mu_2 = 0.0779$ and $p = -0.2454$. The mixed Erlang distribution that is needed to determine the average physical inventory level, has parameter values $k_1 = 1$, $k_2 = 1$, $\mu_1 = 0.9298$, $\mu_2 = 0.0673$ and $p = 0.7869$. The results are summarized in table 5.

P_2	s (tons)	Average physical inventory (tons)
0.95	30	41.17
0.96	32	43.15
0.97	36	47.11
0.98	41	52.07
0.99	50	61.04
0.995	59	70.01
0.999	80	90.99

Table 5: Results adjusted compound renewal model.

Now suppose that the ordinary compound renewal model would have been used, although the condition in section 3.7 is not satisfied. Then (12) and (13) give $\mu_V = 3.7057$ and $\sigma_V^2 = 90.2130$. Hence, $E(V^2) = 103.95$. The results are given in table 6.

Finally, the compound renewal model can be compared with the adjusted compound renewal model. In table 7 the inventory reduction is given if one uses the adjusted model instead of the ordinary model.

Dependent on the required service level, an inventory reduction between 1% and 4% is possible.

P_2	s (tons)	Average physical inventory (tons)
0.95	30	41.55
0.96	33	44.51
0.97	37	48.47
0.98	42	53.42
0.99	52	63.37
0.995	61	72.34
0.999	83	94.31

Table 6: Results compound renewal model.

P_2	Inventory reduction if adjusted model is used (in %)
0.95	0.9
0.96	3.1
0.97	2.8
0.98	2.5
0.99	3.7
0.995	3.2
0.999	3.5

Table 7: Inventory reduction.

6 Conclusions

According to Stoop [6], a lot of money could be saved by implementing the compound renewal model for A-tastes of Mentos. More specifically, an inventory reduction of 17% was possible for A-tastes. This conclusion was based on the results of a simulation study. In this simulation study, the average physical inventory level and the service level that have actually been realized are determined, for every taste separately.

For B-taste items we tested the adjusted compound renewal approach and compared the results with the unadjusted one, knowing that the restriction in section 3.7 was not satisfied, and the renewal formulae (12) and (13) might be doubtful. For several experiments, the adjusted approach appears to give reduction in average inventory of 1 to 4% for specified fill rate service levels, compared to using the unadjusted approach. Given this small reduction and the complexity of the adjusted approach, we believe that the formulae (12) and (13) are more robust than the restriction in section 3.7 suggests.

A Results ABC-analysis

To perform an ABC-analysis, the demand data of the last 28 weeks of 1996 are considered. The results of the ABC-analysis are summarized in table 8.

Taste	Demand (tons)	% of total demand	Cumulative	Class
Mint	7508	38.90	38.90	A
Strawberry	2956	15.31	54.21	
Orange	2294	11.88	66.09	
Lemon	2071	10.73	76.82	
Chlorophylle	1096	5.68	82.50	B
Apple	811	4.20	86.70	
Licorice	664	3.44	90.14	
Grape	583	3.02	93.16	
Reglisse	220	1.14	94.30	C
Strong	207	1.07	95.37	
Peach	200	1.03	96.40	
Citrus-fresh	194	1.01	97.41	
Mini orange	124	0.64	98.05	
Mini apple	120	0.62	98.67	
Mini lemon	117	0.61	99.28	
Mini strawberry	90	0.47	99.75	
Grapefruit	49	0.25	100	

Table 8: Results ABC-analysis.

B The mixed Erlang distribution

A mixed Erlang distribution is a mixture of two Erlang distributions (see e.g. Tijms [7, pp. 358-361]). Variables that are mixed Erlang distributed can be interpreted as a random sum of independent exponential variables. Let X be a mixed Erlang distributed variable. The density f of X is given by

$$f(x) = p\mu_1^{k_1}\frac{x^{k_1-1}}{(k_1-1)!}e^{-\mu_1 x} + (1-p)\mu_2^{k_2}\frac{x^{k_2-1}}{(k_2-1)!}e^{-\mu_2 x},\ x \geqslant 0, \tag{31}$$

where $k_1, k_2 \in \mathbb{N}$.

Let $c_V := \sigma_V/\mu_V$ be the coefficient of variation of V. If the squared coefficient of variation of the demand for a certain taste is larger than 0.5, the following mixed Erlang density can be used:[1]

$$f_V(x) = p\mu_1 e^{-\mu_1 x} + (1-p)\mu_2 e^{-\mu_2 x},\ x \geqslant 0 \tag{32}$$

This is a special case of (31) with $k_1 = k_2 = 1$. According to Janssen [1] the parameters μ_1, μ_2 and p satisfy the following equations:

$$\begin{aligned} \mu_1 &= \frac{2}{\mu_V}\left(1 + \sqrt{\frac{c_V^2 - \frac{1}{2}}{c_V^2 + 1}}\right) \\ \mu_2 &= \frac{4}{\mu_V} - \mu_1 \\ p &= \frac{\mu_1(\mu_2 \mu_V - 1)}{\mu_2 - \mu_1} \end{aligned} \tag{33}$$

Here μ_V is the mean of the mixed Erlang distribution for this specific A- or B-taste item.

If the squared coefficient of variation of the demand for a certain taste is smaller than 0.5, the following mixed Erlang density can be used:

$$f_V(x) = p\mu^{k-1}\frac{x^{k-2}}{(k-2)!}e^{-\mu x} + (1-p)\mu^k \frac{x^{k-1}}{(k-1)!}e^{-\mu x},\ x \geqslant 0, \tag{34}$$

This is a special case of (31) with $k_1 = k - 1$ and $k_2 = k$. The parameters satisfy the following equations:

$$\begin{aligned} k &= \left\lfloor \frac{1}{c_V^2} + 1 \right\rfloor \\ p &= \frac{1}{1 + c_V^2}\left(kc_V^2 - \sqrt{k(1 + c_V^2) - k^2 c_V^2}\right) \\ \mu &= \frac{k - p}{\mu_V}. \end{aligned} \tag{35}$$

C Solving the service equation

C.1 The procedure

In order to solve the service equation (10), one has to deal with integrals of gamma densities, since the demand during the pseudo lead time plus undershoot is assumed to be gamma distributed. Recall that f_Z is the density of the demand during the pseudo lead time plus undershoot. It holds that

$$f_Z(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha - 1} e^{-\lambda x},\ x > 0. \tag{36}$$

[1] Note that the boundary for c_V^2 is not 1 (as in Tijms [7]) but 0.5. This is done as experiments showed that better percentile estimations in the upper tail were obtained then.

The distribution of Z can be approximated by a mixed Erlang distribution, see appendix B, to simplify the integrals in the service equation. The mixed Erlang density f of Z is given by

$$f(x) = p\mu_1^{k_1}\frac{x^{k_1-1}}{(k_1-1)!}e^{-\mu_1 x} + (1-p)\mu_2^{k_2}\frac{x^{k_2-1}}{(k_2-1)!}e^{-\mu_2 x},\ x \geqslant 0, \tag{37}$$

where $k_1, k_2 \in \mathbb{N}$.

Let c_Z^2 be the squared coefficient of variation of Z. If $c_Z^2 \leqslant 0.5$, the density of Z is approximated by the mixed Erlang density

$$f(x) = p\mu^{k-1}\frac{x^{k-2}}{(k-2)!}e^{-\mu x} + (1-p)\mu^{k}\frac{x^{k-1}}{(k-1)!}e^{-\mu x},\ x \geqslant 0, \tag{38}$$

with

$$\begin{aligned} k &= \left\lfloor \frac{1}{c_Z^2} + 1 \right\rfloor \\ p &= \frac{1}{1+c_Z^2}\left(kc_Z^2 - \sqrt{k(1+c_Z^2) - k^2 c_Z^2}\right) \\ \mu &= \frac{k-p}{\mu_Z} \end{aligned} \tag{39}$$

An Erlang distribution is the same as a gamma distribution, except for the fact that the shape parameter must be an integer. Consider the density of the Erlang(r, λ) distribution, with r a positive integer number and $\lambda > 0$:

$$f_{r,\lambda}(x) = \frac{\lambda^r}{(r-1)!}x^{r-1}e^{-\lambda x}. \tag{40}$$

To simplify expressions, define two auxiliary functions, see Valkenburg [8, page 112]:

$$G_\lambda(j) := \frac{(\lambda s)^j}{j!}e^{-\lambda s} \tag{41}$$

$$H_\lambda(r) := \sum_{j=0}^{r} G_\lambda(j) = \frac{1}{\lambda s}\sum_{j=0}^{r+1} jG_\lambda(j) \tag{42}$$

Using these definitions, the integrals in (10) can be simplified:

$$\int_s^\infty (x-s)f(x)\,\mathrm{d}x = \frac{1}{\mu}\sum_{j=0}^{k-1}(k-j-p)G_\mu(j) = \frac{e^{-\mu s}}{\mu}\sum_{j=0}^{k-1}(k-j-p)\frac{(\mu s)^j}{j!} \tag{43}$$

$$\int_{s+Q}^\infty (x-(s+Q))f(x)\,\mathrm{d}x = \frac{e^{-\mu(s+Q)}}{\mu}\sum_{j=0}^{k-1}(k-j-p)\frac{(\mu(s+Q))^j}{j!}. \tag{44}$$

If $c_Z^2 > 0.5$, the density of Z is approximated by the following mixed Erlang density:

$$f_Z(x) = p\mu_1 e^{-\mu_1 x} + (1-p)\mu_2 e^{-\mu_2 x},\ x \geqslant 0 \tag{45}$$

where

$$\begin{aligned}
\mu_1 &= \frac{2}{\mu_Z}\left(1+\sqrt{\frac{c_Z^2-\frac{1}{2}}{c_Z^2+1}}\right) \\
\mu_2 &= \frac{4}{\mu_Z}-\mu_1 \\
p &= \frac{\mu_1(\mu_2\mu_Z-1)}{\mu_2-\mu_1}
\end{aligned} \tag{46}$$

Here μ_Z is the mean of the mixed Erlang distribution. The integrals in (10) can now be simplified as follows:

$$\int_s^\infty (x-s)f(x)\,\mathrm{d}x = p\frac{1}{\mu_1}H_{\mu_1}(0)+(1-p)\frac{1}{\mu_2}H_{\mu_2}(0) = p\frac{1}{\mu_1}e^{-\mu_1 s}+(1-p)\frac{1}{\mu_2}e^{-\mu_2 s} \tag{47}$$

$$\int_{s+Q}^\infty (x-(s+Q))f(x)\,\mathrm{d}x = p\frac{1}{\mu_1}e^{-\mu_1(s+Q)}+(1-p)\frac{1}{\mu_2}e^{-\mu_2(s+Q)}. \tag{48}$$

A derivation of these results can be found in Valkenburg [8]. After substituting these expressions into the service equation, the optimal value of the reorder level can easily be solved.

C.2 An example for an A-taste item

To give an example, let's consider Mentos Mint and assume management has specified a P_2-service level of respectively $95\%, 96\%, \ldots, 99.9\%$. The following additional input parameters with their respective values are needed:

- the review interval $R = 1$ day
- the order quantity $Q = 64.8$ tons
- the average lead time of a production order $\mu_L = 1.208$ days
- the standard deviation of the production lead time $\sigma_L = 0.017$ days
- the average demand quantity $\mu_D = 53.63$ tons
- the standard deviation of the demand quantity $\sigma_D = 9.59$ tons
- the average interarrival time $\mu_A = 1$ day
- the standard deviation of the interarrival times $\sigma_A = 0$.

Now compute the following second moment of the demand quantity:

$$\alpha_2 = \sigma_D^2 + \mu_D^2 = (9.59)^2 + (53.63)^2 = 2968.15.$$

From section 3.4 it follows that the parameters of the demand process have values $\lambda = 0.5831$ and $\alpha = 31.27$. From section 3.6 it follows that $\mu_U \approx 27.67$ and $\sigma_U^2 \approx 286.89$. Furthermore, $\mu_V \approx 37.97$, $\sigma_V^2 \approx 305.63$, $\mu_Z \approx 65.64$, $\sigma_Z^2 \approx 592.52$, $c_Z^2 \approx 0.1375$. The mixed Erlang approximation of Z has parameter values $k = 8$, $p = 0.4860$ and $\mu = 0.1145$. The service equation has now become an equation with only one unknown. It can be solved by a numerical procedure.

For the A-taste item Mentos Mint the production moment (s) and the average physical inventory level are given in table 9 (see appendix D for details about the calculation).

P_2	s (tons)	Average physical inventory (tons)
0.95	87	81.45
0.96	91	85.45
0.97	96	90.44
0.98	102	96.44
0.99	113	107.43
0.995	123	117.43
0.999	146	140.43

Table 9: Production moment and average physical inventory

D The calculation of the average physical stock for an (s, Q) inventory model, using mixed Erlang distributions

The average physical inventory level (see formula (16)) can be approximated using mixed Erlang distributions. Since the coefficients of variation of demand during the lead time will most probably differ for A-tastes and B-tastes, we will separate two cases.

D.1 A-tastes

Let c_V^2 be the squared coefficient of variation of V. Since probably $c_V^2 \leqslant 0.5$ for A-tastes, the density of V is approximated by the mixed Erlang density

$$f(x) = p\mu^{k-1}\frac{x^{k-2}}{(k-2)!}e^{-\mu x} + (1-p)\mu^k\frac{x^{k-1}}{(k-1)!}e^{-\mu x}, \; x \geqslant 0, \tag{49}$$

with

$$k = \left\lfloor \frac{1}{c_V^2} + 1 \right\rfloor$$

$$
\begin{aligned}
p &= \frac{1}{1+c_V^2}\left(kc_V^2 - \sqrt{k(1+c_V^2) - k^2c_V^2}\right) \\
\mu &= \frac{k-p}{\mu_V}
\end{aligned}
$$

Let $f_{r,\lambda}(x)$ be the density of Erlang(r,λ). To simplify expressions, the auxiliary functions are used again, see (41) and (42). Consider three properties of the Erlang(k,μ) distribution:

$$
\begin{aligned}
\int_0^s f(x)\,\mathrm{d}x &= 1 - H_\mu(k-1) && (50)\\
\int_0^s xf(x)\,\mathrm{d}x &= \frac{k}{\mu}\left(1 - H_\mu(k)\right) && (51)\\
\int_0^s x^2f(x)\,\mathrm{d}x &= \frac{(k+1)k}{\mu^2}\left(1 - H_\mu(k+1)\right) && (52)
\end{aligned}
$$

The expression $\int_0^s (s-x)^2 f(x)\,\mathrm{d}x$ in (16) will be simplified using these properties. The derivation is given below.

$$
\begin{aligned}
\int_0^s (s-x)^2 f(x)\,\mathrm{d}x &= \int_0^s (x^2 - 2sx + s^2)\left(pf_{k-1,\mu}(x)\,\mathrm{d}x + (1-p)f_{k,\mu}(x)\,\mathrm{d}x\right) \\
= \; & p\int_0^s (x^2 - 2sx + s^2)f_{k-1,\mu}(x)\,\mathrm{d}x + (1-p)\int_0^s (x^2 - 2sx + s^2)f_{k,\mu}(x)\,\mathrm{d}x \\
= \; & p\frac{k(k-1)}{\mu^2}\left(1 - H_\mu(k)\right) + (1-p)\frac{(k+1)k}{\mu^2}\left(1 - H_\mu(k+1)\right) \\
& -2sp\frac{k-1}{\mu}\left(1 - H_\mu(k-1)\right) - 2s(1-p)\frac{k}{\mu}\left(1 - H_\mu(k)\right) \\
& +s^2p\left(1 - H_\mu(k-2)\right) + s^2(1-p)\left(1 - H_\mu(k-1)\right) \qquad (53)
\end{aligned}
$$

To find a simplified expression of $\int_0^{s+Q} (s+Q-x)^2 f(x)\,\mathrm{d}x$, replace s by $s+Q$ in (53). Note that also in definitions (41) and (42) s should be replaced by $s+Q$. This gives

$$
\begin{aligned}
& \int_0^{s+Q} (s+Q-x)^2 f(x)\,\mathrm{d}x = \\
= \; & p\frac{k(k-1)}{\mu^2}\left(1 - H_\mu(k)\right) + (1-p)\frac{(k+1)k}{\mu^2}\left(1 - H_\mu(k+1)\right) \\
& -2(s+Q)\,p\frac{k-1}{\mu}\left(1 - H_\mu(k-1)\right) - 2(s+Q)(1-p)\frac{k}{\mu}\left(1 - H_\mu(k)\right) \\
& +(s+Q)^2p\left(1 - H_\mu(k-2)\right) + (s+Q)^2(1-p)\left(1 - H_\mu(k-1)\right).
\end{aligned}
$$

After substituting these expressions into (16), the average physical stock can easily be computed, using the optimal value of the reorder point s.

D.2 B-tastes

For B-tastes the squared coefficient of variation of V is probably larger than 0.5. Therefore, the density of V is approximated by the mixed Erlang density

$$f(x) = p\mu_1 e^{-\mu_1 x} + (1-p)\mu_2 e^{-\mu_2 x},\ x \geqslant 0. \tag{54}$$

Similar to the previous section, $\int_0^s (s-x)^2 f(x)\,\mathrm{d}x$ can again be simplified.

$$\begin{aligned}
&\int_0^s (s-x)^2 f(x)\,\mathrm{d}x = \int_0^s (x^2 - 2sx + s^2)\left(p f_{1,\mu_1}(x)\,\mathrm{d}x + (1-p) f_{1,\mu_2}(x)\,\mathrm{d}x\right) \\
= \ & p\int_0^s (x^2 - 2sx + s^2) f_{1,\mu_1}(x)\,\mathrm{d}x + (1-p)\int_0^s (x^2 - 2sx + s^2) f_{1,\mu_2}(x)\,\mathrm{d}x \\
= \ & \frac{2p}{\mu_1^2}\left(1 - H_{\mu_1}(2)\right) - \frac{2sp}{\mu_1}\left(1 - H_{\mu_1}(1)\right) + s^2 p\left(1 - H_{\mu_1}(0)\right) \\
& + \frac{2(1-p)}{\mu_2^2}\left(1 - H_{\mu_2}(2)\right) - \frac{2s(1-p)}{\mu_2}\left(1 - H_{\mu_2}(1)\right) + s^2(1-p)\left(1 - H_{\mu_2}(0)\right).
\end{aligned}$$

Analogously, it follows that

$$\begin{aligned}
&\int_0^{s+Q} (s+Q-x)^2 f(x)\,\mathrm{d}x = \\
= \ & \frac{2p}{\mu_1^2}\left(1 - H_{\mu_1}(2)\right) - \frac{2(s+Q)p}{\mu_1}\left(1 - H_{\mu_1}(1)\right) + (s+Q)^2 p\left(1 - H_{\mu_1}(0)\right) \\
& + \frac{2(1-p)}{\mu_2^2}\left(1 - H_{\mu_2}(2)\right) - \frac{2(s+Q)(1-p)}{\mu_2}\left(1 - H_{\mu_2}(1)\right) \\
& + (s+Q)^2(1-p)\left(1 - H_{\mu_2}(0)\right).
\end{aligned}$$

E Closed form solution

The derivation of the closed form solution of $P(S_k \leqslant L)$ is given below. If the coefficient of variation of S_k is smaller than 0.5, the solution will be as follows.

$$P(S_k \leqslant L) = \int_0^\infty P(S_k \leqslant L \mid L = s) g(s)\,\mathrm{d}s = \int_0^\infty \left(\int_0^s f(x)\,\mathrm{d}x\right) g(s)\,\mathrm{d}s. \tag{55}$$

Recall that $k_1 = k-1,\ k_2 = k,\ \varphi,\ p_1 = p$ and $p_2 = 1-p$ are the parameters of the mixed Erlang distribution of S_k. Furthermore, $l_1,\ l_2,\ \rho,\ q_1 = q$ and $q_2 = 1-q$ are the parameters of the mixed Erlang distribution of L. To find a closed form solution, first determine $\int_0^s f(x)\,\mathrm{d}x$.

$$\int_0^s f(x)\,\mathrm{d}x = \int_0^s \sum_{j=1}^{2} p_j \varphi^{k_j} \frac{x^{k_j-1}}{(k_j-1)!} e^{-\varphi x}\,\mathrm{d}x$$

$$
\begin{aligned}
&= \sum_{j=1}^{2} p_j \int\limits_0^s \varphi^{k_j} \frac{x^{k_j-1}}{(k_j-1)!} e^{-\varphi x}\, \mathrm{d}x \\
&= \sum_{j=1}^{2} p_j \int\limits_0^s f_{k_j,\varphi}(x)\, \mathrm{d}x \\
&= \sum_{j=1}^{2} p_j \left(1 - H_\varphi(k_j-1)\right),
\end{aligned}
$$

where $f_{k_j,\varphi}(x)$ is the density of the Erlang(k_j,φ) distribution and the function H is defined as in (42). Use this result to determine $P(S_k \leqslant L)$. In the derivation, $f_{l_i,\rho}(s)$ is the density of the Erlang(l_i,ρ) distribution.

$$
\begin{aligned}
P(S_k \leqslant L) &= \int\limits_0^\infty \left(\int\limits_0^s f(x)\, \mathrm{d}x \right) g(s)\, \mathrm{d}s \\
&= \int\limits_0^\infty \sum_{j=1}^{2} p_j \left(1 - H_\varphi(k_j-1)\right) g(s)\, \mathrm{d}s \\
&= \int\limits_0^\infty \sum_{j=1}^{2} p_j \left(1 - H_\varphi(k_j-1)\right) \sum_{i=1}^{2} q_i \rho^{l_i} \frac{s^{l_i-1}}{(l_i-1)!} e^{-\rho s}\, \mathrm{d}s \\
&= \sum_{j=1}^{2} \sum_{i=1}^{2} p_j q_i \int\limits_0^\infty \left(1 - H_\varphi(k_j-1)\right) \rho^{l_i} \frac{s^{l_i-1}}{(l_i-1)!} e^{-\rho s}\, \mathrm{d}s \\
&= \sum_{j=1}^{2} \sum_{i=1}^{2} p_j q_i \int\limits_0^\infty \left(1 - H_\varphi(k_j-1)\right) f_{l_i,\rho}(s)\, \mathrm{d}s \\
&= \sum_{j=1}^{2} \sum_{i=1}^{2} p_j q_i \left(\int\limits_0^\infty f_{l_i,\rho}(s)\, \mathrm{d}s - \int\limits_0^\infty H_\varphi(k_j-1) f_{l_i,\rho}(s)\, \mathrm{d}s \right) \\
&= \sum_{j=1}^{2} \sum_{i=1}^{2} p_j q_i \left(1 - \int\limits_0^\infty H_\varphi(k_j-1) f_{l_i,\rho}(s)\, \mathrm{d}s \right) \\
&= \sum_{j=1}^{2} \sum_{i=1}^{2} p_j q_i \left(1 - \frac{\rho^{l_i}}{(l_i-1)!} \int\limits_0^\infty s^{l_i-1} e^{-(\varphi+\rho)s} \sum_{t=0}^{k_j-1} \frac{(\varphi s)^t}{t!}\, \mathrm{d}s \right) \\
&= \sum_{j=1}^{2} \sum_{i=1}^{2} p_j q_i \left(1 - \sum_{t=0}^{k_j-1} \frac{(t+l_i-1)!}{t!(l_i-1)!} \frac{\varphi^t \rho^{l_i}}{(\varphi+\rho)^{t+l_i}} \right) \\
&= \sum_{j=1}^{2} \sum_{i=1}^{2} p_j q_i \left(1 - \sum_{t=0}^{k_j-1} \binom{t+l_i-1}{t} \frac{\varphi^t \rho^{l_i}}{(\varphi+\rho)^{t+l_i}} \right).
\end{aligned} \tag{56}
$$

If the coefficient of variation of S_k is larger than 0.5, the solution will be as follows.

$$P(S_k \leqslant L) = \sum_{j=1}^{2} \sum_{i=1}^{2} p_j q_i \left(1 - \left(\frac{\rho}{\varphi_j + \rho}\right)^{l_i}\right) \tag{57}$$

As before, $\varphi_1,\ \varphi_2,\ p_1 = p$ and $p_2 = 1 - p$ are the parameters of the mixed Erlang distribution of S_k. Furthermore, $l_1,\ l_2,\ \rho,\ q_1 = q$ and $q_2 = 1 - q$ are the parameters of the mixed Erlang distribution of L.

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