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On properness and protectiveness in two person multicriteria games

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Abstract

This paper extends the concepts of proper equilibria, protective behaviour and prudent behaviour to multicriteria games. Three types of proper equilibria based on different types of domination are introduced. It is shown that protective behaviour coincides with prudent behaviour. Possible relations and existence are analyzed.

Keywords: Multicriteria games, proper, protective, prudent.

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1 Introduction

Multicriteria games were first analyzed by Blackwell (1956). Here, contrary to classical strategic form games, players face multiple objectives and the payoffs according to the different objectives of a player can not be compared.

The notion of equilibrium in non-cooperative games in strategic form is introduced by Nash (1951). The set of Nash equilibria can be quite large, or can contain outcomes that are not very suitable. To deal with this problem various refinements of Nash equilibria are given in literature. For example Selten (1975) introduces the notion of perfect equilibria, i.e. equilibria that are resistent against small mistakes in the actions of the players. Proper equilibria are introduced by Myerson (1978). These equilibria are based on the idea that players assure that costly mistakes occur with smaller probability than less costly mistakes. A survey can be found in Van Damme (1987). Fiestras-Janeiro, Borm, and Van Megen (1998) study protective and prudent behaviour in non-cooperative games. They prove that for matrix games the set of protective equilibria coincides with the set of proper equilibria.

Shapley (1959) introduces (Pareto) equilibria for two person multicriteria games and shows the correspondence with Nash equilibria of so called trade-off games. Since payoffs of different criteria cannot be compared, the set of equilibria can be quite large, but rather surprisingly, refinements of Pareto equilibria did not receive much attention in the literature. Borm, Van Megen, and Tijs (1999) extend the notion of perfect equilibria to multicriteria games, prove existence and provide several characterizations. They also hint at an extension of proper equilibria to a multicriteria setting.

In this paper we analyze three extensions of the notion of proper equilibria based on different types of domination. One of these extensions contains the set of proper equilibria of trade-off games. The extensions are illustrated by means of an example, clarifying the differences between the three types of domination. Moreover, we study an extension of protective and prudent behaviour to multicriteria games. We show that protective behaviour coincides with prudent behaviour, and prove existence. Two examples illustrate some special aspects of protective behaviour in a zero-sum like environment.

2 Preliminaries: multicriteria games

We consider mixed extensions of two person finite strategic multicriteria games, in which player one can choose between m and player two between n pure strategies. The sets $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots f_n\}$ contain the pure strategies of player one and two respectively. Player one and two take into account r and s criteria respectively. Let $A = (A_1, \ldots, A_r)$ and $B = (B_1, \ldots, B_s)$ be two vectors of real valued $m \times n$ matrices in which the rows correspond to pure strategies $e_1, \ldots, e_m$ and the columns to pure strategies $f_1, \ldots, f_n$. The matrix A_t (B_t) can be interpreted as the payoff matrix of player one (two) with respect to criterium t. A two person multicriteria game can then be described by a tuple $\Gamma = \langle \Delta_m, \Delta_n, u_1, u_2 \rangle$. Here Δ_m, a shorthand for $\Delta\{e_1, \ldots, e_m\}$, denotes the set of all mixed strategies of player one and Δ_n ($= \Delta\{f_1, \ldots, f_n\}$) is the set of all mixed strategies of player two. The (relative) interior of e.g. Δ_m is denoted by $\mathring{\Delta}_m$: it represents the set of all completely mixed strategies. The functions u_1 and u_2 are the (vector) payoff functions of player one and two. For all $(p, q) \in \Delta_m \times \Delta_n$

$$u_1(p, q) = (pA_1q, \ldots, pA_rq), \; u_2(p, q) = (pB_1q, \ldots, pB_sq),$$

or, in short notation,

$$u_1(p, q) = pAq, \; u_2(p, q) = pBq.$$

We will usually describe a two person multicriteria game by the tuple $\Gamma = \langle \Delta_m, \Delta_n, A, B \rangle$. The set of all $r \times s$ multicriteria games of this type is denoted by $MG(r \times s, m \times n)$.

The (Pareto) equilibria of multicriteria games can be described by the notion of best reply functions. Take strategies $p \in \Delta_m$ and $q \in \Delta_n$. Then p is a best reply to q ($p \in B_1(\Gamma, q)$) if there is no strategy $\bar{p} \in \Delta_m$ such that $pAq < \bar{p}Aq$ (i.e. $(pAq)_t < (\bar{p}Aq)_t$ for all $t \in \{1, \ldots r\}$). Similarly q is a best reply to p ($q \in B_2(\Gamma, p)$)if there is no strategy $\bar{q} \in \Delta_n$ such that $pBq < pB\bar{q}$. The strategy combination (p, q) is an equilibrium of Γ if p is a best reply to q and q is a best reply to p. The set of all equilibria of Γ is denoted by $E(\Gamma)$. Note that we use the notion of strong dominance, which results in weak (Pareto) equilibria.

There is a direct connection between multicriteria games and their corresponding trade off games in which the various criteria of each player are weighted.

Definition 2.1 *Let $\Gamma = \langle \Delta_m, \Delta_n, A, B \rangle \in MG(r \times s, m \times n)$ be a two person multicriteria game and let $\lambda_1 \in \Delta_r$, $\lambda_2 \in \Delta_s$ be trade off (or weight) vectors for player one and two respectively. Denote $\lambda = (\lambda_1, \lambda_2)$. The unicriterium game $\Gamma(\lambda)$ is defined as the bimatrix game with mixed strategy spaces Δ_m and Δ_n and $m \times n$ payoff matrices $A(\lambda_1)$, $B(\lambda_2)$ given by*

$A(\lambda_1)_{ij} = \sum_{t=1}^{r} (\lambda_1)_t (A_t)_{ij}$ and $B(\lambda_2)_{ij} = \sum_{t=1}^{s} (\lambda_2)_t (B_t)_{ij}$. The set of all Nash equilibria of the bimatrix game $\Gamma(\lambda)$ is denoted by $NE(\Gamma(\lambda))$.

The following theorem states that each equilibrium of a multicriteria game Γ can be found as a Nash equilibrium of an unicriterium game, derived from Γ by using a suitable tradeoff vector.

Theorem 2.1 (Shapley (1959)) *Let $\Gamma = \langle \Delta_m, \Delta_n, A, B \rangle$ be a multicriteria game. Then*

$$E(\Gamma) = \{(p, q) \in \Delta_m \times \Delta_n \mid (p, q) \in NE(\Gamma(\lambda)) \text{ for some } \lambda \in \Delta_r \times \Delta_s)\}.$$

Borm, Van Megen, and Tijs (1999) provide a characterization of (Pareto) equilibria points in terms of carriers and efficient best reply sets. Let $(p, q) \in \Delta_m \times \Delta_n$. The carrier of p with respect to $\Gamma = \langle \Delta_m, \Delta_n, A, B \rangle$ is defined as $C(\Gamma, p) := \{i \in \{1, \ldots, m\} \mid p_i > 0\}$. $I \subseteq \{1, \ldots, m\}$ is efficient for player one with respect to q in Γ if for all strategies $p \in \Delta_m$ with $C(\Gamma, p) \subseteq I$ it holds that pAq is undominated in the polytope $P_1(\Gamma, q) := \mathrm{conv}\{e_i Aq \mid i \in \{1, \ldots m\}\} \subset \mathbb{R}^r$ of all possible payoff vectors with respect to q. Note that if $I \subseteq \{1, \ldots, m\}$ is efficient, then each subset $K \subseteq I$ is efficient too. $I \subseteq \{1, \ldots, m\}$ is an *efficient pure best reply set* for player one with respect to q in Γ if I is efficient with respect to q in Γ and there does not exist an efficient set $K \subseteq \{1, \ldots, m\}$ such that $I \subseteq K$ and $I \neq K$. The set of all efficient pure best reply sets of player one with respect to q in Γ is denoted by $\mathcal{E}_1(\Gamma, q)$. In the same way one can define efficient sets and efficient pure best reply sets for player two. The following theorem shows the connection between best replies and efficient pure best reply sets.

Theorem 2.2 (Borm et al. (1999)) *Let $\Gamma = \langle \Delta_m, \Delta_n, A, B \rangle$ be a multicriteria game and $(p, q) \in \Delta_m \times \Delta_n$. Then $p \in B_1(\Gamma, q)$ if and only if $C(\Gamma, p) \subseteq I$ for some $I \in \mathcal{E}_1(\Gamma, q)$ and $q \in B_2(\Gamma, p)$ if and only if $C(\Gamma, q) \subseteq J$ for some $J \in \mathcal{E}_2(\Gamma, p)$.*

3 Proper equilibria

In this section we introduce three types of proper equilibria of multicriteria games.

Borm et al. (1999) introduce perfect equilibria of multicriteria games, generalizing the notion of perfect Nash equilibria introduced by Selten (1975). Take $\varepsilon > 0$. A strategy combination $(p, q) \in \mathring{\Delta}_m \times \mathring{\Delta}_n$ is called an *ε-perfect pair* if there exists an $I \in \mathcal{E}_1(\Gamma, q)$ and an $J \in \mathcal{E}_2(\Gamma, p)$ such that $p_i \leq \varepsilon$, for all $i \notin I$ and $q_j \leq \varepsilon$ for all $j \notin J$.

Definition 3.1 *Let* $\Gamma = \langle \Delta_m, \Delta_n, A, B \rangle$ *be a two person multicriteria game. A strategy combination* (p, q) *is a perfect equilibrium of* Γ *if there exists a sequence* $\{\varepsilon_t\}_{t=1}^{\infty} \subset (0, \infty)$ *converging to zero and a sequence* $\{(p^t, q^t)\}_{t=1}^{\infty} \subset \mathring{\Delta}_m \times \mathring{\Delta}_n$ *converging to* (p, q)*, such that* (p^t, q^t) *is an* ε_t*-perfect pair for all* t*.*

Another refinement of Nash equilibria are the proper equilibria, introduced in Myerson (1978). Let $\Gamma = (\Delta_m, \Delta_n, A, B)$ be an $m \times n$ bimatrix game, i.e. $\Gamma \in MG(1 \times 1, m \times n)$. Take $\varepsilon \in (0, 1)$. Then a strategy combination $(p, q) \in \mathring{\Delta}_m \times \mathring{\Delta}_n$ is an *ε-proper pair* if for all $i, k \in \{1, \ldots, m\}$, $j, l \in \{1, \ldots, n\}$ we have:

$$\begin{aligned} e_i A q < e_k A q \quad &\Rightarrow \quad p_i \leq \varepsilon p_k, \\ pBf_j < pBf_l \quad &\Rightarrow \quad q_j \leq \varepsilon q_l. \end{aligned}$$

A strategy combination $(p, q) \in \Delta_m \times \Delta_n$ is a *proper equilibrium* of Γ if there exist a sequence $\{\varepsilon_t\}_{t=1}^{\infty}$ of positive numbers converging to zero and a sequence $\{(p^t, q^t)\}_{t=1}^{\infty}$ of completely mixed strategy combinations converging to (p, q) such that (p^t, q^t) is an ε_t-proper pair for all $t \in \mathbb{N}$. The set of all proper equilibria of Γ is denoted by $PR(\Gamma)$.

The idea behind proper equilibria is that costly mistakes occur with relatively smaller probabilities than less costly mistakes, whereas in perfect equilibria mistakes are not compared at all. Note that in unicriterium games any two pure strategies can be mutually compared and as a consequence the set of pure strategies can be partitioned into well-defined levels of quality, given a strategy of the opponent. In a multicriteria environment we do not have this completeness and there are several options on how to deal with this. We introduce three types of proper equilibria in multicriteria games each based on another way of comparing pure strategies. To do so, we first introduce some terminology.

Let $\Gamma = \langle \Delta_m, \Delta_n, A, B \rangle \in MG(r \times s, m \times n)$ and take $q \in \Delta_n$. The first level of best replies of player one against q is the set of all pure strategies contained in some efficient pure best reply sets with respect to q. The t'th level of best replies is constructed by considering the best replies with respect to q taking into account all strategies that are not in the first $t - 1$ levels. Formally this boils down to the following definition.

$$\left\{ \begin{array}{rcl} M^1(q) & := & \{1, \ldots, m\}, \quad m^1(q) := m \\ \Gamma_1^1(q) & := & \Gamma = \langle \Delta_m, \Delta_n, A, B \rangle \\ \mathcal{E}_1^1(\Gamma, q) & := & \mathcal{E}_1(\Gamma_1^1(q), q) \\ L_1^1(q) & := & \{i \in M^1(q) \mid i \in I \text{ for some } I \in \mathcal{E}_1^1(\Gamma, q)\}, \end{array} \right.$$

recursively, for each $t \in \mathbb{N}$, $t \geq 2$:

$$\begin{cases} M^t(q) & := & M^{t-1}(q) \backslash L_1^{t-1}(q), \quad m^t(q) := |M^t(q)| \\ \Gamma_1^t(q) & := & \langle \Delta_{m^t(q)}, \Delta_n, A^t, B^t \rangle \\ \mathcal{E}_1^t(\Gamma, q) & := & \mathcal{E}_1(\Gamma_1^t(q), q) \\ L_1^t(q) & := & \{i \in M^t(q) \mid i \in I \text{ for some } I \in \mathcal{E}_1^t(\Gamma, q)\}. \end{cases}$$

Here $\Delta_{m^t(q)} = \Delta(\{e_i\}_{i \in M^t(q)})$ is the restricted mixed strategy space and A^t and B^t are the corresponding vectors of $m^t(q) \times n$ submatrices of the vectors of matrices A and B corresponding to the rows in $M^t(q)$. Note that there is an unique $z \in \mathbb{N}$ such that $M^{z-1}(q) \neq \emptyset$ and $M^z(q) = \emptyset$. In the same way one can define $N^t(p)$, $n^t(p)$, $\Gamma_2^t(p)$, $\mathcal{E}_2^t(\Gamma, p)$ and $L_2^t(p)$ for $p \in \Delta_m$.

Let $\varepsilon \in (0, 1)$ and $\Gamma = \langle \Delta_m, \Delta_n, A, B \rangle$ be a multicriteria game. The completely mixed strategy combination $(p, q) \in \mathring{\Delta}_m \times \mathring{\Delta}_n$ is

- a *pure domination ε-proper pair* of Γ if for all $i, k \in \{1, \ldots, m\}$ such that $e_i Aq < e_k Aq$ it holds that $p_i \leq \varepsilon p_k$ and if for all $j, l \in \{1, \ldots, n\}$ such that $pBf_j < pBf_l$ it holds that $q_j \leq \varepsilon q_l$.
- a *level domination ε-proper pair* if for all $i \in L_1^t(q)$: if there is a $\bar{p} \in \Delta_m$ and $I \in \mathcal{E}_1^{t-1}(\Gamma, q)$ with $C(\bar{p}) \subseteq I$ such that $e_i Aq \leq \bar{p}Aq$, then $p_i \leq \varepsilon p_k$ for all $k \in I$ and if for all $j \in L_2^t(p)$: if there is a $\bar{q} \in \Delta_n$ and $J \in \mathcal{E}_2^{t-1}(\Gamma, p)$ with $C(\bar{q}) \subseteq J$ such that $pBf_j \leq pB\bar{q}$, then $q_j \leq \varepsilon q_l$ for all $l \in J$.
- a *level ε-proper pair* if for all $i, k \in \{1, \ldots, m\}$: if $i \in L_1^t(q)$ and $k \in L_1^{t-1}(q)$, then $p_i \leq \varepsilon p_k$ and if for all $j, l \in \{1, \ldots, n\}$: if $j \in L_2^t(p)$ and $l \in L_2^{t-1}(p)$, then $q_j \leq \varepsilon q_l$.

We now introduce three types of proper equilibria in multicriteria games. The difference between the three types of proper equilibria are due to the different notions of domination between pure strategies used. The first type of proper equilibria is based on standard vector domination in all coordinates (criteria), this leads to *pure domination proper equilibria.* In the second type of domination a strategy e_i dominates e_k, if e_i is contained in an efficient pure best reply set of a lower level such that there exist a mixed strategy (with carrier within this pure best reply set) such that e_k is dominated in all coordinates by this mixed strategy. This second type of proper equilibria is called *level domination proper equilibria.* Finally, in the third type of domination a strategy of a higher level is dominated by all strategies of lower levels. This leads to *level proper equilibria.* Note that in unicriterium games all three concepts of domination coincide. The idea of level proper equilibria is shortly mentioned in Borm et al. (1999).

Definition 3.2 *Let $\Gamma = \langle \Delta_m, \Delta_n, A, B \rangle$ be a multicriteria game. The strategy combination $(p, q) \in \Delta_m \times \Delta_n$ is a pure domination proper equilibrium of Γ if (p, q) is a perfect equilibrium of Γ and there exists a sequence $\{\varepsilon_t\}_{t=1}^{\infty} \subset (0, 1)$ converging to zero and a sequence $\{(p^t, q^t)\}_{t=1}^{\infty} \subset \mathring{\Delta}_m \times \mathring{\Delta}_n$ converging to (p, q), such that (p^t, q^t) is a dominated ε_t-proper pair for all t. Similarly the strategy combination $(p, q) \in \Delta_m \times \Delta_n$ is a level domination proper (level proper) equilibrium of Γ if (p, q) is a perfect equilibrium of Γ and there exists a sequence $\{\varepsilon_t\}_{t=1}^{\infty} \subset (0, 1)$ converging to zero and a sequence $\{(p^t, q^t)\}_{t=1}^{\infty} \in \mathring{\Delta}_m \times \mathring{\Delta}_n$ converging to (p, q), such that (p^t, q^t) is a level domination (level proper) ε_t-proper pair for all t.*

The set of pure domination proper equilibria of a multicriteria game Γ is denoted by $PDP(\Gamma)$. The sets of level domination proper equilibria and level proper equilibria are denoted by $LDP(\Gamma)$ and $LP(\Gamma)$ respectively.

Proposition 3.1 *Let Γ be a multicriteria game. Then $LP(\Gamma) \subseteq LDP(\Gamma) \subseteq PDP(\Gamma)$.*

Proof: It suffices to note that each level ε-proper pair is a level domination ε-proper pair and each level domination ε-proper pair is a pure domination ε-proper pair. $\square$

In the following theorem it is shown that the set of pure domination proper equilibria contains the set of proper equilibria of all trade-off games.

Theorem 3.1 *Let $\Gamma = \langle \Delta_m, \Delta_n, A, B \rangle$ be a multicriteria game. Then we have that*

$$\{(p, q) \in \Delta_m \times \Delta_n \mid \exists \lambda \in \Delta_r \times \Delta_s : (p, q) \in PR(\Gamma(\lambda))\} \subseteq PDP(\Gamma).$$

Proof: Let $\lambda = (\lambda_1, \lambda_2) \in \Delta_r \times \Delta_s$ and $(p, q) \in PR(\Gamma(\lambda))$. We first show that $(p, q) \in PE(\Gamma)$. This follows immediately from Theorem 5.3 in Borm et al. (1999) and the fact that in unicriterium games the set of proper equilibria is contained in the set of perfect equilibria.

Furthermore, since (p, q) is a proper equilibrium in the unicriterium game $\Gamma(\lambda)$, there exist sequences $\{\varepsilon_t\}_{t=1}^{\infty} \subset (0, 1)$ and $\{(p^t, q^t)\}_{t=1}^{\infty} \subset \mathring{\Delta}_m \times \mathring{\Delta}_n$ such that $\lim_{t\to\infty} \varepsilon_t = 0$ and $\lim_{k\to\infty}(p^t, q^t) = (p, q)$ and (p^t, q^t) is ε_t-proper in $\Gamma(\lambda)$ for all $t \in \mathbb{N}$. Let $t \in \mathbb{N}$ and let $i, k \in \{1, \ldots, m\}$ be such that

$$e_i A q^t < e_k A q^t.$$

Clearly we are finished if we can show that $p_i^t \leq \varepsilon_t p_k^t$.

Since $\lambda_1 \geq 0$ it is true that

$$\begin{aligned} e_i A(\lambda) q^t &= \sum_{d=1}^{r} (\lambda_1)_d e_i A_d q^t \\ &< \sum_{d=1}^{r} (\lambda_1)_d e_k A_d q^t \\ &= e_k A(\lambda) q^t. \end{aligned}$$

From the fact that (p^t, q^t) is ε_t-proper in $\Gamma(\lambda)$ it then follows that:

$$p_i^t \leq \varepsilon_t p_k^t.$$

$\square$

Now the existence of pure domination proper equilibria easily follows from the existence of proper equilibria in unicriterium games as proved in Myerson (1978):

Corollary 3.1 *For every multicriteria game, the set of pure domination proper equilibria is non-empty*[1].

The following example illustrates the differences between the three concepts of properness.

Example 3.1 Consider a $(2 \times 1, 5 \times 4)$ multicriteria game Γ with payoff matrix A and B given by

$$A = \begin{pmatrix} (2,3) & (2,3) & (2,3) & (2,3) \\ (0,3) & (0,3) & (0,3) & (0,3) \\ (6,0) & (6,0) & (6,0) & (6,0) \\ (3,2) & (3,2) & (3,2) & (3,2) \\ (2,1) & (2,1) & (2,1) & (2,1) \end{pmatrix},$$

and

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 1 & 2 \end{pmatrix},$$

[1] We conjecture that the sets of level domination proper and level proper equilibria are non-empty as well.

respectively. Note that the vector payoff of player one is independent of the strategy chosen by player two. The set of possible payoff vectors of player one, $\text{conv}\{e_i Aq \mid i \in \{1, \ldots, 5\}\}$, does not depend on q and is drawn in figure 1. From this figure it is immediately clear that for all $q \in \Delta_4$ the set of first level efficient pure best reply sets is given by $\mathcal{E}_1^1(\Gamma, q) = \{\{1,2\},\{1,3\}\}$, second, that $\mathcal{E}_1^2(\Gamma, q) = \{\{4\}\}$ and $\mathcal{E}_1^3(\Gamma, q) = \{\{5\}\}$. It can easily be verified that the set of perfect equilibria equals the set of Nash equilibria:

$$\begin{aligned} E(\Gamma) &= \text{conv}\{e_1, e_3\} \times \text{conv}\{f_1, f_2, f_4\} \cup \\ &\quad \text{conv}\{e_1, e_2\} \times \{f_1\} \cup \\ &\quad \{e_1\} \times \text{conv}\{f_1, f_2, f_3, f_4\} \\ &= PE(\Gamma). \end{aligned}$$

To find all pure domination proper equilibria, note that the strategy e_5 of player one is purely dominated by e_4. Let $\varepsilon > 0$ and (p, q) a pure domination ε-proper pair. Then it holds that f_4 is dominated by at least one strategy: $pBf_1 > pBf_4$ or $pBf_2 > pBf_4$, (or both), from which it follows that for a pure domination proper equilibrium (p, q) it holds that $q_4 = 0$. By choosing suitable sequences, one can show that all remaining equilibria are pure domination proper. Therefore the set of all pure domination proper equilibria equals

$$\begin{aligned} PDP(\Gamma) &= \text{conv}\{e_1, e_3\} \times \text{conv}\{f_1, f_2\} \cup \\ &\quad \text{conv}\{e_1, e_2\} \times \{f_1\} \cup \\ &\quad \{e_1\} \times \text{conv}\{f_1, f_2, f_3\}. \end{aligned}$$

To find all level domination proper equilibria, note that for all $q \in \Delta_4$, e_4 is dominated by $\bar{p} = \frac{7}{10}e_1 + \frac{3}{10}e_3$ and the carrier of $\bar{p}$ equals $\{1, 3\}$. Hence if (p, q) is a level domination ε-proper pair, then it holds that $p_5 \leq \varepsilon p_4$, $p_4 \leq \varepsilon p_1$, $p_4 \leq \varepsilon p_3$. It follows that $pBf_3 \leq pBf_1$ for small ε. If $p_2 = p_4$, both f_1 and f_2 are best replies. The set of level domination proper equilibria becomes

$$\begin{aligned} LDP(\Gamma) &= \text{conv}\{e_1, e_3\} \times \text{conv}\{f_1, f_2\} \cup \\ &\quad \text{conv}\{e_1, e_2\} \times \{f_1\} \cup \\ &\quad \{e_1\} \times \text{conv}\{f_1, f_2\}. \end{aligned}$$

Suppose (p, q) is a level ε-proper pair. Then the following inequalities are true $p_5 \leq \varepsilon p_4$ and $p_4 \leq \varepsilon p_k$, with $k \in \{1, 2, 3\}$. Consequently $pBf_2 < pBf_1$,

$pBf_3 < pBf_1$, $pBf_4 < pBf_1$ and the set of level proper equilibria is given by

$$LP(\Gamma) \quad = \quad \mathrm{conv}\{e_1, e_3\} \times \{f_1\} \cup \mathrm{conv}\{e_1, e_2\} \times \{f_1\}.$$

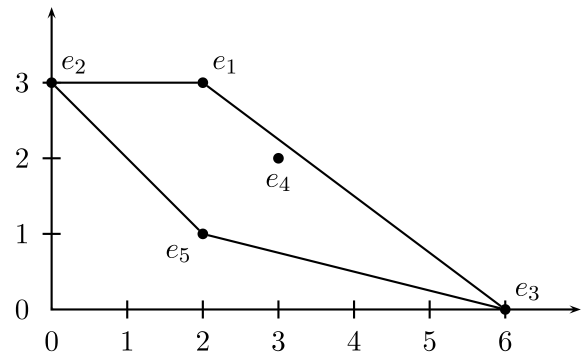

Figure 1: Possible payoff vectors of player one (independent of player 2's strategy choice).

4 Protective behaviour

Protective and prudent strategies for mixed extensions of finite games are introduced in Fiestras-Janeiro, Borm, and Van Megen (1998). Generally speaking a protective strategy of a player maximizes his worst possible payoff with respect to all pure strategy combinations of the other players. In case of inconclusiveness it also searches for minimality in terms of inclusion of the sets of pure strategy combinations which give rise to this worst payoff. A prudent strategy also maximizes the worst possible payoff with respect to all pure strategy combinations of the other players, but on the secondary level it aims to minimize the cardinality of the sets of pure strategy combinations causing this worst payoff.

Fiestras-Janeiro et al. (1998) show that the notions of prudent and protective are in fact equivalent and prove existence. Moreover, it turns out that for matrix games the set of protective strategy combinations coincides with the set of proper equilibria.

In this section we introduce protective and prudent strategies for our setting of (two-person) multicriteria games. The equivalence of protective and prudent strategies is shown and existence is proved. By means of an example we show that even for zero-sum like games protective strategy combinations need not be Pareto equilibria.

Let $\theta : \mathbb{R}^u \to \mathbb{R}^u$ be the map that assigns to any $z \in \mathbb{R}^u$ the vector $\theta(z)$ which orders the coordinates of z in a weakly increasing order.

Let $\Gamma = \langle \Delta_m, \Delta_n, A, B \rangle$ be a multicriteria game. Let $\bar{p} \in \Delta_m$. The pure strategies of player 2 are compared on the basis of a worst case scenario taking all criteria into account with equal importance. Hence the payoffs corresponding to the several criteria are ordered in a weakly increasing order and then compared by using the lexicographic ordering ($\preceq_L$). Recursively, we define the vector $a^t(\bar{p}) \in \mathbb{R}^r$ and the sets $Z^t(\bar{p})$ and $J^t(\bar{p})$ by

(i) for $t = 1$,

$$\begin{aligned} Z^1(\bar{p}) &= \{z \in \mathbb{R}^r \mid \text{there is a } k \in \{1, \ldots, n\} \text{ such that } \bar{p}Af_k = z \\ &\quad\ \text{and such that } \theta(\bar{p}Af_k) \preceq_L \theta(\bar{p}Af_j) \text{ for all } j \in \{1, \ldots, n\}\} \\ a^1(\bar{p}) &=^{2} \theta(z) \text{ for all } z \in Z^1(\bar{p}), \text{ and} \\ J^1(\bar{p}) &= \{j \in \{1, \ldots, n\} \mid \bar{p}Af_j \in Z^1(\bar{p})\}. \end{aligned}$$

(ii) for $t > 1$,

$$\begin{aligned} Z^t(\bar{p}) &= \{z \in \mathbb{R}^r \mid \text{there is a } k \in \{1, \ldots, n\} \setminus \cup_{l=1}^{t-1} J^l(\bar{p}) \text{ such that} \\ &\quad\ \bar{p}Af_k = z \text{ and such that } \theta(\bar{p}Af_k) \preceq_L \theta(\bar{p}Af_j) \text{ for all} \\ &\quad\ j \in \{1, \ldots, n\} \setminus \cup_{l=1}^{t-1} J^l(\bar{p})\} \\ a^t(\bar{p}) &= \theta(z) \text{ for all } z \in Z^t(\bar{p}), \text{ and} \\ J^t(\bar{p}) &= \{j \in \{1, \ldots, n\} \mid \bar{p}Af_j \in Z^t(\bar{p})\}. \end{aligned}$$

If $Z^t(\bar{p}) = \emptyset$, then $J^t(\bar{p}) = \emptyset$ and we define $a^t(\bar{p}) = (\infty, \ldots, \infty)$. Similarly one can define the vector $b^t(\bar{q}) \in \mathbb{R}^s$ and the sets $Z^t(\bar{q})$ and $I^t(\bar{q})$.

Definition 4.1 *Let $\Gamma = \langle \Delta_m, \Delta_n, A, B \rangle$ be a multicriteria game. Let $\bar{p}, p \in \Delta_m$. We say that $\bar{p}$ protectively dominates p, in notation, $\bar{p} \succ_{pro} p$, if there exists an $l \in \mathbb{N}$ such that*

(i) $a^t(\bar{p}) = a^t(p)$ and $J^t(\bar{p}) = J^t(p)$ for all $t \in \mathbb{N}, t < l$, and

(ii) $a^l(\bar{p}) \succ_L a^l(p)$ or, both $a^l(\bar{p}) = a^l(p)$ and $J^l(\bar{p}) \subsetneq J^l(p)$.

[2] Note that this vector is well-defined since the lexicographic minimum within the ordered sets is unique.

A mixed strategy $p \in \Delta_m$ is called protective for player 1 in Γ if there does not exist a mixed strategy $\bar{p} \in \Delta_m$ such that $\bar{p} \succ_{pro} p$. In a similar way one can define protective strategies for player 2.

Even though the protective dominance relation need not to be complete, the next lemma reveals that a protective strategy is dominant, up to payoff equivalence, with respect to the relation $\succ_{pro}$. Here, two strategies are payoff equivalent if they yield the same payoff with respect to all criteria and all pure strategies of the other player. For this we first need a technical lemma.

Lemma 4.1 *Let $\Gamma = \langle \Delta_m, \Delta_n, A, B \rangle$ be a multicriteria game. Consider $\bar{p}, p \in \Delta_m$ and define $\hat{p} = \alpha\bar{p} + (1-\alpha)p$ for some $\alpha \in (0,1)$. Moreover, let $j \in \{1, \dots, n\}$ be such that $\bar{p}Af_j \neq pAf_j$. Then either $\theta(\hat{p}Af_j) \succ_L \theta(\bar{p}Af_j)$ or $\theta(\hat{p}Af_j) \succ_L \theta(pAf_j)$.*

Proof: Let $d \in \{1, \dots, r\}$. Then $(\hat{p}Af_j)_d = \alpha(\bar{p}Af_j)_d + (1-\alpha)(pAf_j)_d$ and hence,

$$(\hat{p}Af_j)_d = (\bar{p}Af_j)_d = (pAf_j)_d \quad \text{if } (\bar{p}Af_j)_d = (pAf_j)_d, \tag{1}$$
$$(\hat{p}Af_j)_d > \min\{(\bar{p}Af_j)_d, (pAf_j)_d\} \quad \text{if } (\bar{p}Af_j)_d \neq (pAf_j)_d. \tag{2}$$

Let $\bar{\sigma}$ be an ordering of the criteria such that $(\bar{p}Af_j)_{\bar{\sigma}(u)} = (\theta(\bar{p}Af_j))_u$ for all $u \in \{1, \dots, r\}$ and $\bar{\sigma}(u) < \bar{\sigma}(v)$ whenever $(\theta(\bar{p}Af_j))_u = (\theta(\bar{p}Af_j))_v$ and $u < v$. In a similar way one can define σ which gives an ordering of the criteria based on $\theta(pAf_j)$.

If $\bar{\sigma}$ and σ are equal, then $\theta(\hat{p}Af_j) = \alpha\theta(\bar{p}Af_j) + (1-\alpha)\theta(pAf_j)$. Since $\bar{p}Af_j \neq pAf_j$, we can assume without loss of generality that $\theta(\bar{p}Af_j)) \succ_L \theta(pAf_j)$. Define v as the smallest number such that $(\theta(\bar{p}Af_j))_v \neq (\theta(pAf_j))_v$. Then equation (1) is valid for all criteria $\sigma(l)$ with $l < v$. Inequality (2) is true for $d = \sigma(v)$, implying that $(\hat{p}Af_j)_{\sigma(v)} > (pAf_j)_{\sigma(v)}$. For all $l > v$: $(\bar{p}Af_j)_{\sigma(l)} \geq (\bar{p}Af_j)_{\sigma(v)} > (pAf_j)_{\sigma(v)}$, together with $(pAf_j)_{\sigma(l)} \geq (pAf_j)_{\sigma(v)}$, this indicates that $(\hat{p}Af_j)_{\sigma(l)} > (pAf_j)_{\sigma(v)}$. It can be concluded that $\theta(\hat{p}Af_j) \succ_L \theta(pAf_j)$.

Assume that $\bar{\sigma}$ and σ are not equal. Let u be the smallest number such that $\bar{\sigma}(u) \neq \sigma(u)$. There are two cases.

In the first case $(\theta(\bar{p}Af_j))_l = (\theta(pAf_j))_l$ for all $l \in \{1, l \dots, u\}$ and hence equation (1) is valid for all $l \in \{1, \dots, u-1\}$, implying that

$$(\hat{p}Af_j)_{\sigma(l)} = (\bar{p}Af_j)_{\sigma(l)} = (pAf_j)_{\sigma(l)} \tag{3}$$

for all $l \in \{1, \ldots, u-1\}$. Without loss of generality we assume that $\sigma(u) < \bar{\sigma}(u)$. It follows from the definition of σ and $\bar{\sigma}$ that $(\bar{p}Af_j)_{\sigma(u)} > (pAf_j)_{\sigma(u)}$. Using inequality (2) we find

$$(\hat{p}Af_j)_{\sigma(u)} > (pAf_j)_{\sigma(u)}. \tag{4}$$

For all $l > u$ it is true that $(\bar{p}Af_j)_{\sigma(l)} \geq (\theta(\bar{p}Af_j))_u = (\theta(pAf_j))_u$ and $(pAf_j)_{\sigma(l)} \geq (\theta(pAf_j))_u$. Hence,

$$(\hat{p}Af_j)_{\sigma(l)} \geq (\theta(pAf_j))_u, \tag{5}$$

and equality holds if and only if $(\bar{p}Af_j)_{\sigma(l)} = (\theta(pAf_j))_u = (pAf_j)_{\sigma(l)}$. We conclude from (3), (4) and (5) that

$$\theta(\hat{p}Af_j) \succ_L \theta(pAf_j).$$

In the second case there exists an $l \leq u$ such that $(\theta(\bar{p}Af_j))_l \neq \theta(pAf_j))_l$. Let v be the smallest number such that $(\theta(\bar{p}Af_j))_v \neq \theta(pAf_j))_v$. Assume without loss of generality that $\theta(\bar{p}Af_j) \succ_L \theta(pAf_j)$. Then for all $l < v$: $(\bar{p}Af_j)_{\sigma(l)} = (pAf_j)_{\sigma(l)}$ and hence equation (1) is true with $d = \sigma(l)$. For all $l \geq v$: $(\bar{p}Af_j)_{\sigma(l)} \geq (\theta(\bar{p}Af_j))_v > (\theta(pAf_j))_v = (pAf_j)_{\sigma(v)}$. Hence, by inequality (2),

$$(\theta(\hat{p}Af_j))_{\sigma(l)} > \min\{(\bar{p}Af_j)_{\sigma(l)}, (pAf_j)_{\sigma(l)}\} > (\theta(pAf_j))_v.$$

It follows that $\theta(\hat{p}Af_j) \succ_L \theta(pAf_j)$. $\square$

Lemma 4.2 *Let $\Gamma = \langle \Delta_m, \Delta_n, A, B \rangle$ be a multicriteria game. Let $\bar{p}$ be a protective strategy for player 1 and let $p \in \Delta_m$. Then, either $\bar{p}$ and p are payoff equivalent for player 1 or $\bar{p} \succ_{pro} p$.*

Proof: Suppose that $\bar{p}$ and p are not payoff equivalent and that $\bar{p}$ does not protectively dominate p. Let $\alpha \in (0, 1)$ and define $\hat{p} := \alpha\bar{p} + (1-\alpha)p$. It is shown below that the strategy $\hat{p}$ protectively dominates $\bar{p}$, yielding a contradiction.

Because of our assumptions, there exists an $l \in \mathbb{N}$ such that

$$a^t(\bar{p}) = a^t(p) \text{ and } J^t(\bar{p}) = J^t(p) \text{ for all } t \in \mathbb{N}, t < l, \text{ and} \tag{6}$$
$$a^l(\bar{p}) = a^l(p), J^l(\bar{p}) \setminus J^l(p) \neq \emptyset \text{ and } J^l(p) \setminus J^l(\bar{p}) \neq \emptyset. \tag{7}$$

We first prove that for all $j \in J^t(\bar{p})$, with $t < l$ it holds that $\bar{p}Af_j = pAf_j$ and hence $\bar{p}Af_j = pAf_j = \hat{p}Af_j$. Suppose not: pick the smallest $t < l$

for which we can find $j \in J^t(\bar{p}) = J^t(p)$ such that $\bar{p}Af_j \neq pAf_j$. Since $j \in J^t(p) = J^t(\bar{p})$, it holds that $\theta(pAf_j) = \theta(\bar{p}Af_j)$. It follows from Lemma 4.1 that $\theta(\hat{p}Af_j) \succ_L \theta(\bar{p}Af_j) = \theta(pAf_j)$. Let $k \in \{1, \ldots, n\} \setminus \cup_{\ell=1}^{t} J^{\ell}(\bar{p})$. Then $\theta(\bar{p}Af_k) \succ_L a^t(\bar{p})$, and $\theta(pAf_k) \succ_L a^t(p) = a^t(\bar{p})$. Using Lemma 4.1 yields: $\theta(\hat{p}Af_k) \succ_L a^t(\bar{p})$. Clearly for all $u < t$: $a^u(\hat{p}) = a^u(\bar{p})$, and $J^u(\hat{p}) = J^u(\bar{p}) = J^u(p)$, furthermore $a^t(\hat{p}) \succ_L a^t(\bar{p})$ or $a^t(\hat{p}) = a^t(\bar{p})$ and $J^t(\hat{p}) \subset J^t(\bar{p})$ (since $j \in J^t(\bar{p})$, but $j \notin J^t(\hat{p})$). We can conclude that $\hat{p} \succ_{prot} \bar{p}$, yielding a contradiction.

Secondly, it can be proved in the same way as above that for all $j \in J^l(\bar{p}) \cap J^l(p)$ it is true that $\bar{p}Af_j = pAf_j$.

There are now two cases: $J^l(\bar{p}) \cap J^l(p) = \emptyset$ and $J^l(\bar{p}) \cap J^l(p) \neq \emptyset$. Assume first that $J^l(\bar{p}) \cap J^l(p) = \emptyset$. Take $j \in J^l(\bar{p})$, then there is an $u > l$ such that $j \in J^u(p)$. Because of Lemma 4.1 $\theta(\hat{p}Af_j) \succ_L \theta(\bar{p}Af_j) = a^l(\bar{p})$. Let $j \in J^u(\bar{p})$, with $u > l$. Then $\theta(\bar{p}Af_j) \succ_L a^l(\bar{p})$ and $\theta(pAf_j) \succeq_L a^l(\bar{p})$. Applying Lemma 4.1 gives $\theta(\hat{p}Af_k) \succ_L a^l(\bar{p})$. We can conclude that

$$\theta(\hat{p}Af_j) \succ_L a^l(\bar{p}) \tag{8}$$

for all $j \in \{1, \ldots, n\} \setminus \cup_{k=1}^{l-1} J^k(p)$. Hence, $a^l(\hat{p}) \succ_L a^l(\bar{p})$ and $\hat{p} \succ_{prot} \bar{p}$.

Now assume $J^l(\bar{p}) \cap J^l(p) \neq \emptyset$. Then there is also a level of $\hat{p}$ with the same vector value, since for all $j \in J^l(\bar{p}) \cap J^l(p) \neq \emptyset$, it holds that $\theta(\hat{p}Af_j) = \theta(\bar{p}Af_j) = \theta(pAf_j) = a^l(\bar{p})$. As we have seen before it follows from Lemma 4.1 that for all $j \notin \cup_{t=1}^{l}(J^t(\bar{p}) \cap J^t(p))$:

$$\theta(\hat{p}Af_j) \succ_L a^l(\bar{p}).$$

Hence $a^l(\hat{p}) = a^l(\bar{p})$ and $J^l(\hat{p}) \subsetneq J^l(\bar{p})$. So $\hat{p} \succ_{prot} \bar{p}$, establishing a contradiction. $\square$

Definition 4.2 *Let $\Gamma = \langle \Delta_m, \Delta_n, A, B \rangle$ be a multicriteria game. Let $p, \bar{p} \in \Delta_m$. We say that $\bar{p}$ prudently dominates p, in notation, $\bar{p} \succ_{pru} p$, if there exists an $l \in \mathbb{N}$ such that*

(i) $a_1^t(\bar{p}) = a_1^t(p)$ and $\left|J^t(\bar{p})\right| = \left|J^t(p)\right|$ for all $t \in \mathbb{N}, t < l$, and

(ii) $a_1^l(\bar{p}) \succ_L a_1^l(\sigma)$ or both $a_1^l(\bar{\sigma}) = a_1^l(\sigma)$ and $\left|J^l(\bar{p})\right| < \left|J^l(p)\right|$.

A mixed strategy $p \in \Delta_m$ is called prudent for player 1 in Γ if there does not exist a mixed strategy $\bar{p} \in \Delta_m$ such that $\bar{p} \succ_{pru} p$.

Notice that any prudent strategy is also protective, since if a strategy is protectively dominated, it is also prudently dominated by the same strategy. The following lemma proves the converse with the use of Lemma 4.2.

Lemma 4.3 *In any multicriteria game, a mixed strategy for player* 1 *is protective if and only if it is prudent.*

Proof: Let $\Gamma = \langle \Delta_m, \Delta_n, A, B \rangle$ be a multicriteria game. It remains to prove that any protective strategy is also prudent. Let $p \in \Delta_m$ be a protective strategy and suppose p is not a prudent strategy. Then, by definition, there exists a strategy $\bar{p}$ such that $\bar{p} \succ_{pru} p$. This means that p and $\bar{p}$ are not payoff equivalent. By Lemma 4.2, $p \succ_{pro} \bar{p}$, from which it follows that $p \succ_{pru} \bar{p}$. This is a contradiction. $\square$

The following theorem guarantees the existence of a prudent (protective) strategy combination in any two person multicriteria game.

Theorem 4.1 *Every multicriteria two-person game has at least one prudent strategy combination.*

Proof: It suffices to prove the existence of a prudent strategy for player 1. Let $\Gamma = \langle \Delta_m, \Delta_n, A, B \rangle$ be a multicriteria game. Define the sets

$M^1 = \{p \in \Delta_m \mid a^1(p) \succeq_L a^1(\bar{p}) \text{ for every } \bar{p} \in \Delta_m\}$,

$P^1 = \{p \in M^1 \mid |J^1(p)| \leq |J^1(\bar{p})| \text{ for any } \bar{p} \in M^1\}$, and

for every $t > 1$,

$M^t = \{p \in P^{t-1} \mid a^t(p) \succeq_L a^t(\bar{p}) \text{ for every } \bar{p} \in P^{t-1}\}$, and

$P^t = \{p \in M^t \mid |J^t(p)| \leq |J^t(\bar{p})| \text{ for any } \bar{p} \in M^t\}$.

It follows from the continuity of the function θ and the compactness of Δ_m that M^1 and P^1 are non-empty compact sets[3]. It follows by induction that $M^t \neq \emptyset$, $P^t \neq \emptyset$ and compact for any $t \in \mathbb{N}$. Moreover, for every $p, \bar{p} \in M^t$ we have $a_1^t(p) = a_1^t(\bar{p})$ and for every $p, \bar{p} \in P^t$ it holds that $a^l(p) = a^l(\bar{p})$ and $|J^l(\bar{p})| = |J^l(p)|$, for all $l \in \{1, \ldots, t\}$. Since player 2 has finitely many pure strategies, there exists a $s \in \mathbb{N}$ such that for all $p \in P^s$: $\cup_{l=1}^s J^l(p) = \{1, \ldots, n\}$ and $|J^s(p)| > 0$. Note that for all $l > s$, $P^l = P^s$, $a^l(p) = (\infty, \ldots, \infty)$ and $J^l(p) = \emptyset$ for all $p \in P^l$. By definition the set P^s is precisely the set of prudent strategies for player 1. $\square$

In the following two examples the notion of protective strategy combinations is illustrated in a zero-sum like environment. In the first example it is

[3] In fact P^1 is a finite union of finite intersections of inverse images of compact sets, hence P^1 is compact itself.

shown that in a game with a large set of equilibria protectiveness picks the intuitively logic strategic combination. In the second example it is shown that a protective strategy combination need not necessarily be a (Pareto) equilibrium.

Example 4.1 Consider the two person multicriteria game $(\Delta_2, \Delta_2, A, B)$, where $B = -A$ and

$$A = \begin{pmatrix} (0,1) & (1,0) \\ (1,0) & (0,1) \end{pmatrix}$$

The set of Pareto equilibria equals $\Delta_2 \times \Delta_2$, so each strategy combination is an equilibrium. Let $\bar{p} = (p, 1-p)$, with $p \in [0,1]$ be a strategy of player one. Then $\theta(\bar{p}Af_1) = \theta(\bar{p}Af_2) = (p, 1-p)$ if $p \leq \frac{1}{2}$ and $\theta(\bar{p}Af_1) = \theta(\bar{p}Af_2) = (1-p, p)$ if $p \geq \frac{1}{2}$. Hence for player one, only the strategy $\left(\frac{1}{2}, \frac{1}{2}\right)$ is protective. Obviously, also $\left(\frac{1}{2}, \frac{1}{2}\right)$ is the only protective strategy of player 2.

Example 4.2 Consider the two person multicriteria game $(\Delta_2, \Delta_2, A, B)$, where

$$A = \begin{pmatrix} (0,0) & (3,-2) \\ (1,1) & (5,-3) \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & -3 \\ -1 & -5 \end{pmatrix},$$

Clearly, the only protective strategy of player one is e_1. the only protective strategy of player two is f_1. The unique protective strategy combination (e_1, f_1) is not an equilibrium, since player one can deviate to e_2, achieving a higher payoff in both criteria.

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