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Observational Equivalence of Discrete String Models and Market Models

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ABSTRACT

In this paper we show that, contrary to the claim made in Longstaff, Santa-Clara, and Schwartz (2001a) and Longstaff, Santa-Clara, and Schwartz (2001b), discrete string models are not more parsimonious than market models. In fact, they are found to be observationally equivalent. We derive that, for the estimation of both a K-factor discrete string model and a K-factor Libor market model for N forward rates the number of parameters that needs to be estimated equals $NK - K(K-1)/2$ and not $K(K+1)/2$ and NK, respectively.

Key words: String model, market model

JEL codes: G12

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I. Introduction

In this paper we discuss the discrete string model as used by Longstaff, Santa-Clara, and Schwartz (2001a) and Longstaff, Santa-Clara, and Schwartz (2001b) (LSS papers) and the Libor market model (LMM) as introduced by Miltersen, Sandmann, and Sondermann (1997), Brace, Gatarek, and Musiela (1997), and Jamshidian (1997). The LSS papers use a discrete version of the string model of Santa-Clara and Sornette (2001) applied to forward Libor rates. In their papers, LSS claim to blend the string model and the LMM. In this paper it is shown that the discrete version of the string model that they use is observationally equivalent to the LMM.

The structure of the paper is as follows. In Section II, the discrete version of the string model as used in the LSS papers and the LMM are described. Section III gives observational equivalence of the two models. Subsequently, in Section IV the number of parameters to be estimated in both models is derived. Section V concludes.

II. Description of models

First, the discrete version of the string model as used in the LSS papers is described. Second, a description of the Libor market model is given.

A. Discrete string model

Kennedy (1994) introduced the idea to model the evolution of the term structure of forward rates as a stochastic string. His analysis has been generalized in Kennedy (1997), Goldstein (2000), and Santa-Clara and Sornette (2001). By construction, the string model is high-dimensional (infinite dimensional if we model a continuum of forward rates), since each rate has its own perturbation. Here, we describe the string model based on a finite number of forward Libor rates. First, we define a finite set of dates, the so-called tenor structure

$$T_0 < T_1 < T_2 < \ldots < T_{N+1}. \tag{1}$$

We denote the current time as T_0 and $T_1, ..., T_{N+1}$ as the forward tenor dates. This gives a spot Libor rate (for $[T_0, T]$) and N forward Libor rates from (for $[T_i, T_{i+1}]$, $i = 1, ..., N$). We define $\delta_i = T_{i+1} - T_i$ as the so-called daycount fractions, which are determined by the maturity of the Libor rate and are most often equal to 3 or 6 months. Let the forward Libor rate from T_i to T_{i+1} at time T_0 be denoted by $F(T_0, T_i, T_{i+1})$ which is defined as

$$F(T_0, T_i, T_{i+1}) \equiv \frac{1}{\delta_i}\left(\frac{D(T_0, T_i) - D(T_0, T_{i+1})}{D(T_0, T_{i+1})}\right), \tag{2}$$

where $D(T_0, T)$ denotes the value of a discount bond at time T_0 with maturity T. For notational convenience, we define $F_i(T_0) \equiv F(T_0, T_i, T_{i+1})$. The string model specifies the following dynamics for the N individual forward rates

$$\frac{dF_i(t)}{F_i(t)} = \alpha_i^M(t)\,dt + \sigma_i dZ_i^M(t). \tag{3}$$

for $i = 1, ..., N$ where $Z_i^M(t)$ are (correlated) Wiener processes under probability measure M with

$$d\left[Z_i^M, Z_j^M\right](t) = \rho_{ij}\,dt.$$

The drift term $\alpha_i^M(t)$ is left unspecified and depends on the probability measure M used in (3). Let Q^{i+1}. denote the probability measure associated with the numeraire $D(\bullet, T_{i+1})$[1]. From the first fundamental theorem of asset pricing (see, for example, Delbaen and Schachermayer (1994)) we know that to exclude arbitrage possibilities $\alpha_i^M(t)$ equals 0 under Q^{i+1}. The volatility functions σ_i (and the correlation parameters ρ_{ij}) do not depend on the probability measure M and are taken to be constant for ease of exposition, but can easily be extended to be deterministic functions of time.

We can stack the individual Wiener processes in a vector $Z^M = \left[\begin{array}{ccc} Z_1^M & \cdots & Z_N^M \end{array}\right]'$. The correlation matrix Ψ of Z is given by

$$\Psi = \left[\begin{array}{ccc} 1 & \cdots & \rho_{1N} \\ \vdots & \ddots & \vdots \\ \rho_{N1} & \cdots & 1 \end{array}\right]. \tag{4}$$

The volatility functions σ_i together with the correlations of the Wiener processes determine the covariance matrix of the forward rate changes. Therefore, we have to estimate $N(N+1)/2$ parameters (N volatilities σ_i and $N(N-1)/2$ correlation parameters

[1] Under Q^{i+1} the value of every tradable asset, say X, satisfies

$$\frac{X(t)}{D(t, T_{i+1})} = \mathbb{E}^{Q^{i+1}}\left[\frac{X(T)}{D(T, T_{i+1})}|\mathcal{F}_t\right]$$

for $T \leq T_{i+1}$.

ρ_{ij} $(\rho_{ij} = \rho_{ji})$). We can write the model in matrix notation as follows

$$\begin{bmatrix} \frac{dF_1(t)}{F_1(t)} \\ \vdots \\ \frac{dF_N(t)}{F_N(t)} \end{bmatrix} = \begin{bmatrix} \alpha_1^M(t) \\ \vdots \\ \alpha_N^M(t) \end{bmatrix} dt + \begin{bmatrix} \sigma_1 & & \emptyset \\ & \ddots & \\ \emptyset & & \sigma_N \end{bmatrix} dZ^M(t) \tag{5}$$

The covariance matrix of the log forward rate changes is then given by

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \cdots & \rho_{1N}\sigma_1\sigma_N \\ \vdots & \ddots & \vdots \\ \rho_{1N}\sigma_1\sigma_N & \cdots & \sigma_N^2 \end{bmatrix}. \tag{6}$$

B. Libor market model

The Libor market model as introduced by Miltersen, Sandmann, and Sondermann (1997), Brace, Gatarek, and Musiela (1997), and Jamshidian (1997) is usually specified in the following form:

$$\frac{dF_i(t)}{F_i(t)} = \mu_i^M(t)\,dt + \Gamma_i \cdot dW^M(t) \tag{7}$$

where W^M denotes an K-dimensional standard Wiener process ($K \leq N$) under probability measure M, $W^M = \begin{bmatrix} W_1^M & \cdots & W_K^M \end{bmatrix}'$ and Γ_i is an K-dimensional volatility function $\Gamma_i = \begin{bmatrix} \Gamma_{i1} & \cdots & \Gamma_{iK} \end{bmatrix}$. As in the discrete string model the drift term $\mu_i^M(t)$ is left unspecified and depends on the probability measure M used in (7). Again the first fundamental theorem of asset pricing gives that to exclude arbitrage possibilities $\mu_i^{Q^{i+1}}(t)$ equals 0. Therefore, we have $\alpha_i^{Q^{i+1}}(t) = \mu_i^{Q^{i+1}}(t)$. To show that $\alpha_i^M(t) = \mu_i^M(t)$ for every M, we need to show that a Γ_i exists such that $\mathcal{L}(\Gamma_i \cdot W^M(t)) = \mathcal{L}(\sigma_i Z_i^M(t))$[2]. In this case a change of measure has the same change of drift in (3) and (7). This is done in Section III. Putting the LMM in matrix notation gives

$$\begin{bmatrix} \frac{dF_1(t)}{F_1(t)} \\ \vdots \\ \frac{dF_N(t)}{F_N(t)} \end{bmatrix} = \begin{bmatrix} \mu_1^M(t) \\ \vdots \\ \mu_N^M(t) \end{bmatrix} dt + \begin{bmatrix} \Gamma_{11} & \cdots & \Gamma_{1K} \\ \vdots & \ddots & \vdots \\ \Gamma_{N1} & \cdots & \Gamma_{NK} \end{bmatrix} dW^M(t). \tag{8}$$

[2] $\mathcal{L}(X)$ denotes the law of X. For example, $\mathcal{L}(X) = \mathcal{N}(0,1)$ has the same meaning as $X \sim \mathcal{N}(0,1)$.

The covariance matrix of the log forward rate changes is given by

$$\Gamma\Gamma' = \begin{bmatrix} \|\Gamma_1\|^2 & \cdots & \Gamma_1 \cdot \Gamma_N \\ \vdots & \ddots & \vdots \\ \Gamma_N \cdot \Gamma_1 & \cdots & \|\Gamma_N\|^2 \end{bmatrix}. \tag{9}$$

III. Observational equivalence

To show observational equivalence[3] of the discrete string model and the LMM we only have to show that their covariance matrices of the log forward rate changes are the same. As pointed out in the previous section this implies an equivalence of the drift terms.

We start by decomposing Ψ in (4) as $\Psi = AA'$. This can be done in the following way. The spectral decomposition of Ψ is given by $\Psi = UDU'$, where U is a matrix of orthonormal eigenvectors and D an ordered diagonal matrix with the eigenvalues on the diagonal[4]. We have

$$UD^{\frac{1}{2}} = \left[\sqrt{\lambda_1}u_1 \dots \sqrt{\lambda_N}u_N\right]$$

where u_i denotes the orthonormal eigenvector corresponding to λ_i. In case of a K-factor model, $\lambda_{K+1} = ... = \lambda_N = 0$. We take

$$A = \left[\sqrt{\lambda_1}u_1 \dots \sqrt{\lambda_K}u_K\right], \tag{10}$$

which gives $\Psi = AA'$. Taking

$$\Gamma = \begin{bmatrix} \sigma_1 & & \emptyset \\ & \ddots & \\ \emptyset & & \sigma_N \end{bmatrix} A \tag{11}$$

gives $\Sigma = \Gamma\Gamma'$. Thus,

$$\mathcal{L}\left(Z\left(t\right)\right) = \mathcal{L}\left(AW\left(t\right)\right) \tag{12}$$

and

$$\mathcal{L}\left(\begin{bmatrix} \sigma_1 & & \emptyset \\ & \ddots & \\ \emptyset & & \sigma_N \end{bmatrix} Z\left(t\right)\right) = \mathcal{L}\left(\Gamma W\left(t\right)\right). \tag{13}$$

[3] Observational equivalence means that for every specification in the class of discrete string models one can find a specification in the class of market models with the same properties and vice versa.

[4] Without loss of generality, we can take (λ_i, u_i) to denote the i^{th} largest (eigenvalue, eigenvector) pair.

The discrete string model is therefore only a reformulation of the Libor market model. Where the Libor market model decomposes the covariance matrix of log forward rate changes as $\Sigma = \Gamma\Gamma'$ the discrete string model does this as

$$\Sigma = \begin{bmatrix} \sigma_1 & & \emptyset \\ & \ddots & \\ \emptyset & & \sigma_N \end{bmatrix} \begin{bmatrix} 1 & \cdots & \rho_{1N} \\ \vdots & \ddots & \vdots \\ \rho_{N1} & \cdots & 1 \end{bmatrix} \begin{bmatrix} \sigma_1 & & \emptyset \\ & \ddots & \\ \emptyset & & \sigma_N \end{bmatrix}.$$

As shown above, this does not have any influence on the properties of the model. The advantage of the discrete string model is that it has a neater economic interpretation.

The observational equivalence shown above is not restricted to the market model setting, but also holds with regards to the (more) general HJM setting. In the HJM setting one needs to specify the discrete string model for the instantaneous forward rates instead of the forward Libor rates. The discrete string model is therefore always just a convenient way to model term structure dynamics when the correlation structure is an input to the model.

IV. Parsimony of the model

In the LSS papers it is claimed that a K-factor Libor market models needs NK parameters, while the discrete string model only needs $K(K+1)/2$ parameters. Note, however, that as a consequence of the observational equivalence of the two models, it necessarily follows that they must need the same number of parameters. Below we demonstrate that in fact both models are specified by $NK - K(K-1)/2$ parameters. As a simple example demonstrating that the LSS claim is incorrect for the string model, note that the $K = 1$ dimensional discrete string model requires $N\,(> K(K+1)/2 = 1)$ parameters to specify the volatility functions. Further, we demonstrate below that there are some hidden restrictions that reduce the number of free parameters in the market model from NK to $NK - K(K-1)/2$.

The LSS papers represent the correlation matrix Ψ of Z in its spectral decomposition[5]

$$\Psi = UDU' = AA' = \sum_{i=1}^{K} \lambda_i u_i u_i' \qquad (14)$$

where U is a matrix with orthonormal eigenvectors and D is an ordered diagonal matrix with

[5] The spectral decomposition can also be performed on the covariance matrix Σ. This would lead to different eigenvalues and eigenvectors. The number of parameters that need to be estimated is the same (see Basilevsky (1995)).

Table I
Restrictions on the eigenvectors $\{u_i\}_{i=1}^{K}$ of the spectral decomposition in (14).

	u_1	u_2	$\cdots$	u_K
u_1	$\|u_1\|^2 = 1$			
u_2	$u_1 \cdot u_2 = 0$	$\|u_2\|^2 = 1$		
$\vdots$	$\vdots$	$\vdots$	$\ddots$	
u_K	$u_1 \cdot u_K = 0$	$u_2 \cdot u_K = 0$	$\cdots$	$\|u_K\|^2 = 1$

the eigenvalues of Ψ[6]. At first it seems that NK parameters are necessary. A is, however, not unique. Consider a $K \times K$ orthonormal matrix T. Then using $A^* = AT$ and $W^* = T'W$ gives the same dynamics as using A and W[7]. The number of necessary parameters to be estimated can be found using (14). We have K unknown eigenvalues $\{\lambda_i\}_{i=1}^{K}$. Further, we have K N-dimensional eigenvectors $\{u_i\}_{i=1}^{K}$ which gives an additional NK unknown parameters. These eigenvectors $\{u_i\}_{i=1}^{K}$ need to be orthonormal which leads to $K(K+1)/2$ restrictions as can be seen from Table I. Using

$$\text{number of parameters} = \text{degrees of freedom} + \text{number of restrictions} \tag{15}$$

we find that the degrees of freedom equals $NK - K(K+1)/2$. Adding the K eigenvalues $\{\lambda_i\}_{i=1}^{K}$ we have $NK - K(K-1)/2$ parameters to estimate. Therefore, by suitable rotation of A we get a A^* such that the first K rows and columns form a lower triangular matrix, i.e.,

$$A^* = \begin{bmatrix} A^*_{11} & & \emptyset \\ \vdots & \ddots & \\ A^*_{K1} & \cdots & A^*_{KK} \\ \vdots & \ddots & \vdots \\ A^*_{N1} & \cdots & A^*_{NK} \end{bmatrix}. \tag{16}$$

V. Conclusion

In this paper, we show that contrary to the claim made in Longstaff, Santa-Clara, and Schwartz (2001a) and Longstaff, Santa-Clara, and Schwartz (2001b), that discrete string models are not more parsimonious than market models. In fact, they are found to be observationally equivalent. We derive that for the estimation of both a K-factor discrete string

[6] Note that $\lambda_{K+1} = ... = \lambda_N = 0$.
[7] W^* is also a standard Wiener process, since $\mathcal{L}(T'W(t)) = \mathcal{N}(0, T'Tt) = \mathcal{N}(0, It)$.

model and a K-factor Libor market model for N forward rates the number of parameters needed to be estimated equals $NK - K(K-1)/2$ and not $K(K+1)/2$ and NK, respectively.

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