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Abstract

In this paper we construct sets of marginal vectors of a TU game with the property that if the marginal vectors from these sets are core elements, then the game is convex. This approach leads to new upperbounds on the number of marginal vectors needed to characterize convexity. Another result is that the relative number of marginals needed to characterize convexity converges to zero.

Keywords: convexity, marginal vectors, TU game

1 Introduction

This paper shows that if specific sets of marginal vectors of some TU game are core elements, then the game is convex. Convexity is an appealing property in cooperative game theory. For convex games it is established that the Shapley value is the barycentre of the core *(Shapley (1971))*, the bargaining set and the core coincide, the kernel coincides with the nucleolus *(Maschler et al. (1972))* and the τ-value can easily be calculated *(Tijs (1981))*.

It is well-known that a game is convex if and only if all marginal vectors are core elements *(Shapley (1971), Ichiishi (1981))*. This result is strengthened in *Rafels, Ybern (1995)*. They showed that if all even marginal vectors are core elements, then all odd marginal vectors are core elements as well, and vice versa. Hence, if all even or all odd marginal vectors are core elements, then the game is convex.

In this paper we establish new bounds for the number of marginal vectors needed to characterize convexity and we present a convergence theorem which states that the relative number of marginal vectors needed to characterize convexity converges to zero.

2 Preliminaries

In this section we recall some notions from cooperative game theory and introduce some notation.

A *cooperative TU game* is a pair (N, v) where $N = \{1, \ldots, n\}$ is a finite (player) set and v, the characteristic function, is a map $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. The map v assigns to each subset $S \subset N$, called a coalition, a real number $v(S)$, called the worth of S. The *core* of a game (N, v) is the set

$$C(v) = \{x \in \mathbb{R}^N | \sum_{j \in S} x_j \geq v(S) \text{ for every } S \subset N, \sum_{j \in N} x_j = v(N)\}.$$

Intuitively, the core of a game can be interpreted as the set of payoff vectors for which no coalition has an incentive to leave the grand coalition N. Note that the core of a game can be empty. A game (N, v) is called *convex* if for all $i, j \in N$ and $S \subset N \backslash \{i, j\}$ it holds that

$$v(S \cup \{i\}) - v(S) \leq v(S \cup \{i\} \cup \{j\}) - v(S \cup \{j\}). \tag{1}$$

Hence, for convex games the marginal contribution of a player to any coalition is less than his marginal contribution to a larger coalition.

Before we introduce marginal vectors, we first introduce orders. For any $T \subset N$, an *order* σ of T is a bijection $\sigma : \{1, \ldots, |T|\} \to T$. This order is denoted by $\sigma(1) \cdots \sigma(|T|)$, where $\sigma(i) = j$ means that with respect to σ, player j is in the i-th position. The *i-th neighbour* of σ is the order σ^i which is obtained by switching the players at the i-th and $(i+1)$-th position of σ, i.e. $\sigma^i = \sigma(1) \cdots \sigma(i-1)\sigma(i+1)\sigma(i)\sigma(i+2) \cdots \sigma(|T|)$. An order is called *even* if it can be turned into the identity order e_T by an even number of neighbourswitches, where e_T is such that for every $i, j \in \{1, \ldots, |T|\}$ with $i < j$, it holds that $e_T(i) < e_T(j)$. An order which is not even is called *odd*. Note that every neighbour of an even order is odd and vice versa and that the set of all orders S_T contains as many even orders as odd orders.

Let (N, v) be a game. For any $\sigma \in S_N$, the *marginal vector* $m^\sigma(v)$ is defined by

$$m_i^\sigma(v) := v([\sigma, i]) - v((\sigma, i)) \text{ for all } i \in N$$

where $[\sigma, i] = \{j \in N : \sigma^{-1}(j) \leq \sigma^{-1}(i)\}$ is the set of predecessors of i with respect to σ including i, and $(\sigma, i) = \{j \in N : \sigma^{-1}(j) < \sigma^{-1}(i)\}$ is the set of predecessors of i excluding i. A marginal vector is called even (odd) if the corresponding order is even (odd). Furthermore, if τ is the i-th neighbour of σ, then $m^\tau(v)$ is called the i-th neighbour of $m^\sigma(v)$.

Let $\{A_1, \ldots, A_k\}$ be a partition of N and let $\sigma_i \in S_{A_i}$ for each $i \in \{1, \ldots, k\}$. Then the combined order $\sigma_1\sigma_2 \cdots \sigma_k$ of N is that order which begins with the players in A_1 ordered according to σ_1, followed by the players in A_2 ordered according to σ_2, etcetera, i.e.

$$\sigma_1\sigma_2 \cdots \sigma_k := \sigma_1(1) \cdots \sigma_1(|A_1|)\sigma_2(1) \cdots \sigma_2(|A_2|) \cdots \sigma_k(1) \cdots \sigma_k(|A_k|).$$

The set $A_1|A_2|\ldots|A_k$ contains those orders which begin with the players in A_1, followed by the players in A_2, etcetera, i.e. $A_1|A_2|\ldots|A_k := \{\sigma_1 \cdots \sigma_k : \sigma_i \in S_{A_i} \text{ for every } i \in \{1, \ldots, k\}\}$.

Example 1 Let $N = \{1, 2, 3, 4, 5\}$, $A_1 = \{1, 5\}$, $A_2 = \{3\}$ and $A_3 = \{2, 4\}$. If $\sigma_1 = 51$, $\sigma_2 = 3$ and $\sigma_3 = 24$, then $\sigma_1\sigma_2\sigma_3 = 51324$, and $15|3|24 = \{15324, 15342, 51324, 51342\}$.

3 Characterizing convexity using core marginals

In this section we present our main results. First we recall the theorem of *Shapley (1971)* and *Ichiishi (1981)* which states that a game is convex if and only if all marginal vectors are core elements. We recall a strengthening of this theorem by *Rafels, Ybern (1995)* and we prove this strengthening in an alternative way. We obtain sharper bounds for the number of marginal vectors needed to characterize convexity and we prove a convergence theorem.

Theorem 1 ***(Shapley (1971),Ichiishi (1981))*** *Let (N, v) be a game. Then v is convex if and only if $m^\sigma(v) \in C(v)$ for all $\sigma \in S_N$.*

The strengthening posed in *Rafels, Ybern (1995)* states that if all even marginal vectors are core elements, then all odd marginal vectors are core elements as well, and vice versa. Hence, Theorem 2 states that if $\frac{n!}{2}$ specific marginal vectors are core elements, then the game is convex. Therefore, Theorem 2 provides an upperbound of $\frac{n!}{2}$ for the number of marginal vectors needed to characterize convexity.

Theorem 2 ***(Rafels, Ybern (1995))*** *Let (N,v) be a game. Then the following statements are equivalent:*

1. *(N,v) is convex*
2. *$m^\sigma(v) \in C(v)$ for all even $\sigma \in S_N$*
3. *$m^\sigma(v) \in C(v)$ for all odd $\sigma \in S_N$.*

Rafels, Ybern (1995) proved this theorem by showing that if all even or all odd marginal vectors are core elements, then (1) is satisfied for every $i,j \in N$ and every $S \subset N\backslash\{i,j\}$. The following lemma states that if for a marginal vector two consecutive neighbours are core elements, then this marginal vector is a core element as well.

Lemma 1 *Let (N,v) be a game with $|N| \geq 3$, and let $\sigma \in S_N$. Suppose there is an $h \in \{1,\dots,n-2\}$ such that $m^{\sigma^h}(v), m^{\sigma^{h+1}}(v) \in C(v)$. Then $m^\sigma(v) \in C(v)$.*

Proof: Without loss of generality we assume that $\sigma = 12\cdots n$. Then $\sigma^h = 1\cdots h-1\ h+1\ h\ h+2\cdots n$, and $\sigma^{h+1} = 1\cdots h\ h+2\ h+1\ h+3\cdots n$. To show that $m^\sigma(v) \in C(v)$, we need to show that $\sum_{i\in S} m_i^\sigma(v) \geq v(S)$ for every $S \subset N$. So let $S \subset N$, we distinguish between four cases.

`Case 1:` $h, h+1 \notin S$.

Then it follows that $\sum_{j\in S} m_j^\sigma(v) = \sum_{j\in S} m_j^{\sigma^h}(v) \geq v(S)$, where the inequality holds because $m^{\sigma^h}(v) \in C(v)$.

`Case 2:` $h \notin S$ and $h+1 \in S$.

Let $T = \{1,\dots,h\} = [\sigma,h]$. It follows that

$$\sum_{j\in T} m_j^{\sigma^h}(v) = v([\sigma,h+1]) - v([\sigma,h-1]\cup\{h+1\}) + v([\sigma,h-1]) \geq v(T) = v([\sigma,h]), \tag{2}$$

where the inequality holds because $m^{\sigma^h}(v) \in C(v)$. Therefore

$$\sum_{j\in S} m_j^\sigma(v) = \sum_{j\in S} m_j^{\sigma^h}(v) - m_{h+1}^{\sigma^h}(v) + m_{h+1}^\sigma(v)$$

$$= \sum_{j\in S} m_j^{\sigma^h}(v) - (v([\sigma,h-1]\cup\{h+1\}) - v([\sigma,h-1]))$$

$$+(v([\sigma,h+1]) - v([\sigma,h])) \geq v(S),$$

where the inequality follows from $m^{\sigma^h}(v) \in C(v)$ and (2).

`Case 3:` $h \in S$ and $h+1 \notin S$. We distinguish between two subcases.

`Subcase 3a:` $h+2 \notin S$.

Then it follows that $\sum_{j\in S} m_j^\sigma(v) = \sum_{j\in S} m_j^{\sigma^{h+1}}(v) \geq v(S)$.

`Subcase 3b:` $h+2 \in S$.

By taking $T = \{1,\dots,h+1\}$ and using that $m^{\sigma^{h+1}}(v) \in C(v)$, it follows similar to case 2 that $\sum_{j\in S} m_j^\sigma(v) \geq v(S)$.

`Case 4:` $h, h+1 \in S$.

Then it follows that $\sum_{j\in S} m_j^\sigma(v) = \sum_{j\in S} m_j^{\sigma^h}(v) \geq v(S)$. □

Example 2 Consider a game (N, v) with $N = \{1, 2, 3\}$. Then 213 and 132 are the first and second neighbour of 123. From Lemma 1 it follows that if $m^{213}(v), m^{132}(v) \in C(v)$, then $m^{123}(v) \in C(v)$.

By applying Lemma 1 we have, for $n \geq 3$, an alternative proof of Theorem 2 by using that each neighbour of an even marginal vector is odd and vice versa. In fact, using Lemma 1, even sharper bounds than $\frac{n!}{2}$ can be obtained. To obtain these bounds we introduce the operator $b : 2^{S_T} \to 2^{S_T}$ for every $T \subset N$. For $|T| = 1, 2$, we define $b(A) := A$ for all $A \subset S_T$. For $|T| \geq 3$, we define

$$b(A) := A \ \cup \ \{\sigma \in S_T : \text{there is an } i \in \{1, \ldots, |T| - 2\} \text{ such that } \sigma^i, \sigma^{i+1} \in A\}.$$

By applying Lemma 1 it follows that if $m^\sigma(v) \in C(v)$ for all $\sigma \in A \subset S_N$, then $m^\tau(v) \in C(v)$ for all $\tau \in b(A)$. Hence b can be used to generate larger sets of core marginals.

Example 3 Let $A = \{213, 132\}$, and let $\sigma = 123$. Then $\sigma^1 = 213 \in A$ and $\sigma^2 = 132 \in A$. Therefore, $123 \in b(A)$.

The following example shows that by repetetive appliance of b larger sets of core marginals can be obtained.

Example 4 Let $A = \{2134, 1324, 1423\}$, $\sigma = 1234$ and $\tau = 1243$. It follows that $\sigma^1 = 2134 \in A$, and $\sigma^2 = 1324 \in A$. Therefore it follows that $\sigma \in b(A)$. Note that because $\tau \notin A$, $\tau^1 = 2143 \notin A$ and $\tau^3 = 1234 \notin A$ it follows that $\tau \notin b(A)$. However $\tau^2 = 1423 \in b(A)$ and $\tau^3 = 1234 \in b(A)$. Therefore, $\tau \in b(b(A)) = b^2(A)$.

Let $T \subset N$. Define the *closure* of $A \subset S_T$, denoted by $b^*(A)$, to be the largest set of orders that can be obtained by repetetive appliance of b, i.e. $b^*(A) = b^k(A)$ for $k \in \mathbb{N}$ such that $b^k(A) = b^{k+1}(A)$. Let $C \subset S_T$. If $A \subset C$ is such that $C \subset b^*(A)$, then A is called *complete in* C. If $A \subset S_T$ is complete in S_T, then A is called *complete*. From Lemma 1 it follows that if $A \subset S_N$ and $m^\sigma(v) \in C(v)$ for every $\sigma \in A$, then $m^\tau(v) \in C(v)$ for every $\tau \in b^*(A)$. Hence, if A is complete, then $m^\sigma(v) \in C(v)$ for every $\sigma \in A$ implies that v is convex. The following lemma provides a set of orders which is complete in $T_1| \ldots |T_k$.

Lemma 2 *Let $\{T_1, \ldots, T_k\}$ be a partition of N and let $A_i \subset S_{T_i}$. If A_i is complete in S_{T_i} for each $i \in \{1, \ldots, k\}$, then $A = \{\tau_1 \cdots \tau_k : \tau_i \in A_i \text{ for each } i \in \{1, \ldots, k\}\}$ is complete in $T_1| \ldots |T_k$.*

Proof: Let $\sigma_1 \cdots \sigma_k \in T_1| \cdots |T_k$. We use induction to show that $\sigma_1 \cdots \sigma_k \in b^*(A)$.

Let $j \in \{1, \ldots, k+1\}$ be such that $\sigma_1 \cdots \sigma_{j-1}\tau_j \cdots \tau_k \in b^*(A)$ for all $(\tau_j, \ldots, \tau_k) \in A_j \times \ldots \times A_k$. Note that $j = 1$ satisfies this property.

Now let $(\tau_{j+1}, \ldots, \tau_k) \in A_{j+1} \times \ldots \times A_k$ and let $C = \{\sigma_1 \cdots \sigma_{j-1}\tau\tau_{j+1} \cdots \tau_k | \ \tau \in A_j\}$. By assumption we have $C \subset b^*(A)$. Because $\sigma_j \in b^*(A_j)$ it follows that $\sigma_1 \cdots \sigma_{j-1}\sigma_j\tau_{j+1} \cdots \tau_k \in b^*(C) \subset b^*(A)$. We conclude that $\sigma_1 \cdots \sigma_k \in b^*(A)$. □

Example 5 Let $N = \{1, \ldots, 6\}$, $T_1 = \{124\}$ and $T_2 = \{3, 5, 6\}$. Let $A_1 = \{124, 241, 412\}$ and $A_2 = \{356, 563, 635\}$. Note that A_1 and A_2 are complete sets. It follows from Lemma 2 that $A = \{\sigma\tau : \sigma \in A_1, \tau \in A_2\}$ is complete in $124|356$.

We now focus on the cardinality of complete sets. A set of orders $A \subset S_T$ is called *minimum complete* if A is a complete set and for every other complete set B it holds that $|B| \geq |A|$, i.e. there exists no complete set with smaller cardinality. Because of symmetry, the cardinality of a minimum complete set only depends on the cardinality $t = |T|$, and not on T. Therefore, let the *neighbour number* Q_t denote the cardinality of a minimum complete set in S_T, i.e. $Q_t :=$

$\min_{A \subset S_T : A \text{ is complete}} |A|$. By definition it holds that $Q_1 = 1$ and $Q_2 = 2$. From Theorem 2 it follows that $Q_n \leq \frac{n!}{2}$ for each $n \geq 3$.

In the final part of this section we establish upperbounds and lowerbounds for Q_n. Moreover, it is shown that the *relative neighbour number* $F_n := \frac{Q_n}{n!} \to 0$ if $n \to \infty$. The following theorem, which is a direct consequence of Lemma 2, gives a strengthening of the bound obtained from Theorem 2.

Theorem 3 *Let $n_1, \ldots, n_k, n \in \mathbb{N}$, be such that $\sum_{i=1}^{k} n_i = n$. Then $Q_n \leq \frac{n!}{n_1! \cdots n_k!} \prod_{i=1}^{k} Q_{n_i}$.*

Proof: The set S_N can be partitioned into $\frac{n!}{n_1! \cdots n_k!}$ sets of the form $T_1| \ldots |T_k$ with $|T_i| = n_i$ for each $i \in \{1, \ldots, k\}$. Now let $\{T_1, \ldots, T_k\}$ be a partition of N for which $|T_i| = n_i$ for each $i \in \{1, \ldots, k\}$. According to Lemma 2 there is a set A containing $\prod_{i=1}^{k} Q_{n_i}$ elements such that A is complete in $T_1| \ldots |T_k$. Therefore $Q_n \leq \frac{n!}{n_1! \cdots n_k!} \prod_{i=1}^{k} Q_{n_i}$. □

From Theorem 3 it follows that $Q_{n+1} \leq \frac{(n+1)!}{n!1!} Q_n Q_1 = (n+1)Q_n$. Hence it follows that $F_{n+1} \leq F_n$ for each n, i.e. the relative neighbour number F_n is nonincreasing. The following lemma gives a lowerbound on the neighbour number.

Lemma 3 *If n is even then $Q_n \geq n! \frac{1}{2^{\frac{n-2}{2}}}$, and if n is odd then $Q_n \geq n! \frac{1}{2^{\frac{n-1}{2}}}$.*

Proof: Let n be even, and let $k = \frac{n+2}{2}$.

The set S_N can be partitioned into $\frac{n!}{1!2! \cdots 2!1!} = \frac{n!}{2^{k-2}}$ sets of the form $T_1| \ldots |T_k$ with $|T_1| = |T_k| = 1$ and $|T_i| = 2$ for every $i \in \{2, \ldots, k-1\}$. Let $C \subset S_N$ be one of those sets, i.e. there exists a partition $\{T_1, \ldots, T_k\}$ of N with $|T_1| = |T_k| = 1$ and $|T_i| = 2$ for every $i \in \{2, \ldots, k-1\}$ such that $C = T_1| \ldots |T_k$.

It follows that for every $\sigma \in C$ it holds that $C = \sigma(1)|\sigma(2)\sigma(3)| \ldots |\sigma(n)$. For every even $i \in \{1, \ldots, n-1\}$ it holds that $\sigma^i \in C$. Now let $A \subset S_N$ such that $A \cap C = \emptyset$. Because for every $\sigma \in C$ there is no $h \in \{1, \ldots, n-2\}$ with $\sigma^h, \sigma^{h+1} \in A$, it follows that $\sigma \notin b(A)$. This holds for every $\sigma \in C$, hence $b(A) \cap C = \emptyset$. By repetition it follows that $b^*(A) \cap C = \emptyset$.

We conclude that if $A \subset S_N$ is complete, then $|A \cap C| \geq 1$. This holds for every C such that $C = T_1| \ldots |T_k$ with $|T_1| = |T_k| = 1$, $|T_i| = 2$ for every $i \in \{2, \ldots, k-1\}$. It follows that $|A| \geq n! \frac{1}{2^{\frac{n-2}{2}}}$.

Now let n be odd, and let $k = \frac{n+1}{2}$. The set S_N can be partitioned into $\frac{n!}{1!2! \cdots 2!2!} = \frac{n!}{2^{k-1}}$ sets of the form $T_1| \ldots |T_k$ with $|T_1| = 1$ and $|T_i| = 2$ for every $i \in \{2, \ldots, k\}$. Now let $C \subset S_N$ be such that $C = T_1| \ldots |T_k$ with $|T_1| = 1$ and $|T_i| = 2$ for every $i \in \{2, \ldots, k\}$. Similar to the case where n is even it follows that if $A \subset S_N$ is complete, then $A \cap C \neq \emptyset$. This holds for every C such that $C = T_1| \ldots |T_k$ with $|T_1| = 1$, $|T_i| = 2$ for every $i \in \{2, \ldots, k\}$. It follows that $|A| \geq n! \frac{1}{2^{\frac{n-1}{2}}}$. Hence, $|A| \geq n! \frac{1}{2^{\frac{n-1}{2}}}$. □

From Theorem 2 and Lemma 3 it follows that $Q_3 = 3$ and $Q_4 = 12$. We also obtain from Theorem 2 that $Q_5 \leq 60$ and from Lemma 3 we obtain that $Q_5 \geq 30$. Therefore $Q_5 \in [30, 60]$. However, using ad hoc methods, we established that $Q_5 = 30$. The proof of this lemma is given in the appendix.

Lemma 4 *The neighbour number $Q_5 = 30$.*

Now, taking $n_1 = n_2 = 3$ and using Theorem 3, it follows that $Q_6 \leq \frac{6!}{3!} 3! \cdot Q_3 \cdot Q_3 = 180$. From Lemma 3 we derive that $Q_6 \geq 6! \frac{1}{2^2} = 180$, and hence $Q_6 = 180$. Similar, by taking $n_1 = 3$ and $n_2 = 4$, it follows that $Q_7 \leq 1260$. From Lemma 3 it follows that $Q_7 \geq 630$, and therefore $Q_7 \in [630, 1260]$. Hence for Q_7 we do not have a sharp bound. However, for Q_8 we have a sharp bound. By taking $n_1 = 3$ and $n_2 = 5$ we obtain that $Q_8 \leq 5040$. From Lemma 3 it follows that

n	3	4	5	6	7	8	9	10	11	12
$n!$	6	24	120	720	5040	40320	362880	3628800	39916800	479001600
$\frac{n!}{2}$	3	12	60	360	2520	20160	181440	1814400	19958400	239500800
Q_n	3	12	30	180	[630,1260]	5040	[22680,45360]	22680	[1247400,2494800]	[14968800,29937600]
F_n	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$[\frac{1}{8}, \frac{1}{4}]$	$\frac{1}{8}$	$[\frac{1}{16}, \frac{1}{8}]$	$\frac{1}{16}$	$[\frac{1}{32}, \frac{1}{16}]$	$[\frac{1}{32}, \frac{1}{16}]$

Table 1: New bounds

$Q_8 \geq 5040$, and hence $Q_8 = 5040$. Some other new bounds are given in Table 1.

The following theorem states that the relative neighbour number F_n converges to 0.

Theorem 4 *The relative neighbour number $F_n \to 0$ if $n \to \infty$.*

Proof: Let $k \in \mathbb{N}$, let $n = 3k$ and let $n_i = 3$ for every $i \in \{1, \ldots, k\}$. From Theorem 3 we deduce that

$$Q_{3k} \leq \frac{(3k)!}{(3!)^k} 3^k,$$

by using that $Q_3 \leq 3$. Therefore $F_{3k} \leq (\frac{1}{2})^k$ for every $k \in \mathbb{N}$. It follows that $F_{3k} \to 0$ if $k \to \infty$. It follows from Theorem 3 that $F_{n+1} \leq F_n$ for all $n \in \mathbb{N}$. Hence, $F_n \to 0$ as $n \to \infty$. □

4 Appendix

Proof of Lemma 4: Let $N = \{1, \ldots, 5\}$ and let

$$\begin{aligned} A = \{&13245, 21345, 31452, 41523, 52134, \\ &14235, 21453, 31524, 42135, 53124, \\ &14352, 21534, 32145, 42513, 53241, \\ &15234, 24351, 32451, 43125, 54123, \\ &15342, 25341, 32514, 43251, 54231, \\ &15423, 25413, 35412, 43512, 54312\}. \end{aligned}$$

We will show that A is a complete set. Consider the sets

$$\begin{aligned} B = \{&13452, 24513, 31245, 41235, 52341, \\ &14523, 25134, 35124, 42351, 53412\} \end{aligned}$$

and

$$C = \{15324, 21435, 32541, 43152, 54213\}.$$

Let $\sigma \in B$. Then $\sigma^1, \sigma^2 \in A$. Hence $\sigma \in b(A)$. Let $\tau \in C$. Then $\tau^3, \tau^4 \in A$. Hence $\tau \in b(A)$. It follows that $(A \cup B \cup C) \subset b(A)$.

Now let

$$D = \{13524, 24135, 35241, 41352, 52413\}.$$

Let $\sigma \in D$. Then $\sigma^1 \in A \subset b(A)$ and $\sigma^2 \in C \subset b(A)$. It follows that $\sigma \in b(b(A))$. Hence $(A \cup B \cup C \cup D) \subset b^2(A)$.

Now let

$$E = 13|245 \cup 14|235 \cup 24|135 \cup 25|134 \cup 35|124 \cup 1|5|234 \cup 2|1|345 \cup 3|2|145 \cup 4|3|125 \cup 5|4|123.$$

For each $\sigma \in E$, it holds that either that $\sigma \in A \cup B \cup C \cup D$ or that $\sigma^3, \sigma^4 \in A \cup B \cup C \cup D$. Hence $\sigma \in b(A \cup B \cup C \cup D) \subset b(b^2(A))$. Now let

$$F = S_N \backslash E = 1|2|345 \cup 2|3|145 \cup 3|4|125 \cup 4|5|123 \cup 5|1|234.$$

Then for all $\sigma \in F$ it holds that $\sigma^1, \sigma^2 \in E$. Hence it follows that E is a complete set. From this we conclude that A is a complete set. Because $|A| = 30$ it follows that $Q_5 \leq 30$. □

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