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SIMULATING AN (R,s,S) INVENTORY SYSTEM

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Scope and Purpose — Existing inventory model solutions generally need several assumptions regarding the demand distribution and control parameters, limiting the use of these models in practice (cf. [1]). This paper shows that approximating an inventory system by simulation can produce accurate reorder points or fill rates without these restrictions. Both practitioners and researchers may benefit from this simulation approach.

Abstract — A fast and efficient algorithm is presented for simulating an (R, s, S) inventory control system, where both demand and lead time may be stochastic. It can be used to calculate performance measures like the fill rate, or to find parameter values leading to a prescribed service level. The algorithm was applied in Moors & Strijbosch [2]: for Erlang distributed demand and a deterministic lead time the results agreed with theory. This agreement is shown here to hold even for (a simple case of) stochastic lead time.

Keywords: inventory control, simulation model, reorder point, fill rate
JEL-codes: C150, C444

1 Introduction

The process of implementing an inventory control system in practice can be split into several steps. For a single item, single location inventory system the following four step scheme is usually followed.

1) Choosing the inventory policy to be used. A common classification is: continuous review ((s, S), (s, Q)) and periodic review ((R, s, S), (R, s, Q)), in which s denotes

the reorder point, S the order-up-to level, Q the order size and R the length of each review period.

2) Choosing the performance criterion. This may be either cost-based: minimizing total costs (including stockout costs), or service-based: reaching a prespecified service level.

3) Estimating the stochastic features in the system, in particular demand.

4) Finding the control parameter values so that 2) is satisfied. This implies solving an (expected costs) minimization problem, or solving a service equation.

An exact solution in step 4) can only be obtained if the probability distribution in 3) is fully known and has a simple analytical form. For stationary normally distributed demand with known parameters, many classical handbooks like Silver *et al.* [1] give exact solutions. Moors & Strijbosch [2] present an exact solution for gamma distributed demand. However, even if the type of distribution is known, but the parameters have to be estimated from past observations, these solutions lead to underperformance of the control system; compare Silver and Rahnama [3,4] and Strijbosch and Moors [5].

Approximate analytic solutions exist for more general situations. We only mention Tijms & Groenevelt [6], Shore [7], Schneider & Ringuest [8] and Platt *et al.* [9]. But of course these approximations are only valid if the assumed conditions are satisfied: often high service levels are needed and/or $S - s$ should be sufficiently large.

In practice, however, demand distributions are unknown and do not belong to a nice analytic class; besides, they will change in time. A fast simulation program then may be very useful. It will give approximate - but very accurate - solutions for any type of demand distribution. Besides, it is useful in analyzing the past: *a posteriori* control parameter values can be found that would have led to the desired performance. If no drastic changes in demand will occur, these values are important for future use.

This paper presents such an algorithm for an (R, s, S) control system; it is organized as follows. In Section 2 the (R, s, S) control system is described and the algorithm for deterministic lead time presented. This algorithm is applied in Section 3 to Erlang distributed demand: to illustrate its use, we show the distribution of net stock just before each potential delivery moment, calculate the fill rate and present values of the reorder point s that lead to prescribed fill rates. Comparison with the theoretical results in [2] shows a nearly perfect agreement.

Section 4 discusses the case of stochastic lead time; under certain conditions, the algorithm can be extended to cover this case as well. For a very simple situation - lead time having a two-point distribution - again full agreement with theory is achieved. The final Section 5 summarizes the general applicability of the extended simulation algorithm and indicates some directions for further research.

2 Simulation algorithm for *deterministic* lead time

The (R, s, S) policy is an inventory control method often encountered in practice. Inventory is checked at review moments, R time-units apart; the interval between two successive review moments is called a review period. Only if the inventory position at a review moment is at or below the reorder point s, an order up to level S is placed. R and $q = S - s$ are assumed to be known (set by management). Backlogging of excess demand is assumed. In this section we assume lead time to be deterministic: orders are delivered after a fixed delay of length l. Consequently, the time between two subsequent order moments (and between two subsequent deliveries) is a multiple KR of R; in other words, KR is the length of this replenishment cycle (RC).

Consider the delivery moment at the end of a certain RC; denote the net stock just **before** delivery by NS_- and just **after** delivery by NS. Writing $x^+ = \max(0, x)$, a possible shortage U at the end of a random RC is given by

$$U = [-NS_-]^+ - [-NS]^+ \tag{1}$$

The values of these three quantities only depend on (stochastic) demand during review period plus lead time. This demand is simulated by random drawing from either a standard statistical distribution, like normal or gamma, or from a frequency distribution based on past observations. Note that non-stationary demand distributions are allowed.

Table 1 first presents some additional notation. In the algorithm any review interval i is split into two parts, the first part starting at the review moment and ending at the next potential delivery moment, and the second consisting of the remainder of the review period. The corresponding demands x_i and y_i are drawn separately and independently. Table 2 presents the core of the algorithm. Starting at an inventory position of S at the beginning of the first review interval, just before and after each subsequent review moment the current inventory position is determined and just before and after each subsequent potential delivery moment the current net stock is established. Demand during leadtime DL_i is (depending on the value of l) always the accumulated demand

Table 1. *Notation*

ε	a very small positive number
i	index for review moment, $i \geq 1$
r_i	i-th review moment
l	lead time
g	quotient of lead time and review time; $g = \frac{l}{R}$
$[r_i, r_{i+1})$	i-th review period
m_i	potential delivery moment; $m_i = r_i + l$
x_i	demand during interval $[r_{i+\lfloor g-\varepsilon \rfloor}, m_i)$ between delivery of an order placed on r_i and the last review moment before delivery
y_i	demand during interval $[m_i, r_{i+\lfloor g-\varepsilon \rfloor+1})$ between delivery of an order placed on r_i and the first review moment after delivery
z_i	demand during review interval $[r_{i+\lfloor g-\varepsilon \rfloor}, r_{i+\lfloor g-\varepsilon \rfloor+1})$
IP_{i-}	Inventory Position immediately before r_i
IP_i	Inventory Position on r_i, immediately after review
NS_{i-}	Net Stock immediately before m_i
NS_i	Net Stock on m_i, immediately after delivery, if any
O_i	dummy indicating whether an order is placed on r_i
DL_i	demand during (potential) leadtime $[r_i, m_i)$

Table 2. *Core of algorithm*

$$IP_{1-} = s + q$$
$$x_i = \text{random drawing from demand distribution}$$
$$y_i = \text{random drawing from demand distribution}$$
$$z_i = x_i + y_i$$
$$IP_{i-} = IP_{i-1} - z_{i-1}, \ (i > 1)$$
$$O_i = \begin{cases} 1 \text{ if } IP_{i-} < s: \text{ do order} \\ 0 \text{ if } IP_{i-} \geq s: \text{ don't order} \end{cases}$$
$$IP_i = \begin{cases} s + q & \text{if } \ O_i = 1 \\ IP_{i-} & \text{if } \ O_i = 0 \end{cases}$$
$$DL_i = \sum_{j=i-\lfloor g-\varepsilon \rfloor}^{i-1} z_j + x_i$$
$$NS_{i-} = IP_{i-} - DL_i$$
$$NS_i = NS_{i-} + (s + q - IP_{i-})\, O_i$$

over zero or more review periods plus demand over the interval between delivery and the last review moment before delivery (x_i). Substracting ε from g is necessary only for $g \in \mathbb{N}$: it guarantees that delivery is accounted for just before review.

To illustrate the possible applications of our algorithm, we use the familiar fill rate β as a performance measure; by definition, β is the fraction of total demand that can be satisfied immediately from stock at hand. Denoting the expected value of shortage at the end of a RC by $E(U)$, the expected length (in units of R) of a RC by $E(K)$, and mean demand in a review period by μ_R, β is given by

$$\beta = 1 - \frac{E(U)}{\mu_R E(K)} \tag{2}$$

After running the algorithm over a certain horizon (which should be sufficiently large to obtain a desired accuracy), an estimate of the fill rate can be calculated by

$$\widehat{\beta} = 1 - \frac{\sum \left[(-NS_{i-})^+ - (-NS_i)^+\right]}{\sum z_i} \tag{3}$$

Of course, frequency distributions of NS_{i-}, NS_i and U_i will be of interest in themselves.

3 Exact and simulated outcomes for *deterministic* lead time

Our algorithm was applied in [2], where we supposed demand to follow a (stationary) gamma-process $\Gamma(\lambda, \rho t)$, meaning that

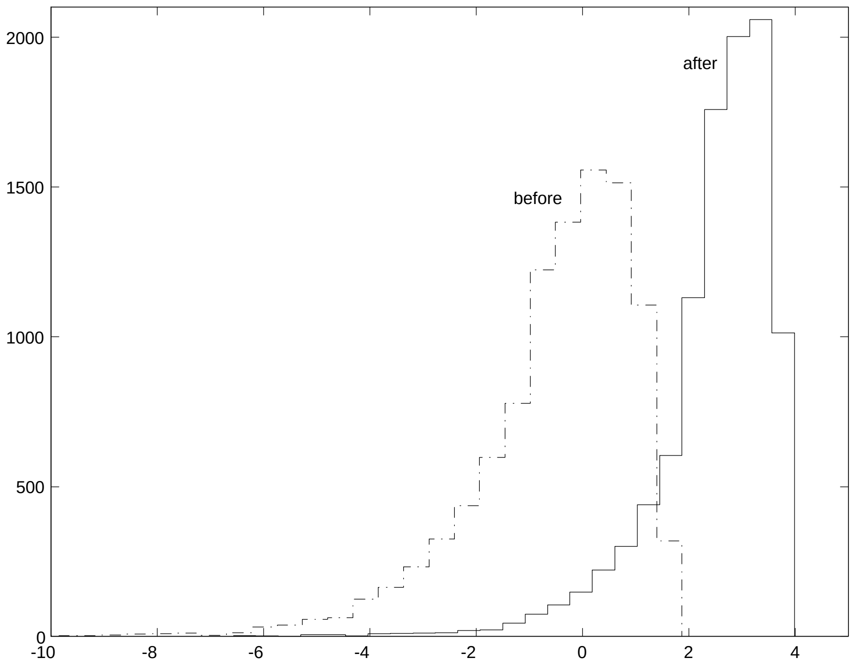
- demand during any interval of length t has distribution $\Gamma(\lambda, \rho t)$,
- demands during disjoint time intervals are independent.

Since $1/\lambda$ is simply a scale-parameter, the simplifying assumption $\lambda = 1$ could be used. The density and the cumulative distribution function of $Y \sim \Gamma(1, b)$ are denoted by f_b and F_b where

$$f_b(y) = \frac{y^{b-1}}{\Gamma(b)} e^{-y}, y \geq 0 \tag{4}$$

First, the shape parameter ρt was assumed to take only integer values: $b \in \mathbb{N}$ for demand during a review period, and $d \in \mathbb{N}$ for demand during lead time l. So, in fact Erlang instead of gamma distributions are considered. Our simulation algorithm was applied to produce 10,000 review periods for this situation. All combinations of the parameter values $b, d, s \in \{1, 2, ..., 10\}$ and $q \in \{0, 1, ..., 20\}$ were considered; for every pair (b, d) a unique demand series $\{x_i, y_i\}$ was generated. For all these 21,000 combinations the estimated fill rate $\widehat{\beta}$ was calculated. The total CPU-time on a Pentium III computer was 109 seconds, which illustrates the speed and efficiency of the algorithm. Figure 1 shows as a typical result the distribution of the net stock before and after delivery.

Figure 1. *Net stock before and after delivery;* $b = 1$, $d = s = q = 2$



From the frequency distribution of NS_{-}, it can be deduced immediately that in 56% of the 10,000 simulated review periods net stock was negative immediately before potential delivery; after potential delivery this was reduced to 3.9%.

For these Erlang distributions, [2] gave the following expressions:

$$E(U) = E_d(U) = e^{-q} \sum_{j=1}^{b} \left(\sum_{k=1}^{\infty} \frac{q^{kb-j}}{(kb-j)!} \right) v_{d+j}(s) - v_d(S) \tag{5}$$

where

$$v_a(x) = \int_x^{\infty} (w - x) f_a(w) dw \tag{6}$$

and

$$E(K) = \sum_{k=1}^{\infty} kP(K = k) = \sum_{k=1}^{\infty} kp_k = \sum_{k=1}^{\infty} \sum_{j=1}^{b} \frac{kq^{kb-j}}{(kb-j)!} e^{-q} \tag{7}$$

Combining (2), (5) and (7), the following exact expression for the fill rate β was obtained:

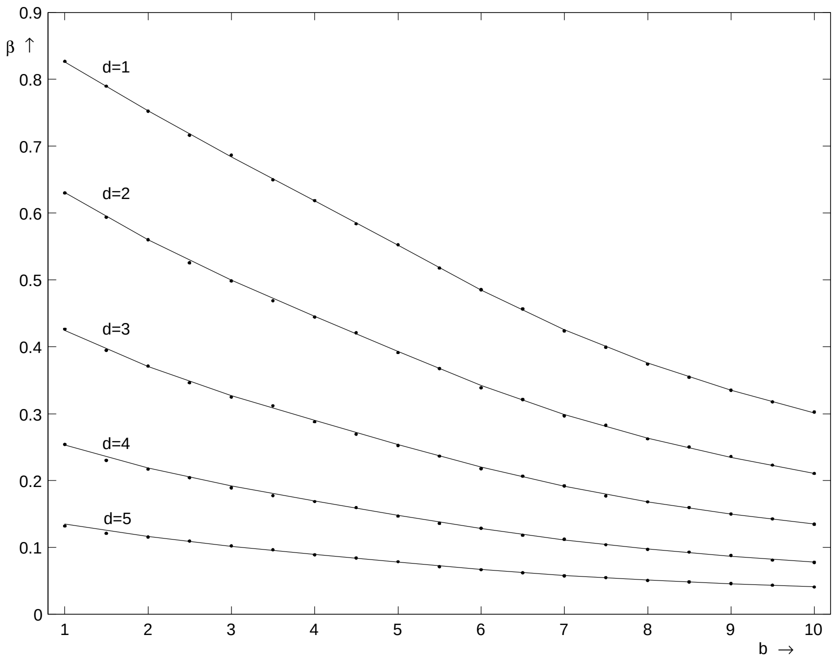
$$\beta = 1 - \frac{\sum_{j=1}^{b} \alpha_j v_{d+j}(s) - v_d(S)}{q + \sum_{j=1}^{b} j\alpha_j} \tag{8}$$

where

$$\alpha_j = e^{-q} \sum_{k=1}^{\infty} \frac{q^{kb-j}}{(kb-j)!} \tag{9}$$

Formula (8) is the main result of [2]; in fact, it gives the function $\beta\,(b, d, s, q)$.

Figure 2. *Simulated and exact fill rates; $q = s = 2$, $d = 1(1)5$*

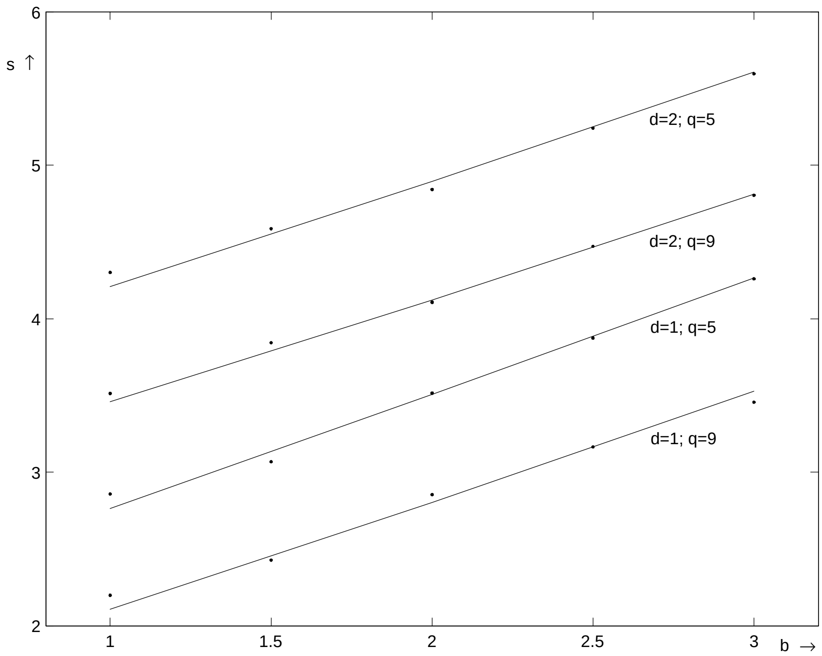


So, a comparison is possible between the simulation outcomes and the theoretical values of $E(U)$ in (5), $E(K)$ in (7) and, particularly, of β in (8) enabling a mutual confirmation. But of course, the algorithm is applicable for real-valued shape parameters as well. Figure 2 shows a typical example of our results. For specific combinations of (q, s, d), exact fill rates β for $b = 1(1)10$ are represented by (invisible) points in Figure 2; to improve readibility, they are connected by line segments for fixed d-values. The dots represent simulated fill rates $\widehat{\beta}$ for $b = 1(0.5)10$; we used 30,000 review periods for this simulation. The agreement between exact and simulated fill rates is quite good; the maximum deviation $|\widehat{\beta} - \beta|$ observed in Figure 2 is 0.0038. See Table 1 in [2] for a more detailed comparison between exact results and simulation.

Finally, control parameter values can be calculated, leading to a prescribed fill rate. As illustration, we calculated the reorder point for $\beta = 0.95$, keeping (b, d, q) fixed.

From the function $\beta(b, d, s, q)$ exact values s can be found for Erlang distributions; for $b = 1, 2, 3$ they are shown in Figure 3, again connected by line segments. For $b = 1(0.5)3$ simulated reorder points $\widetilde{s}$ were found by repeated application of the algorithm in combination with Newton-Raphson techniques; in Figure 3 they are shown as dots.

Figure 3. *Simulated and exact reorder points;* $\beta = 0.95$



The maximum deviation $|\widetilde{s} - s|$ in this figure was 0.14; the maximum relative deviation $|1 - \widetilde{s}/s|$ was 0.044. Applying $\widetilde{s}$ lead to a simulated fill rate that deviated at most 0.0039 from 0.95. More detailed versions of Figure 3 can be used as nomograms to find the reorder point for given demand parameters.

4 Exact and simulated fill rates for *stochastic* lead time

The algorithm described in the Section 2 may be easily adapted for more general use. We show this for the case of stochastic leadtime with given distribution. We need some adapted definitions, see Table 3. Note that the only difference with Table 1 is that l and - consequently - g are now replaced by l_i and g_i. The main change in Table 2 is that an additional line is needed, giving the random drawing l_i from its distribution; a further minor change occurs in the calculation of demand during lead time: it now reads

$$DL_i = \sum_{j=i-\lfloor g_i-\varepsilon \rfloor}^{i-1} z_j + x_i \tag{10}$$

Table 3. *Additional notation*

l_i	length i-th lead time
m_i	potential delivery moment; $m_i = r_i + l_i$
g_i	quotient of lead time i-th and review time; $g_i = \frac{l_i}{R}$
x_i	demand during interval $[r_{i+\lfloor g_i-\varepsilon \rfloor}, m_i)$
y_i	demand during interval $[m_i, r_{i+\lfloor g_i-\varepsilon \rfloor+1})$
z_i	demand during review interval $[r_{i+\lfloor g_i-\varepsilon \rfloor}, r_{i+\lfloor g_i-\varepsilon \rfloor+1})$
DL_i	demand in $[r_i, m_i)$

With these changes our algorithm remains valid, provided the two following complications do not occur. Since leadtime is stochastic now, it may happen that

1. $L_i - L_{i+j} > jR$ for certain $i \geq 1$ and $j \geq 1$ (that is, the order at review moment i wil be delivered later than the order at review moment $i + j$), or

2. Two or more deliveries fall in the same review period.

The first possibility can be excluded by requiring that the distribution of L has the property

$$P\left(L_i - L_{i+j} > jR\right) = 0 \tag{11}$$

The second problem can be dealt with by carefully sampling demand when the number of deliveries in a review period is larger than 1. For example, suppose that we have two deliveries in review period 3, e.g.: $r_3 < m_2 < m_3 < r_4$. Suppose further (again) that demand during a review period follows a stationary gamma-proces $\Gamma(1, b)$. Then x_2 is a random drawing from the distribution $\Gamma(1, b(m_2 - r_3)/R)$, x_3 is obtained by adding x_2 to a drawing from $\Gamma(1, b(m_3 - m_2)/R)$, and z_3 by adding x_3 to a drawing from $\Gamma(1, b(r_4 - m_3)/R)$.

The correctness of this algorithm was verified by simulating a few simple situations with stochastic lead times for which we can determine the exact fill rates as well. Let the distribution of L be given by

$$L = \begin{cases} d_1 R/b & \text{with probability} \quad p \\ d_2 R/b & \text{with probability} \quad 1-p \end{cases} \tag{12}$$

then it is easy to see that the expected shortage can be obtained from (5):

$$E(U) = pE_{d_1}(U) + (1-p)E_{d_2}(U) \tag{13}$$

Figure 4: *Exact fill rates for lead times betweeen 1 and 2; s=2.*

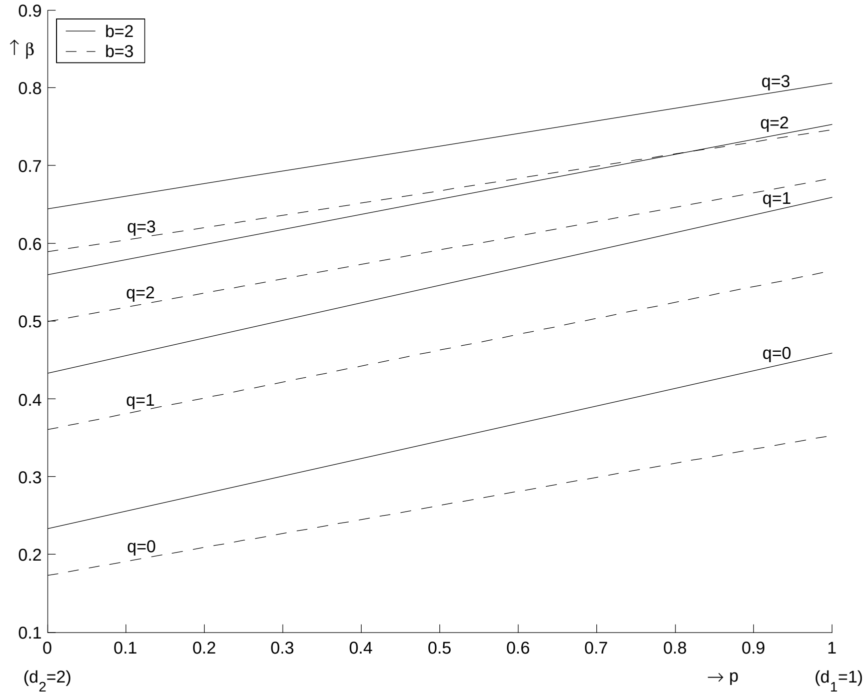
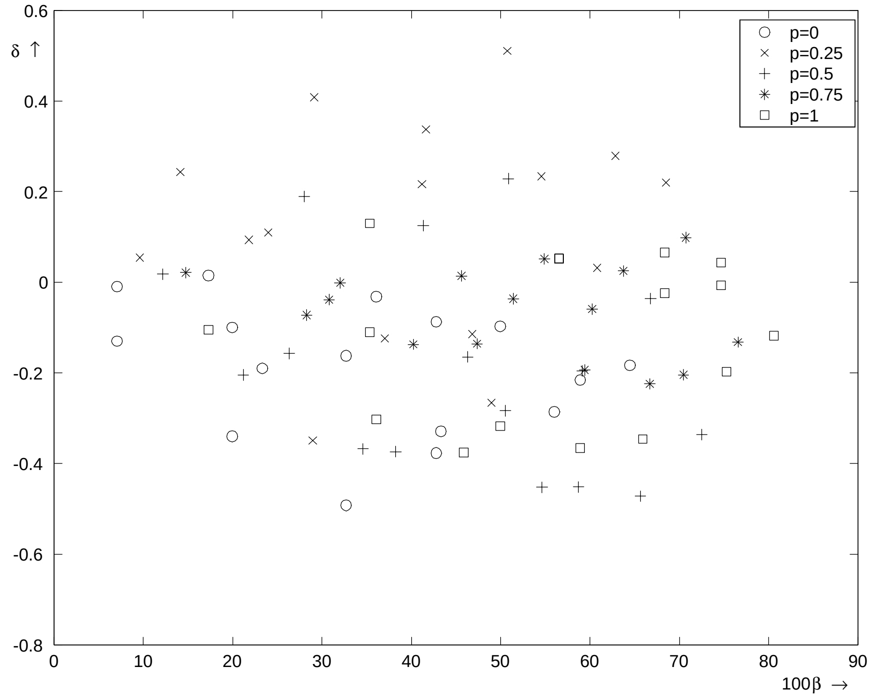


Figure 5: *Difference δ between exact and simulated fill rates for various values of p.*



Formula (7) does not change, so plugging (7) and (13) into (2) yields exact fill rates for this case of stochastic demand. They are shown in Figure 4 as function of p for the case $s = 2$; the other parameter values are indicated in the graph. Our simulation program was applied for $p \in \{0, 1/4, 1/2, 3/4, 1\}$; the control parameter values from Figure 4 were used, extended with the combinations $(3, 1, 3)$ and $(3, 2, 3)$ for $(b,\, d_1, d_2)$. Note that the two problems mentioned before can not occur if we require that $P\,(L \leq R) = 1$, that is $b \geq \max{(d_1, d_2)}$; since this is the case here, our program is applicable without further ado. For all combinations again 30,000 review periods were used. In Figure 5 the simulation results are compared with the exact outcomes. The horizontal axis shows the exact fill rate and the vertical axis the procentual difference $\delta = 100(\hat{\beta} - \beta)$. The deviation from the exact value is at most 0.5%.

5 Conclusion and discussion

In this paper we presented - the core of - a simulation algorithm to determine the fill rate in a (R, s, S)-inventory system; the full Delphi program can be obtained from the first author. Demand is stochastic; simulations may be based on historical demand data or on random drawings. Lead times may be stochastic; note that even the reorder point may change from review to review. The algorithm proved to be extremely fast.

For Erlang distributed demand analytical results are available. Hence, our algorithm could be tested by a comparison between simulated and exact outcomes. The precision of the simulated fill rates proved to be quite acceptable: by using 30,000 review periods, fill rates are estimated with an maximum error of 0.5%; reorder points needed to achieve a prescribed fill rate showed a relative error of 4.4% at most. In general, the algorithm enables accurate determination of fill rates in many situations. Both practitioners and researchers may benefit from the algorithms. Practitioners may use it as an instrument to estimate the effect of setting certain reorder points. Researchers may use it (or variants of it) to study the general behaviour of the fill rate under various circumstances. Besides, it seems to be very simple to adjust the algorithms for simulating other periodic review policies such as an (R, s, Q) or (R, s, nQ)-system.

A subject of further research will be the extension of the presented algorithm to 2-echelon inventory systems. There exist very much literature on 2-echelon inventory systems. However, most of these papers are based again on very specific assumptions, like an analytical demand distribution, and lack general applicability. It is simpler to use a simulation algorithm that can be generally applied. For example, demand distributions and lead time distributions which are based on empirical data may be used; applying forecasting can be incorporated into the algorithm very easily.

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