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On the Connectedness of Coincidences and Zero Points of Mappings[1]

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Abstract

We establish the following generalization of Ky Fan's coincidence theorem. Let X be a non-empty compact and convex set in the n-dimensional Euclidean space $\mathbb{R}^n$, let c be an arbitrary non-zero vector in $\mathbb{R}^n$, and let ϕ and ψ be two upper semi-continuous mappings from X to the collection of non-empty, compact, convex subsets of $\mathbb{R}^n$. Suppose that for every $x \in X$ and every $d \in \mathbb{R}^n$ satisfying both $c^\top d = 0$ and $d^\top x = \max\{d^\top y \mid y \in X, c^\top y = c^\top x\}$, there exist $v \in \phi(x)$ and $w \in \psi(x)$ such that $c^\top v = c^\top w$ and $d^\top v \geq d^\top w$. Then there exists a connected set C of coincidences in X, i.e., $\phi(x) \cap \psi(x) \neq \emptyset$ for every $x \in C$, such that both $C \cap X^- \neq \emptyset$ and $C \cap X^+ \neq \emptyset$, where $X^- = \{x \in X \mid c^\top x \leq c^\top y, \ \forall y \in X\}$ and $X^+ = \{x \in X \mid c^\top x \geq c^\top y, \ \forall y \in X\}$. Several similar results on the existence of a continuum of fixed and zero points and of optima and of solutions to non-linear variational inequality problems are also established. We develop a simplicial algorithm to compute the connected set of solutions. This leads to a constructive proof for our existence theorems.
Keywords: coincidence, fixed point, zero point, continuum, optimum, variational inequality, upper semi-continuity

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JEL classifications: C62, C63

1 Introduction

Several recent studies on equilibrium and fair allocation problems have indicated that there is a growing demand for stronger fixed point theorems beyond Brouwer's and Kakutani's, which guarantee the existence of a continuum of fixed points, stationary points, coincidences or zero points of mappings on arbitrary non-empty compact and convex sets in the n-dimensional Euclidean space $\mathbb{R}^n$; see Herings (1998), Herings, van der Laan and Talman (2001), Herings, Talman and Yang (1996, 2001), Sun and Yang (2001), Talman and Yamamoto (2001), and Yang (1999, 2003). In this paper we introduce a general existence theorem, Theorem 2.1, for a continuum of zero points of a convex-valued, compact-valued upper-semicontinuous mapping on a non-empty compact and convex set X. The continuum is a connected set containing two different points in the boundary of X. At one of these points a linear function, $c^\top x$ for some given non-zero vector c, is minimized on X and at the other point this function is maximized. We show that the theorem extends and unifies existing existence theorems such as Browder's fixed point theorem, see Browder (1960). Browder's theorem is a continuum version of Brouwer's theorem and states that for any continuous function from $X \times [0, 1]$ to X there exists a continuum of fixed points connecting the zero-level with the one-level. Here, the variable lying in $[0, 1]$ can be seen as a homotopy parameter and at each value of this homotopy parameter there exists a solution of the homotopy mapping. As another special case of Theorem 2.1, we establish the existence of a continuum of coincidences of two mappings, thereby generalizing Ky Fan's coincidence theorem (see Fan, 1972) to a connected set of such points. A coincidence is a point at which two images of two different mappings have a non-empty intersection. It is well known from Ky Fan's theorem under which conditions a coincidence exists. However, in the literature there have been no results for the existence of a continuum of such points. Furthermore, we establish several results on the existence of a continuum of fixed points, zero points, optima, and solutions to non-linear variational inequality problems.

In contrast to various approaches used in the literature such as Browder's and Ky Fan's, one prominent feature of our approach is its constructive nature and simplicity in its arguments. To be precise, we demonstrate our main result Theorem 2.1 by means of a simplicial algorithm. This type of algorithm has its root in the work of Scarf (1973). The interested reader could refer to Allgower and Georg (1990), Todd (1976), and Yang (1999) for a comprehensive treatment on the subject. In our approach we first embed the set X of interest into an elaborately-designed full-dimensional rectangular P and then we develop a simplicial algorithm on P. The set P has two distinct facets P^- and P^+ which contain the faces X^- and X^+ of the set X, respectively, that are designated to contain a solution. Given a simplicial subdivision for the set P, the algorithm generates a finite of sequence of adjacent simplices, starting at an arbitrarily chosen point in P^- and ending with a simplex on P^+. It is shown that when the mesh size of the triangulation is small enough, the sequence of simplices induces a path of approximate zero points of the point-to-set mapping of interest linking the facets P^- and P^+. By a limit argument, we will show that there exists a connected set of zero points of the mapping linking the two distinct facets X^- and X^+ of X.

This paper is organized as follows. In Section 2 we present the main existence theorem on an arbitrary non-empty convex and compact set. In Section 3 we propose the simplicial algorithm which will be used to approximate a continuum of solutions and we prove its convergence. In Section 4 we give the constructive proof of the main existence theorem. In Section 5 we show how the main theorem implies the continuum coincidence theorem and several existence theorems of a continuum of zero points, optima, stationary points, and fixed points, including Browder's fixed point theorem.

2 The Main Existence Theorem

Consider an arbitrary non-empty, convex and compact set X in the n-dimensional Euclidean space $\mathbb{R}^n$. Let c be an arbitrary non-zero vector in $\mathbb{R}^n$. Define

$$\begin{array}{rcl} X^+ & = & \{\, x \in X \mid c^\top x \geq c^\top y \text{ for all } y \in X \,\}, \\ X^- & = & \{\, x \in X \mid c^\top x \leq c^\top y \text{ for all } y \in X \,\}, \\ t^+ & = & c^\top x \text{ for any } \; x \in X^+, \\ t^- & = & c^\top x \text{ for any } \; x \in X^-. \end{array}$$

Throughout the paper we assume that $t^+ > t^-$ and thus $X^- \cap X^+ = \emptyset$. For t, $t^- \leq t \leq t^+$, we define $X(t) = \{\, x \in X \mid c^\top x = t \,\}$. Notice that for every t, $t^- \leq t \leq t^+$, the set $X(t)$ is a non-empty, convex, compact set in $\mathbb{R}^n$, $X(t^-) = X^-$, and $X(t^+) = X^+$.

Let Y be an arbitrary non-empty set in $\mathbb{R}^n$. For $x \in Y$, the set

$$N(Y, x) = \{y \in \mathbb{R}^n \mid (x - x')^\top y \geq 0 \text{ for all } x' \in Y\}$$

denotes the normal cone of Y at x and its polar cone

$$T(Y, x) = \{z \in \mathbb{R}^n \mid z^\top y \leq 0 \text{ for all } y \in N(Y, x)\}$$

denotes the tangent cone of Y at x. If Y is compact and convex, $N(Y, \cdot)$ is an upper semi-continuous, convex-valued and closed-valued mapping on Y and $T(Y, \cdot)$ is a convex-valued and closed-valued mapping on Y and, for every $y \in Y$, both $N(Y, y)$ and $T(Y, y)$ are non-empty.

The notion $\mathbb{N}$ denotes the set of all positive integers and I_k denotes the set of the first k positive integers. The notions 0^n, 1^n and $E(n)$ stand for the vector of zeros and ones of dimension n and the $n \times n$ identity matrix, respectively. Given a set D, bd$D$ and int$D$ represent the sets of (relative) boundary and interior points of D, respectively, and co(D) represents the convex hull of D. For an $n \times n$ matrix R and a subset Y of $\mathbb{R}^n$ we define $RY = \{z \in \mathbb{R}^n \mid z = Ry,\ y \in Y\}$. Furthermore, define $B = \{x \in \mathbb{R}^n \mid ||x||_2 \leq 1,\ c^\top x = 0\}$ and $C(v) = \{y \in \mathbb{R}^n \mid y = \mu v + \beta c, \mu \geq 0, \beta \in \mathbb{R}\}$ for every $v \in B$.

A topological space W is said to be connected if the only subsets of W both open and closed are $\emptyset$ and W. A subset of W is called a *connected set* if it is connected as a subspace of W. Given an element $y \in W$, the union of all connected subsets of W containing y is called *the component* of y in W. See Dugundji (1970).

Let ϕ be an upper semi-continuous mapping from X to the collection of non-empty convex and compact subsets of $\mathbb{R}^n$. A point $x^* \in X$ is called a *zero point* of ϕ if $0^n \in \phi(x^*)$, a *fixed point* of ϕ if $x^* \in \phi(x^*)$, a *coincidence* of ϕ and some other mapping ψ on X if $\phi(x^*) \cap \psi(x^*) \neq \emptyset$, and a *stationary point* of ϕ or a solution to the *non-linear variational inequality problem* for ϕ on X if there exists $f^* \in \phi(x^*)$ satisfying $(x^* - x)^\top f^* \geq 0$ for all $x \in X$. Notice that x^* is a stationary point of ϕ if and only if $\phi(x^*) \cap N(X, x^*) \neq \emptyset$. Without any further conditions on ϕ there may not exist any solution at all on X for some of the solution concepts, not to mention a continuum of solutions.

In this paper we are interested in conditions on the mapping ϕ, under which there exists a connected set of solutions of ϕ in X having a non-empty intersection with both X^- and X^+. Since the intersection of X^- and X^+ is empty this implies that the connected set of solutions contains a continuum of points. A solution could be a zero point, fixed point, stationary point, or coincidence with some other mapping on X.

Now we may state the main continuum zero point existence theorem of this paper.

Theorem 2.1 *Let ϕ be an upper semi-continuous mapping from X to the collection of non-empty convex and compact subsets of $\mathbb{R}^n$. Suppose that there exists an upper semi-continuous mapping π from B to the collection of non-empty convex and closed subsets of $\mathbb{R}^n$, and that for every $x \in X$ there exists a non-singular $n \times n$ matrix $A(x)$ being continuous in x, such that for every $x \in X$ and every $v \in N(X(t), x) \cap B$ with $t = c^\top x$, the following two conditions hold:*

1. *The set $A(x)\phi(x) \cap \pi(v) \cap C(v)$ is either empty or contains 0^n;*

2. *The set $A(x)\phi(x) \cap \pi(v) \neq \emptyset$.*

Then there exists a connected set C of zero points of ϕ in X such that $X^- \cap C \neq \emptyset$ and $X^+ \cap C \neq \emptyset$.

The theorem says that the mapping ϕ has a continuum of zero points on X connecting X^- and X^+, if there exists a continuous regular matrix mapping A and an upper semi-continuous, convex-valued and closed-valued mapping π on B satisfying that, for every element v in B of the normal cone of $X(c^\top x)$ at any point x of X with length at most one, the set $A(x)\phi(x)$ and $\pi(v)$ intersect, but this intersection may have no points in common with the two-dimensional cone $C(v)$ determined by the vectors v, c and $-c$, unless the origin is contained in this intersection.

The matrix $A(x)$ translates the image $\phi(x)$ in a linear way, so that $A(x)\phi(x)$ has the same convexity properties as $\phi(x)$ has. Due to the regularity of the matrix $A(x)$ at any x in X, a point x^* is a zero point of ϕ if and only if x^* is a zero point of $A(x)\phi(x)$. The use of the linear mapping $A(\cdot)$ expands the cases to which our result applies. For example, consider $X = B^n$, $c = 1^n$ and the function $f : B^n \to \mathbb{R}^n$ defined by $f(x) = x - (c^\top x)c/n$, where B^n is the n-dimensional unit ball. Then there is no mapping π that satisfies both conditions 1 and 2 with $A(x) = E(n)$, although $f(\beta 1^n) = 0^n$ for any feasible β, connecting X^- and X^+. However, when we take $A(x) = -E(n)$ for all $x \in B^n$, conditions 1 and 2 are satisfied for $\pi(v) = \mathbb{R}^n$ for any $v \in B$.

3 A Simplicial Algorithm

In this section we propose a simplicial algorithm which will lead to a constructive proof of Theorem 2.1. For $t \in \mathbb{R}$, let $H(t) = \{y \in \mathbb{R}^n \mid c^\top y = t\}$, and let H be the union of $H(t)$ over t, $t^- \leq t \leq t^+$. Let $H^- = H(t^-)$ and $H^+ = H(t^+)$. For $x \in H$, let $p(x)$ be the orthogonal projection of x on $X(t)$, where $t = c^\top x$. Since X is a non-empty, compact, convex set, p is a continuous function on H. Moreover, $x - p(x) \in N(X(t), p(x))$ for $x \in H(t)$, $t^- \leq t \leq t^+$. For t, $t^- \leq t \leq t^+$, the set $Q(t)$ is defined by

$$Q(t) = \{q \in H(t) \mid \| q - p(q) \|_2 \leq 1\}.$$

The union of $Q(t)$ over t, $t^- \leq t \leq t^+$, is denoted by Q.

Lemma 3.1 *The set Q is a full-dimensional, compact, convex subset of H.*

Proof: Clearly, Q is a full-dimensional set in $\mathbb{R}^n$ and a subset of H. Since X is compact, Q is also compact. To prove convexity of Q, take any $q^1, q^2 \in Q$ and $0 \leq \lambda \leq 1$ and let

$$q(\lambda) = \lambda q^1 + (1 - \lambda)q^2$$

and

$$p(\lambda) = \lambda p(q^1) + (1 - \lambda)p(q^2).$$

Since X is convex, we have that $p(\lambda) \in X$. Let $t = c^\top p(\lambda)$, i.e. $p(\lambda) \in H(t)$. Note that $c^\top q^1 = c^\top p(q^1)$ and $c^\top q^2 = c^\top p(q^2)$. Then we have $c^\top q(\lambda) = t$, i.e. $q(\lambda) \in H(t)$. Moreover,

$$\| q(\lambda) - p(\lambda) \|_2 \quad \leq \lambda \| q^1 - p(q^1) \|_2 + (1-\lambda) \| q^2 - p(q^2) \|_2 \quad \leq 1.$$

Therefore, $q(\lambda) \in Q$, i.e., Q is a convex set. □

For $q \in Q$, let $v(q) = q - p(q)$. By construction, $v(q) \in B$ for every $q \in Q$, $\| v(q) \|_2 = 1$ if and only if $q \in \text{bd}Q$, and $v(q) = 0^n$ if and only if $q \in X$.

Lemma 3.2 *For every $q \in Q$ it holds that $N(Q(t), q) = C(v(q))$ and $N(Q(t), q) \subseteq N(X(t), p(q))$, where $t = c^\top q$.*

Proof: Since Q is full-dimensional, for $q \in \text{int}Q(t)$ it holds that $N(Q(t), q) = C(0^n)$ and, hence, also $N(Q(t), q) \subseteq N(X(t), p(q))$. Take any point $q \in \text{bd}Q(t)$. Since $p(q)$ is the projection of q on $X(t)$, we must have that $N(Q(t), q)$ contains $C(v(q))$. Let $B(p(q)) = \{x \in H(t) \mid \| x - p(q) \|_2 \leq 1\}$. Clearly, $B(p(q)) \subseteq Q(t)$ and $q \in \text{bd}B(p(q))$, and therefore $N(Q(t), q) \subseteq N(B(p(q)), q)$. However, $q \in \text{bd}B(p(q))$ implies $N(B(p(q)), q) = C(q - p(q))$. Since $v(q) = q - p(q)$, we obtain $N(Q(t), q) \subseteq C(v(q))$. Hence, $N(Q(t), q) = C(v(q))$. Since $q - p(q) \in N(X(t), p(q))$, it follows that $C(v(q)) \subseteq N(X(t), p(q))$. □

Let $a^1, \ldots, a^{n-1}$ be an orthogonal basis for the $(n-1)$-dimensional subspace $H^0 = \{y \in \mathbb{R}^n \mid c^\top y = 0\}$, i.e.,

- $H^0 = \{y \in \mathbb{R}^n \mid y = \sum_{i=1}^{n-1} \lambda_i a^i, \lambda_i \in \mathbb{R} \text{ for all } i\}$;
- $c^\top a^i = 0$ for all i;
- $a^{i\top} a^i = 1$ for all i;
- $a^{i\top} a^j = 0$ for all $j \neq i$.

Without loss of generality we also assume that $c^\top c = 1$. Take any x^0 in X^- and $M > 0$. Then the $(n-1)$-dimensional cube P^- in H^- is defined by

$$P^- = \{x \in H^- \mid -M \leq a^{i\top}(x - x^0) \leq M, \; i = 1, \ldots, n-1\}.$$

and the rectangular P in H is defined by

$$P = \{x \in H \mid x = x^- + \lambda c, \; 0 \leq \lambda \leq t^+ - t^-, \; x^- \in P^-\}.$$

For t, $t^- \leq t \leq t^+$, let $P(t) = P \cap H(t)$. We can choose M so large that for every t, $t^- \leq t \leq t^+$, the cube $P(t)$ contains $Q(t)$ in its relative interior. For $i = 1, \ldots, n-1$, define $a^{-i} = -a^i$ and let $I = \{-(n-1), \ldots, -1, 1, \ldots, n-1\}$. Then P can be reformulated as

$$P = \{x \in \mathbb{R}^n \mid a^{i\top} x \leq M - a^{i\top} x^0 \text{ for all } i \in I, c^\top x \leq t^+, -c^\top x \leq t^-\}.$$

Let $\bar{I} = I \cup \{-n, n\}$, $a^n = c$, $a^{-n} = -c$, $b_i = M - a^{i\top} x^0$ for every $i \in I$, $b_n = t^+$, and $b_{-n} = t^-$. Then $P^- = P(t^-)$ and P can be further rewritten as

$$P^- = \{x \in \mathbb{R}^n \mid a^{i\top} x \leq b_i \text{ for all } i \in I \ \text{ and } \ c^\top x = t^-\}$$

and

$$P = \{x \in \mathbb{R}^n \mid a^{i\top} x \leq b_i \text{ for all } i \in \bar{I}\}.$$

Similarly, we may write $P^+ = P(t^+)$ as

$$P^+ = \{x \in \mathbb{R}^n \mid a^{i\top} x \leq b_i \text{ for all } i \in I \ \text{ and } \ c^\top x = t^+\}.$$

Notice that P is a simple full-dimensional polytope, and no constraints are redundant. Let $\mathcal{I}$ be the collection of subsets J of I such that $|J| \leq n-1$ and $j \notin J$ whenever $-j \in J$. For each $J \in \mathcal{I}$, define

$$F(J) = \{x \in P^- \mid a^{i\top} x = b_i, \text{ for all } i \in J\}.$$

Clearly, $F(J)$ is a face of P^- and $F(\emptyset) = P^-$.

Let v be any point in P^-. The point v will be the starting point of the algorithm to be described below. Let $vF(J)$ be the convex hull of the point v and a face $F(J)$ not containing v. Now we first describe a simplicial subdivision or triangulation of the polytope P which underlies the algorithm.

For a nonnegative integer t, a t-dimensional simplex or t-simplex, denoted by σ, is defined by the convex hull of $t+1$ affinely independent points $x^1, \cdots, x^{t+1}$ in $\mathbb{R}^n$. We often write $\sigma = \sigma(x^1, \cdots, x^{t+1})$ and call $x^1, \cdots, x^{t+1}$ the vertices of σ. A $(t-1)$-simplex being the convex hull of t vertices of σ is said to be a facet of σ. The facet $\tau(x^1, \cdots, x^{i-1}, x^{i+1}, \cdots, x^{t+1})$ is called the facet of $\sigma(x^1, \cdots, x^{t+1})$ opposite to the vertex x^i. For k, $0 \leq k \leq t$, a k-simplex being the convex hull of $k+1$ vertices of σ is said to be a k-face or face of σ. A finite collection $\mathcal{T}$ of n-simplices is a triangulation of the polytope P if

(i) P is the union of all simplices in $\mathcal{T}$;

(ii) The intersection of any two simplices of $\mathcal{T}$ is either the empty set or a common face of both.

The diameter of a simplex $\sigma(x^1, \cdots, x^{n+1})$ is the maximum Euclidean distance between any two points in σ and is denoted by $diam(\sigma)$. The mesh size of a triangulation $\mathcal{T}$ is defined as

$$mesh(\mathcal{T}) = \max_{\sigma \in \mathcal{T}} \{diam(\sigma)\}.$$

Let $\mathcal{T}$ be a triangulation of P such that every subset $vF(J)$ of P^- is subdivided into t-simplices, where $t = n - |J|$. For example, we may take the V-triangulation introduced by Talman and Yamamoto (1989) for triangulating a polytope. Since $\mathcal{T}$ is finite and P is compact, every facet τ of an n-simplex σ on P either lies on the boundary of P and is only a facet of σ or does not lie on the boundary of P and is a facet of exactly one other n-simplex in $\mathcal{T}$. Similarly, a facet of a t-simplex σ on $vF(J)$, where $t = n - |J|$, either lies on the boundary of $vF(J)$ and is only a facet of σ or does not lie on the boundary of $vF(J)$ and is a facet of exactly one other t-simplex on $vF(J)$.

Now we consider the point-to-set mapping $\bar{\phi} : P \mapsto \mathbb{R}^n$ defined by

$$\bar{\phi}(x) = \begin{cases} \{p(x) - x\}, & \text{if } x \in P \setminus Q, \\ \mathrm{co}(\{p(x) - x\} \cup [A(p(x))\phi(p(x)) \cap \pi(x - p(x))]), & \text{if } x \in \mathrm{bd}Q, \\ A(p(x))\phi(p(x)) \cap \pi(x - p(x)), & \text{if } \ x \in \mathrm{int}Q. \end{cases}$$

Recall that for $x \in Q$ the vector $x - p(x)$ is an element of $N(X(t), p(x)) \cap B$, where $t = c^\top x$, so that according to condition 2 of Theorem 2.1 $\bar{\phi}(x) \neq \emptyset$ for all $x \in P$. One can easily verify that $\bar{\phi}$ is an upper semi-continuous mapping with convex and compact values. Let x be a vertex of a simplex of $\mathcal{T}$, then we assign to x the vector label $f(x)$, where $f(x)$ is an arbitrarily chosen element in $\bar{\phi}(x)$. Now we extend f piecewise linearly on each simplex of $\mathcal{T}$, i.e., f is affine on each simplex of $\mathcal{T}$. We call f *the piecewise linear approximation of* $\bar{\phi}$ *with respect to* $\mathcal{T}$.

A row vector is *lexicopositive* if it is a non-zero vector and its first non-zero entry is positive. A matrix is said to be *lexicopositive* if all its rows are lexicopositive. A matrix is said to be *semi-lexicopositive* if each row except possibly the last row is lexicopositive.

Definition 3.3 *Let $\tau(x^1, \cdots, x^t)$ be a facet of a t-simplex on $vF(J)$, where $J \in \mathcal{I}$ with $J = \{j_{t+1}, \cdots, j_n\}$, $t = n - |J|$. The $(n+1) \times (n+1)$ matrix*

$$A_{\tau,J} = \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ -f(x^1) & \cdots & -f(x^t) & a^{j_{t+1}} & \cdots & a^{j_n} & c \end{bmatrix}$$

is the label matrix of τ with respect to J. The simplex τ is J-complete if $A_{\tau,J}^{-1}$ exists and is semi-lexicopositive.

Definition 3.4 *Let $\tau(x^1, \cdots, x^n)$ be a facet of an n-simplex on P. The $(n+1) \times (n+1)$ matrix*

$$A_\tau = \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 \\ -f(x^1) & -f(x^2) & \cdots & -f(x^n) & c \end{bmatrix}$$

is the label matrix of τ. The simplex τ is complete if A_τ^{-1} exists and is semi-lexicopositive.

Notice that if for a J-complete (complete) simplex τ we change the ordering of the first n columns of the matrix $A_{\tau,J}$ (A_τ), the inverse of the resulting matrix still exists and is semi-lexicopositive. Clearly, if, for some $J \in \mathcal{I}$, a $(t-1)$-simplex $\tau(x^1, \ldots, x^t)$ is a J-complete facet of a simplex $\sigma(x^1, \ldots, x^{t+1})$ on $vF(J)$, then the system of $n+1$ linear equations with $n+2$ variables

$$\sum_{i=1}^{t+1} \lambda_i \begin{pmatrix} 1 \\ -f(x^i) \end{pmatrix} + \sum_{j \in J} \mu_j \begin{pmatrix} 0 \\ a^j \end{pmatrix} + \beta \begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0^n \end{pmatrix} \qquad (*)$$

has a solution $(\lambda, \mu, \beta) = (\lambda_1, \ldots, \lambda_{t+1}, (\mu_j)_{j\in J}, \beta)$ satisfying $\lambda_i \geq 0$ for all $i \in I_{t+1}$ and $\mu_j \geq 0$ for all $j \in J$, with $\lambda_{t+1} = 0$. Let x be defined by $x = \sum_{i=1}^{t+1} \lambda_i x^i$ at a solution (λ, μ, β) of $(*)$. Then x lies in σ and $f(x) = \sum_{j\in J} \mu_j a^j + \beta c$. Similarly, if an $(n-1)$-simplex $\tau(x^1, \ldots, x^n)$ is a complete facet of a simplex $\sigma(x^1, \ldots, x^{n+1})$ on P, then the system of $n+1$ linear equations with $n+2$ variables

$$\sum_{i=1}^{n+1} \lambda_i \begin{pmatrix} 1 \\ -f(x^i) \end{pmatrix} + \beta \begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0^n \end{pmatrix} \qquad (**)$$

has a solution $(\lambda, \beta) = (\lambda_1, \ldots, \lambda_{n+1}, \beta)$ satisfying $\lambda_i \geq 0$ for all $i \in I_{n+1}$, with $\lambda_{t+1} = 0$. Let x be defined by $x = \sum_{i=1}^{n+1} \lambda_i x^i$ at a solution (λ, β) of $(**)$. Then x lies in σ and $f(x) = \beta c$ lies in $C(0^n)$.

Here we recall the following result from Fujishige and Yang (1998) which will be used below.

Lemma 3.5 *Consider any polytope* V *given by* $\mathrm{V} = \{x \in \mathbb{R}^n \mid c^{i\top} x \leq d_i,\ i \in I_n$ *and* $c^{0\top} x = d_0\}$. *Suppose that* V *is an* $(n-1)$*-dimensional simple polytope with no redundant constraints. For any* $g \in \mathbb{R}^n$, *there exists a unique subset* $\{j_1, \cdots, j_{n-1}\}$ *of* I_n *such that the inverse of the matrix*

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ g & c^{j_1} & c^{j_2} & \cdots & c^{j_{n-1}} & c^0 \end{bmatrix}$$

exists and is semi-lexicopositive.

We now show that $\{v\}$ is a J-complete 0-simplex for a unique index set $J \in \mathcal{I}$ containing $n-1$ indices.

Lemma 3.6 *There exists a unique subset* $J = \{j_1, \cdots, j_{n-1}\}$ *of* $\mathcal{I}$ *such that* $\{v\}$ *is a* J*-complete* 0*-simplex.*

Proof: Note that the set

$$P^- = \{x \in \mathbb{R}^n \mid a^{i\top} x \leq b_i \text{ for all } i \in I \ \text{ and } \ c^\top x = t^-\}$$

is an $(n-1)$-dimensional simple polytope with no redundant constraints. It follows immediately from Lemma 3.5 that there exists a unique subset $J = \{j_1, \cdots, j_{n-1}\}$ of $\mathcal{I}$ such that the inverse of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -f(v) & a^{j_1} & a^{j_2} & \cdots & a^{j_{n-1}} & c \end{bmatrix}$$

exists and is semi-lexicopositive. This means that the 0-simplex $\{v\}$ is J-complete. By definition, $\{v\}$ lies on $vF(J)$. □

The following lemma is well-known in linear programming theory and can easily be proved. Let R be a matrix. We denote its i-th row by $R_{i.}$ and its j-th column by $R_{.j}$.

Lemma 3.7 *Let $R = (R_{.1}, \cdots, R_{.n+1})$ be any non-singular $(n+1)\times(n+1)$ matrix and let x be any vector in $\mathbb{R}^{n+1}$. Let $k \in I_{n+1}$ and $\bar{R} = (R_{.1}, \cdots, R_{.k-1}, x, R_{.k+1}, \cdots, R_{.n+1})$. Then either $(R^{-1}x)_k = 0$ and $\bar{R}$ is singular, or $(R^{-1}x)_k \neq 0$, $\bar{R}$ is non-singular and $\bar{R}^{-1}$ is given by*

$$\bar{R}^{-1} = \begin{bmatrix} (R^{-1})_{1.} - \frac{(R^{-1}x)_1}{(R^{-1}x)_k}(R^{-1})_{k.} \\ \vdots \\ (R^{-1})_{k-1.} - \frac{(R^{-1}x)_{k-1}}{(R^{-1}x)_k}(R^{-1})_{k.} \\ \frac{1}{(R^{-1}x)_k}(R^{-1})_{k.} \\ (R^{-1})_{k+1.} - \frac{(R^{-1}x)_{k+1}}{(R^{-1}x)_k}(R^{-1})_{k.} \\ \vdots \\ (R^{-1})_{n+1.} - \frac{(R^{-1}x)_{n+1}}{(R^{-1}x)_k}(R^{-1})_{k.} \end{bmatrix}.$$

Using this lemma, the following lemmas will be proved.

Lemma 3.8 *Let σ be a t-simplex on $vF(J)$ where $J \in \mathcal{I}$, $t = n - |J|$ and $J = \{j_{t+1}, \cdots, j_n\}$. If σ has a J-complete facet τ, then exactly one of the following three cases occurs:*

(1) The simplex σ is a complete simplex on P^-;

(2) The simplex σ is a $\bar{J}$-complete simplex on $vF(\bar{J})$ where $\bar{J} = J \setminus \{j\}$ for precisely one index $j \in J$;

(3) The simplex σ has exactly one other J-complete facet $\bar{\tau}$.

Proof: Let x^{t+1} be the vertex of σ opposite to τ, and let $y = A_{\tau,J}^{-1}(1, -f(x^{t+1})^\top)^\top$. Notice that $y \neq 0^{n+1}$. Let $K = \{i \in I_n \mid y_i > 0\}$. We first prove $|K| > 0$. Since $A_{\tau,J}y = (1, -f(x^{t+1})^\top)^\top$, we have $\sum_{i=1}^t y_i = 1$. This implies that there exists at least one index $i \in I_t$ such that $y_i > 0$. Hence K is non-empty.

Consider the ratio vectors $(1/y_j)(A_{\tau,J}^{-1})_{j\cdot}$ for all $j \in K$. Choose $k \in K$ such that the k-th ratio vector is the minimum in the lexicographic order over all such ratio vectors. Since $A_{\tau,J}^{-1}$ is regular, k is uniquely determined. Now we consider the following two cases (i) and (ii).

(i) If $k \in I_n \setminus I_t$, then let $l = i_k$ and $\bar{J} = J \setminus \{l\}$. If $\bar{J} = \emptyset$, then σ is a complete simplex on P^-. Otherwise, $\bar{J} \in \mathcal{I}$ and σ is on $vF(\bar{J})$. Let R be the matrix obtained from $A_{\tau,J}$ by replacing its k-th column by $(1, -f(x^{t+1})^\top)^\top$. It follows from Lemma 3.7 that R^{-1} exists and is semi-lexicopositive. By reordering the columns of R we get $A_{\sigma,\bar{J}}$ whose inverse exists and is semi-lexicopositive. So, σ is $\bar{J}$-complete.

(ii) If $k \in I_t$, then let $\bar{\tau}$ be the facet of σ opposite to the vertex x^k. Using Lemma 3.7, it follows from the choice of k that $A_{\bar{\tau},J}^{-1}$ exists and is semi-lexicopositive. Hence $\bar{\tau}$ is a J-complete $(t-1)$-simplex on $vF(J)$.

It follows immediately from Lemma 3.7 that if any column other than the k-th column is replaced, then the inverse of the resulting matrix is not semi-lexicopositive. □

Lemma 3.9 *Let σ be a J-complete $(t-1)$-simplex on $vF(J)$ where $J \in \mathcal{I}$, $t = n - |J|$ and $J = \{j_{t+1}, \cdots, j_n\}$. If σ is on $vF(\bar{J})$ where $\bar{J} = J \cup \{l\} \in \mathcal{I}$ for some $l \in I_m \setminus J$, then exactly one of the following two cases occurs:*

(1) There exists a unique set $J' \in \mathcal{I}$ with $|J'| = |J|$ and $J' \neq J$ so that σ is on $vF(J')$ and is J'-complete.

(2) There exists exactly one facet τ of σ which is on $vF(\bar{J})$ and is $\bar{J}$-complete.

Proof: Let $x = (0, a^{l\top})^\top$ and $y = A_{\sigma,J}^{-1} x$. Note that $y \neq 0^{n+1}$. Let $K = \{i \in I_n \mid y_i > 0\}$. Note that $A_{\sigma,J} y = (0, a^{l\top})^\top$. We first show that K is non-empty. Suppose that $y_i = 0$ for all $i \in I_t$. Then there must exist some parameters y_i for $i = t+1,\ t+2,\ \cdots,\ n$, such that $a^l = \sum_{i=t+1}^{n} y_i a^{j_{i-t}} + y_{n+1} c$, and y_i must be non-zero for some i. This implies that the vectors c, a^l, a^j for all $j \in J$ are linearly dependent. This is impossible by assumption. Hence there exists at least one index $i \in I_t$ such that $y_i \neq 0$. If there exists an index $j \in I_t$ such that $y_j < 0$, then there must exist an index $i \in I_t$ such that $y_i > 0$ since $\sum_{k=1}^{t} y_i = 0$. Hence K is non-empty.

Consider the ratio vectors $(1/y_j)(A_{\sigma,J}^{-1})_{j\cdot}$ for all $j \in K$. Choose $k \in K$ such that the k-th ratio vector is the minimum in the lexicographic order over all such ratio vectors. Since $A_{\tau,J}^{-1}$ is regular, k is uniquely determined. Now we consider the following two cases (1) and (2).

(1) If $k \in I_n \setminus I_t$, then let $p = i_k$ and $J' = J \cup \{l\} \setminus \{p\}$. Clearly, $J' \in \mathcal{I}$, $J' \neq J$, $|J'| = |J|$ and σ is on $vF(J')$. Let R be the matrix obtained from $A_{\sigma,J}$ by replacing its

k-th column by x. It follows from Lemma 3.7 that R^{-1} exists and is semi-lexicopositive. It is clear that $A_{\sigma,J'} = R$. So, σ is a J'-complete $(t-1)$-simplex on $vF(J')$.

(2) If $k \in I_t$, then let τ be the facet of σ opposite to the vertex x^k. Clearly, τ is a $(t-2)$-simplex on $vF(\bar{J})$. Let R be the matrix obtained from $A_{\sigma,J}$ by replacing its k-th column by x. It follows from Lemma 3.7 that R^{-1} exists and is semi-lexicopositive. By reordering the columns of R we get $A_{\tau,\bar{J}}$ whose inverse also exists and is semi-lexicopositive. So, τ is a $\bar{J}$-complete $(t-2)$-simplex on $vF(\bar{J})$.

Again it follows from Lemma 3.7 that if any other column is replaced, then the new matrix is no longer semi-lexicopositive. □

Lemma 3.10 *Let $\tau(x^1, \cdots, x^n)$ be a complete $(n-1)$-simplex on $vF(\{i\})$ for some $i \in I$. Then there exists exactly one n-simplex $\sigma(x^1, \cdots, x^{n+1})$ on P which has τ as one of its facets and σ has exactly one other complete facet $\bar{\tau}$. Furthermore, τ has exactly one facet τ^* which is $\{i\}$-complete.*

Proof: Let x^{t+1} be the vertex of σ opposite to τ, and let $y = A_\tau^{-1}(1, -f(x^{t+1})^\top)^\top$. Notice that $y \neq 0^{n+1}$. Let $K = \{i \in I_n \mid y_i > 0\}$. We first prove $|K| > 0$. Since $A_\tau y = (1, -f(x^{t+1})^\top)^\top$, we have $\sum_{i=1}^t y_i = 1$. This implies that there exists at least one index $i \in I_t$ such that $y_i > 0$. Hence K is non-empty.

Consider the ratio vectors $(1/y_j)(A_\tau^{-1})_{j\cdot}$ for all $j \in K$. Choose $k \in K$ such that the k-th ratio vector is the minimum in the lexicographic order over all such ratio vectors. Since A_τ^{-1} is regular, k is uniquely determined. Note that $J = \emptyset$. Let $\bar{\tau}$ be the facet of σ opposite to the vertex x^k. Using Lemma 3.7, it follows from the choice of k that $A_{\bar{\tau}}^{-1}$ exists and is semi-lexicopositive. Hence $\bar{\tau}$ is a complete facet of σ.

It follows immediately from Lemma 3.7 that if any column other than the k-th column is replaced, then the inverse of the resulting matrix is not semi-lexicopositive.

To prove the second part of the lemma, let $x = (0, a^{i\top})^\top$ and $y = A_\tau^{-1}x$. Note that $y \neq 0^{n+1}$. Let $K = \{i \in I_n \mid y_i > 0\}$. Note that $A_\tau y = (0, a^{i\top})^\top$. We will show that K is non-empty. Suppose that $y_i = 0$ for all $i \in I_n$. Then there must exist some parameter β such that $\beta c = a^i$. This implies that the vectors c and a^i are linearly dependent. This is impossible by assumption. Hence there exists at least one index $i \in I_n$ such that $y_i \neq 0$. If there exists an index $j \in I_n$ such that $y_j < 0$, then there must exist an index $i \in I_n$ such that $y_i > 0$ since $\sum_{i=1}^n y_i = 0$. Hence K is non-empty.

Consider the ratio vectors $(1/y_j)(A_\tau^{-1})_{j\cdot}$ for all $j \in K$. Choose $k \in K$ such that the k-th ratio vector is the minimum in the lexicographic order over all such ratio vectors. Since A_τ^{-1} is regular, k is uniquely determined. Now let τ^* be the facet of τ opposite to the vertex x^k. Let R be the matrix obtained from A_τ by replacing its k-th column by x. It follows from Lemma 3.7 that R^{-1} exists and is semi-lexicopositive. By reordering the columns of R we get $A_{\tau^*,\{i\}}$ whose inverse also exists and is semi-lexicopositive. So, τ^* is an $\{i\}$-complete $(n-2)$-simplex on $vF(\{i\})$.

Again it follows from Lemma 3.7 that if any other column is replaced, then the new matrix is no longer semi-lexicopositive. $\Box$

By repeating the first part of the proof of Lemma 3.10, one can show the following lemma.

Lemma 3.11 *If an n-simplex $\sigma(x^1, \cdots, x^{n+1})$ on P has a complete facet $\tau(x^1, \cdots, x^n)$, then σ has exactly one other complete facet $\bar{\tau}$.*

In the following we will show that starting at v there exists a finite sequence of adjacent J-complete or complete simplices for varying J, $J \in \mathcal{I}$, which leads to a complete $(n-1)$-simplex σ^+ on P^+. First we show that a J-complete facet can not lie on the boundary of P^-.

Lemma 3.12 *If $\tau(x^1, \cdots, x^t)$ is a J-complete $(t-1)$-simplex on $vF(J)$ where $t = n - |J|$, then τ does not lie on the boundary of P^-.*

Proof: Suppose to the contrary that τ lies on the boundary of P^-. Then τ must be a subset of $F(J)$, so $f(x^i) = p(x^i) - x^i$ for all $i = 1, \cdots, t$, and $a^{j\top} x^i = b_j$ for all $j \in J$ and $i = 1, \cdots, t$. Since $a^{j\top} p(x^i) < b_j$ we obtain that $a^{j\top} f(x^i) < 0$ for all $j \in J$ and all $i = 1, \cdots, t$. Because τ is J-complete, we have

$$\sum_{i=1}^{t} \lambda_i f(x^i) = \sum_{j \in J} \mu_j a^j + \beta c \tag{3.1}$$

for some $\lambda_i \geq 0$, $i = 1, \cdots, t$, $\mu_j \geq 0$ for all $j \in J$, and $\beta \in \mathbb{R}$, with $\sum_{i=1}^{t} \lambda_i = 1$. By premultiplying equation (3.1) with any vector a^i, $i \in J$, we obtain

$$\begin{aligned} 0 &> \textstyle\sum_{h=1}^{t} \lambda_h a^{i\top} f(x^h) \\ &= \textstyle\sum_{j \in J} \mu_j a^{i\top} a^j + \beta a^{i\top} c \\ &= \mu_i \\ &\geq 0, \end{aligned}$$

yielding a contradiction. The assumptions imposed on the a^is and c imply the above equalities. $\Box$

The next lemma shows that every complete simplex not on P^- or P^+ can not lie on the boundary of P.

Lemma 3.13 *If $\tau(x^1, \cdots, x^n)$ is a complete $(n-1)$-simplex on bd$P$, then either τ lies on P^- or τ lies on P^+.*

Proof: Suppose to the contrary that τ is a subset of the set $P \setminus (intP^- \cup intP^+)$, namely, there exists some $i \in I$ such that $a^{i\top} x^h = b_i$ for all $h = 1, \cdots, n$. Then we have $f(x^h) = p(x^h) - x^h$ for all $h = 1, \cdots, n$, and so $a^{i\top} f(x^h) < 0$ for all $h = 1, \cdots, n$. Because τ is complete, we have

$$\sum_{h=1}^{n} \lambda_h f(x^h) = \beta c \tag{3.2}$$

for some $\lambda_h \geq 0$, $h = 1, \cdots, n$, and some β, with $\sum_{h=1}^{n} \lambda_h = 1$. Premultiplying equation (3.2) by the vector a^i yields a contradiction, namely,

$$\begin{aligned} 0 &> \textstyle\sum_{h=1}^{n} \lambda_h a^{i\top} f(x^h) \\ &= \beta a^{i\top} c \\ &= 0. \end{aligned}$$

□

We construct a graph $G = (N, E)$ where N denotes a set of nodes and E denotes a set of edges. A simplex σ is called *a node* if it is an J-complete $(n - |J| - 1)$-simplex for some $J \in \mathcal{I}$ or it is a complete $(n-1)$-simplex. Two nodes σ_1 and σ_2 are said to be *adjacent* if both σ_1 and σ_2 are facets of an n-simplex, or if σ_1 and σ_2 are J-complete and are facets of an $(n - |J|)$-simplex on $vF(J)$, or if σ_1 is J-complete and σ_2 is J'-complete and σ_1 is a facet of σ_2 and σ_2 is an $(n - |J|)$-simplex on $vF(J)$, or if σ_1 is $\{j\}$-complete and σ_2 is a complete $(n-1)$-simplex on $vF(\{j\})$ and σ_1 is a facet of σ_2. The notion $e = \{\sigma_1, \sigma_2\}$ is called *an edge* if the two nodes σ_1 and σ_2 are adjacent. The degree of a node σ in G is defined to be the number of nodes connected with it, denoted by $deg(\sigma)$. A finite sequence of adjacent simplices in G from σ_0 to σ_l is defined as $(\sigma_0, \sigma_1, \cdots, \sigma_l)$, where σ_0, σ_1, $\cdots$, σ_l are nodes in G and $e_i = \{\sigma_{i-1}, \sigma_i\}$ are edges in G for all $i \in I_l$.

Theorem 3.14 *Let $\mathcal{T}$ be a triangulation of P. Then there exists a finite sequence of adjacent J-complete or complete simplices for varying $J \in \mathcal{I}$ from $\{v\}$ to a complete $(n-1)$-simplex σ^+ on P^+.*

Proof: From Lemma 3.6 it follows that $\{v\}$ is a J-complete 0-simplex on $vF(J)$ for some unique set $J \in \mathcal{I}$ with $|J| = n - 1$. Since $\{v\}$ lies on the boundary of $vF(J)$, there exists a unique 1-simplex σ on $vF(J)$ having $\{v\}$ as its facet. By Lemma 3.8, either σ is a $\bar{J}$-complete simplex on $vF(\bar{J})$ where $\bar{J} = J \setminus \{j\}$ for some unique $j \in J$, or σ has exactly one other J-complete facet τ. Hence there exists a unique node connecting $\{v\}$. That is, $deg(\{v\}) = 1$.

Let τ be any node on P^+. Then we know that τ is a complete $(n-1)$-simplex on the boundary of P. This implies that there is a unique n-simplex σ in $\mathcal{T}$ having τ as its facet.

By Lemma 3.11 we know that σ has exactly one other complete facet $\bar{\tau}$ which is a node by definition. Hence we have $deg(\tau) = 1$.

In all other cases, we will show that $deg(\tau) = 2$ if τ is a node. We need to address several cases. (1) If τ is a complete $(n-1)$-simplex and does not lie on P^- or on P^+, then according to Lemma 3.13 τ does not lie on the boundary of P. Hence, there exist exactly two n-simplices σ_1 and σ_2 on P sharing τ as their common facet. It follows from Lemma 3.11 that there are two nodes adjacent to τ. Thus, $deg(\tau) = 2$. (2) If τ is a complete $(n-1)$-simplex and lies on P^-, then it follows from Lemma 3.10 that $deg(\tau) = 2$. (3) If τ is a J-complete $(n-|J|-1)$-simplex on $vF(J)$ for some $J \in \mathcal{I}$, either τ does not lie on the boundary of $vF(J)$ or τ lies on the boundary of $vF(J)$. If τ does not lie on the boundary of $vF(J)$, then τ is a facet of precisely two $(n-|J|)$-simplices on $vF(J)$. It follows from Lemma 3.8 that τ is adjacent to exactly two nodes. If τ lies on the boundary of $vF(J)$, then there exists exactly one $(n-|J|)$-simplex σ on $vF(J)$ having τ as its facet. By Lemma 3.8 either σ is a $\bar{J}$-complete $(n-|\bar{J}|-1)$-simplex on $F(\bar{J})$ for some unique $\bar{J} \in \mathcal{I}$ with $|\bar{J}| = |J| - 1$ and has no other J-complete facets, or σ has exactly one other J-complete facet. This yields one adjacent node to τ. On the other hand, since τ lies on the boundary of $vF(J)$, it follows from Lemma 3.12 that τ does not lie on the boundary of P^-. Hence, τ lies on $vF(\tilde{J})$ for some unique set $\tilde{J} \in \mathcal{I}$ with $|\tilde{J}| = |J| + 1$. By Lemma 3.9 either τ is J'-complete for some unique set $J' \in \mathcal{I}$ with $|J'| = |J|$ and $J' \neq J$, or τ has exactly one $\tilde{J}$-complete facet. It follows again that in both these cases there exists exactly one node adjacent to τ. This concludes that τ has exactly two adjacent nodes. That is, we have $deg(\tau) = 2$.

As shown above, the degree of each node in the graph $G = (N, E)$ is at most two. Since the number of simplices on P is finite, the number of nodes in G is finite. Since $deg(\{v\}) = 1$, there exists a finite sequence of adjacent nodes starting from $\{v\}$. The end node of this sequence must be a node of degree 1 and different from $\{v\}$. The only possibility is that this node is a complete simplex on P^+. $\square$

The algorithm is such that it generates the sequence of adjacent simplices described in the theorem. From the theorem it follows that starting at the point v, the algorithm generates a finite sequence of adjacent J-complete or complete simplices for varying $J \in \mathcal{I}$, which leads to a complete $(n-1)$-simplex σ^+ on P^+. After leaving the set P^-, the algorithm may return to P^- to generate again J-complete simplices on P^- for varying $J \in \mathcal{I}$. In this way, the algorithm may generate an (odd) number of complete $(n-1)$-simplices on P^- before it leaves P^- forever and terminates with a complete $(n-1)$-simplex σ^+ on P^+. Let σ^- be the last complete $(n-1)$-simplex generated by the algorithm on P^-. Then it is clear that from σ^- on, the algorithm generates a finite sequence of adjacent complete $(n-1)$-simplices from σ^- on P^- to σ^+ on P^+. We summarize this in the following corollary.

Corollary 3.15 *Let $\mathcal{T}$ be a triangulation of P. Then there exists a finite sequence of adjacent complete $(n-1)$-simplices from a complete $(n-1)$-simplex σ^- on P^- to a complete $(n-1)$-simplex σ^+ on P^+.*

Given a function $g : P \mapsto \mathbb{R}^n$, a point $x \in P$ is called *a stationary point with respect to c* if $g(x)$ is an element of $N(P(t), x)$ with $t = c^\top x$. Such a solution is also called *a parametrized stationary point* of g. From Corollary 3.15 and the system of equations $(**)$ we see that every simplex generated by the algorithm from σ^- to σ^+ contains a stationary point of the piecewise linear approximation function f with respect to the vector c. By taking the straight line segments between the parametrized stationary points of any two adjacent complete simplices, we obtain in P a piecewise linear path of parametrized stationary points of f connecting the simplices σ^- and σ^+.

Corollary 3.16 *Let $\mathcal{T}$ be a triangulation of P. Then with respect to the vector c there exists a piecewise linear path $\rho([0,1])$ in P of parametrized stationary points of the piecewise linear approximation f of $\bar{\phi}$ with respect to $\mathcal{T}$ and this path connects the point $\rho(0) \in \sigma^-$ in P^- and $\rho(1) \in \sigma^+$ in P^+. No point of this path lies on the boundary of $P(t)$ for some t, $t^- \leq t \leq t^+$.*

Proof: The first part is obvious. The second part follows from the proof of Lemma 3.13. □

From the last corollary it follows that for every $q \in \rho([0,1])$ it holds that $f(q) = \beta c$ for some $\beta \in \mathbb{R}$, i.e., $f(q) \in C(0^n)$. In the next section we show, by taking a sequence of triangulations of P with mesh size going to zero, that there exists a connected set of zero points of ϕ in X having a non-empty intersection with both X^- and X^+.

4 A Constructive Existence Proof

By making use of the results obtained in Section 3 we will give a constructive proof for Theorem 2.1. To achieve this, a sequence of triangulations $\mathcal{T}^r$, $r \in \mathbb{N}$, with mesh size converging to zero is taken. According to Corollary 3.16, for every $r \in \mathbb{N}$, there exists a piecewise linear function $\rho^r : [0,1] \mapsto P$ with image set $\rho^r([0,1])$ connecting P^- and P^+ and satisfying that any $q^r \in \rho^r([0,1])$ is a parametrized stationary point of the piecewise linear approximation f^r of $\bar{\phi}$ with respect to $\mathcal{T}^r$. In the next lemma we show by a limit argument that if the sequence $(q^r)_{r\in\mathbb{N}}$ converges to some q^*, then $p(q^*)$ is a zero point of ϕ. Recall that $p(\cdot)$ is a continuous function.

Lemma 4.1 *Let $\phi : X \mapsto \mathbb{R}^n$ be a point-to-set mapping satisfying the conditions in Theorem 2.1. For $r \in \mathbb{N}$, let $\mathcal{T}^r$ be a triangulation of P with mesh size smaller than $\frac{1}{r}$ and* *let $\rho^r([0,1])$ be the piecewise linear path as constructed in Corollary 3.16. Then, for every convergent sequence $(q^r)_{r\in\mathbb{N}}$ with limit q^* satisfying $q^r \in \rho^r([0,1])$ for all $r \in \mathbb{N}$, it holds that $x^* = p(q^*)$ is a zero point of ϕ in X.*

Proof: Since $q^r \in P$ for all $r \in \mathbb{N}$ and P is a closed set, we have $q^* \in P$. Moreover, since the mesh size of the sequence of triangulations of P converges to zero and $\bar{\phi}$ is upper semi-continuous and both compact-valued and convex-valued, the system of equations $(**)$ at q^r will reduce in the limit for r going to infinity, after taking subsequences if necessary, to

$$f^* = \beta^* c$$

for some $\beta^* \in \mathbb{R}$ and some $f^* \in \bar{\phi}(q^*)$. Let $t^* = c^\top q^*$ and $v^* = q^* - p(q^*)$. Clearly, $v^* \in N(X(t^*), p(q^*)) \cap B$ whenever $q^* \in P$. We have to consider the following cases.

i) In case $q^* \in P \setminus Q$, we have $f^* = p(q^*) - q^* = \beta^* c$. Since $c^\top p(q^*) = c^\top q^* = t^*$ and so $c^\top f^* = 0$, we obtain $\beta^* = 0$ and therefore $p(q^*) = q^*$. Since q^* is not in X, we obtain a contradiction.

ii) In case $q^* \in bdQ$, we have $f^* = \mu^*(p(q^*) - q^*) + (1-\mu^*)f = \beta^* c$ for some $0 \leq \mu^* \leq 1$ and some $f \in A(p(q^*))\phi(p(q^*)) \cap \pi(v^*)$. For $\mu^* = 1$ this case reduces to case i). For $\mu^* < 1$, we obtain $f \in C(v^*)$. According to condition 1 of Theorem 2.1 $x^* = p(q^*)$ is a zero point of ϕ.

iii) In case $q^* \in Q \setminus X$ and $q^* \notin bdQ$, we have $f^* \in A(p(q^*))\phi(p(q^*)) \cap \pi(v^*)$ and $f^* = \beta^* c \in C(v^*)$. According to condition 1 of Theorem 2.1 $x^* = p(q^*)$ is a zero point of ϕ.

iv) In case $q^* \in X$, we have $q^* = p(q^*)$. This implies that $f^* \in A(q^*)\phi(q^*) \cap \pi(0^n)$. Since $f^* = \beta^* c \in C(0^n)$, we obtain $A(q^*)\phi(q^*) \cap \pi(0^n) \cap C(0^n) \neq \emptyset$. According to condition 1 of Theorem 2.1, $x^* = q^*$ is a zero point of ϕ. $\square$

Let $Z = \{x \in X \mid 0^n \in \phi(x)\}$ be the set of zero points of ϕ in X. For a non-empty, compact set $S \subset \mathbb{R}^n$, define the distance function $d_S : \mathbb{R}^n \to \mathbb{R}$ by $d_S(x) = \min\{\|x-y\|_2 \mid y \in S\}$. It is well known that d_S is continuous.

PROOF OF THEOREM 2.1: From Lemma 4.1 it immediately follows that $Z \cap X^- \neq \emptyset$ and $Z \cap X^+ \neq \emptyset$, also Z is compact. For $x \in Z$, let Z_x be the component of x in Z. We know that Z_x is connected and compact. The collection of all distinct components in Z forms a partition of Z. Define $Z^- = \cup_{x\in Z\cap X^-} Z_x$ and $Z^+ = Z \setminus Z^-$. Now let $V^- = X^- \cup Z^-$ and $V^+ = X^+ \cup Z^+$ and $V = Z \cup X^- \cup X^+$. Suppose the theorem is false. Then V^- and V^+ are non-empty, disjoint, compact sets. Hence, there exists $\varepsilon > 0$ such that $\min\{\|q^0 - q^1\|_2 \mid q^0 \in V^-,\ q^1 \in V^+\} \geq \varepsilon$. For $r \in \mathbb{N}$, let $\mathcal{T}^r$ be a triangulation of P with mesh size smaller than $\frac{1}{r}$ and let $\rho^r : [0,1] \mapsto P$ be the corresponding continuous function with image set connecting P^- and P^+, as constructed in Corollary 3.16. Define $g^r : [0,1] \mapsto \mathbb{R}$ by

$$g^r(t) = d_{V^-}(p(\rho^r(t))) - d_{V^+}(p(\rho^r(t))), \quad \forall t \in [0,1].$$

Since g^r is continuous, $g^r(0) \leq -\varepsilon$, and $g^r(1) \geq \varepsilon$, there exists a point $t^r \in (0,1)$ such that $g^r(t^r) = 0$. Hence, $d_{V^-}(p(\rho^r(t^r))) = d_{V^+}(p(\rho^r(t^r))) = d_V(p(\rho^r(t^r))) \geq \frac{1}{2}\varepsilon$. Without loss of generality we assume that $(\rho^r(t^r))_{r\in\mathbb{N}}$ converges to a point $q^* \in P$. Hence,

$$d_V(p(q^*)) = d_V(\lim_{r\to\infty} p(\rho^r(t^r))) = \lim_{r\to\infty} d_V(p(\rho^r(t^r))) \geq \frac{1}{2}\varepsilon > 0.$$

By Lemma 4.1, we have $d_Z(p(q^*)) = 0$. Because $p(q^*) \in Z \subseteq V$, it follows that $d_V(p(q^*)) = 0$, yielding a contradiction. □

From Lemma 4.1 and the proof of Theorem 2.1 above it follows that the projection on X of any point of the piecewise linear path of parametrized stationary points of f^r being generated by the algorithm can be seen as an approximate zero point of ϕ in the sense that every convergent sequence of parametrized stationary points of f^r, for a sequence of triangulations $(\mathcal{T}^r)_{r\in\mathbb{N}}$ with mesh size converging to zero, converges to a zero point of ϕ when r goes to infinity. The approximation typically improves when the mesh size of the triangulation becomes smaller. The fact that for every triangulation in the sequence the algorithm generates a path of such points from P^- to P^+ guarantees that there exists a continuum of zero points of ϕ connecting X^- and X^+.

5 Continuums of Coincidences, Fixed Points, and Optima

In this section we derive several results from Theorem 2.1 about the existence of a continuum of zero points, coincidences, fixed points, optima and stationary points. Unless otherwise stated, we maintain the notations and assumptions used in Section 2. In particular, c is a non-zero vector in $\mathbb{R}^n$, $B = \{x \in \mathbb{R}^n \mid ||x||_2 \leq 1,\ c^\top x = 0\}$, and ϕ is an upper semi-continuous point-to-set mapping from the non-empty convex and compact set X in $\mathbb{R}^n$ to the collection of non-empty, convex and compact subsets of $\mathbb{R}^n$.

Theorem 5.1 *Suppose that for every $x \in X$ and every $v \in N(X(t), x)$ with $c^\top v = 0$, there is a $y \in \phi(x)$ satisfying $c^\top y = 0$ and $v^\top y \leq 0$, where $t = c^\top x$. Then there exists a connected set C of zero points of ϕ in X such that $C \cap X^- \neq \emptyset$ and $C \cap X^+ \neq \emptyset$*

Proof: Take $A(x) = E(n)$ for every $x \in X$ and $\pi(v) = \{y \in \mathbb{R}^n \mid c^\top y = 0, v^\top y \leq 0\}$ for $v \in B$. Clearly, π is an upper semi-continuous mapping from B to the collection of non-empty convex and closed subsets of $\mathbb{R}^n$. Since for every $v \in B$ it holds that $\pi(v) \cap C(v) = \{0^n\}$, condition 1 of Theorem 2.1 is satisfied. The condition in the theorem also implies that condition 2 of Theorem 2.1 is satisfied. Hence, according to Theorem 2.1 there exists a connected set of zero points of ϕ in X intersecting with both X^- and X^+. $\square$

By taking $A(x) = -E(n)$ for every $x \in X$ the same result holds when for every $x \in X$ and every $v \in N(X(t), x)$ with $c^\top v = 0$ there is a $y \in \phi(x)$ satisfying $c^\top y = 0$ and $v^\top y \geq 0$.

From Theorem 5.1 we immediately obtain the following results. The first result gives a sufficient condition for the existence of a continuum of coincidences and therefore generalizes Fan's coincidence theorem; see Fan (1972) in which the existence of a single coincidence is proved.

Theorem 5.2 *Let ϕ and ψ be two upper semi-continuous mappings from X to the collection of non-empty compact and convex subsets of $\mathbb{R}^n$. Suppose that for every $x \in X$ and every $d \in \mathbb{R}^n$ satisfying both $c^\top d = 0$ and $d^\top x = \max\{d^\top y \mid y \in X,\ c^\top y = c^\top x\}$, there exist $u \in \phi(x)$ and $w \in \psi(x)$ such that $c^\top u = c^\top w$ and $d^\top u \geq d^\top w$. Then there exists a connected set C of points in X such that $\phi(x) \cap \psi(x) \neq \emptyset$ for every $x \in C$, $C \cap X^- \neq \emptyset$, and $C \cap X^+ \neq \emptyset$.*

Proof: Define the mapping γ on X by $\gamma(x) = \psi(x) - \phi(x)$ for all $x \in X$. Since $d^\top x = \max\{d^\top y \mid y \in X,\ c^\top y = c^\top x\}$ implies $d \in N(X(t), x)$, where $t = c^\top x$, $\gamma(\cdot)$ satisfies the conditions of Theorem 5.1. Hence, there exists a connected set of zero points of γ in X intersecting with both X^- and X^+. By construction, every zero point of the mapping γ is a coincidence of the mappings ϕ and ψ. $\square$

The next result can be seen as a generalization of Browder's fixed point theorem for point-to-set mappings.

Theorem 5.3 *Suppose that for every $x \in X$ it holds that $\phi(x) \cap X(t) \neq \emptyset$, where $t = c^\top x$. Then there exists a connected set of fixed points of ϕ in X intersecting with both X^- and X^+.*

Proof: For any $x \in X$, take some $y \in \phi(x) \cap X(t)$, where $t = c^\top x$. Since $y \in X(t)$ we have that $c^\top y = c^\top x = t$ and $v^\top y \leq v^\top x$ for all $v \in N(X(t), x)$. Hence, the mapping ψ on X defined by $\psi(x) = \phi(x) - \{x\}$ for all $x \in X$ satisfies the conditions of Theorem 5.1. Therefore, there exists a connected set of zero points of ψ in X intersecting with both X^- and X^+. Clearly, a zero point of ψ is a fixed point of ϕ in X. $\square$

Notice that in this theorem we only require that $\phi(x) \cap X(t) \neq \emptyset$ for every $x \in X(t)$. The image $\phi(x)$ may contain elements outside the set $X(t)$. Clearly, when ϕ is a fixed point mapping in the sense that for every $x \in X$ it holds that $\phi(x) \subseteq X(t)$, where $t = c^\top x$, then Theorem 5.3 implies that there exists a continuum of fixed points connecting X^- and X^+.

Corollary 5.4 *Suppose that $\phi(x) \subseteq X(t)$ whenever $x \in X(t)$. Then there exists a connected set of fixed points of ϕ in X having a non-empty intersection with both X^- and X^+.*

Browder's fixed point theorem for mappings is now an immediate consequence of this corollary. Browder (1960) proved the continuous function case and Mas-Colell (1974) extended the result to the upper semi-continuous point-to-set mapping case. For a mapping ϕ from $X \times [0,1]$ to X a point $(x,t) \in X \times [0,1]$ is called a fixed point of ϕ if $x \in \phi(x,t)$.

Corollary 5.5 *Let ϕ be an upper semi-continuous mapping from $X \times [0,1]$ to the collection of non-empty convex and compact subsets of X, where X is a non-empty, convex and compact set in $\mathbb{R}^n$. Then there exists a connected set of fixed points of ϕ in $X \times [0,1]$ intersecting with both $X \times \{0\}$ and $X \times \{1\}$.*

Proof: Define $S = X \times [0,1]$ and the mapping ψ on S by $\psi(s,t) = \phi(s,t) \times \{t\}$ for all $(s,t) \in S$. Take $c = (0^{n\top}, 1)^\top$. Then S and ψ satisfy the conditions of Corollory 5.4 with respect to the non-zero vector c. Hence, there exists a connected set of fixed points of ψ in S intersecting with both $S^- = X \times \{0\}$ and $S^+ = X \times \{1\}$. Clearly, a fixed point of ψ in S is a fixed point of ϕ in $X \times [0,1]$. □

By generalizing Theorem 3.4 of Herings, Talman and Yang (2001) on polytopes, Talman and Yamamoto (2001) establish the following existence theorem on a continuum of zero points by using the concept of tangent cone. We will show that this result also follows from Theorem 5.1. Recall that for $x \in X$ the tangent cone of $X(t)$ at x with $t = c^\top x$ equals the set

$$T(X(t), x) = \{z \in \mathbb{R}^n \mid y^\top z \leq 0 \text{ for all } y \in N(X(t), x)\}.$$

Corollary 5.6 *Suppose that for every $x \in X$ it holds that $\phi(x) \cap T(X(t), x) \neq \emptyset$ with $t = c^\top x$. Then there exists a connected set of zero points of ϕ in X intersecting with both X^- and X^+.*

Proof: For any $x \in X$ it holds that $T(X(t), x) \subseteq \{y \in \mathbb{R}^n \mid c^\top y = 0, v^\top y \leq 0\}$ for every $v \in N(X(t), x)$. Hence, ϕ satisfies the conditions of Theorem 5.1. □

When in Theorem 2.1 $\pi(v) = \mathbb{R}^n$ for all $v \in B$, we may derive the following result, which is a generalization of both Theorem 3.2 of Herings, Talman and Yang (2001) on polytopes and of Theorem 3.1 of Talman and Yamamoto (2001) on non-empty compact and convex sets.

Theorem 5.7 *Suppose there exists a continuous and non-singular matrix map A on X such that for every $x \in X$, $A(x)\phi(x) \cap N(X(t), x)$ is either empty or contains 0^n, where $t = c^\top x$. Then there exists a connected set of zero points of ϕ in X intersecting with both X^- and X^+.*

Proof: Take $\pi(v) = \mathbb{R}^n$ for all $v \in B$. Clearly, π is an upper semi-continuous mapping from B to the collection of non-empty convex and closed subsets of $\mathbb{R}^n$. Condition 2 of Theorem 2.1 is satisfied because $\phi(x) \neq \emptyset$ for all $x \in X$. Concerning condition 1 of Theorem 2.1 take any $x \in X$ and $v \in N(X(t), x) \cap B$. From Lemma 3.2 we obtain that $C(v) \subseteq N(X(t), x)$. Hence, $A(x)\phi(x) \cap N(X(t), x) = \emptyset$ implies that $A(x)\phi(x) \cap \pi(v) \cap C(v) = \emptyset$. Moreover, since both $N(X(t), x)$ and $C(v)$ contain 0^n, we have that if $A(x)\phi(x) \cap N(X(t), x)$ contains 0^n then also $A(x)\phi(x) \cap \pi(v) \cap C(v)$ contains 0^n. □

The theorem states that if every image of a point can be transformed in a continuously linear way such that every transformed image has an empty intersection with the normal cone at that point, unless it contains the origin, then there exists a connected set of zero points. Special cases are when $A(x) = E(n)$, or $A(x) = -E(n)$ for all $x \in X$ or more generally when $A(x)$ is a diagonal matrix with non-zero diagonal elements depending in a continuous way on x for all $x \in X$.

When in Theorem 2.1 $\pi(v) = \{y \in \mathbb{R}^n \mid v^\top y \leq 0\}$, the following result is obtained.

Theorem 5.8 *Suppose that for every $x \in X$ and every $v \in N(X(t), x)$, where $t = c^\top x$, the following two conditions hold:*

(i) The set $\phi(x) \cap \{y \mid v^\top y \leq 0\} \cap C(v)$ is either empty or contains 0^n;

(ii) The set $\phi(x) \cap \{y \mid v^\top y \leq 0\} \neq \emptyset$.

Then there exists a connected set C of zero points of ϕ in X such that $X^- \cap C \neq \emptyset$ and $X^+ \cap C \neq \emptyset$.

Proof: Take $A(x) = E(n)$ for all $x \in X$ and $\pi(v) = \{y \mid v^\top y \leq 0\}$ for all $v \in B$. Clearly, π is an upper semi-continuous mapping from B to the collection of non-empty convex and closed subsets of $\mathbb{R}^n$. Moreover, for every $x \in X$ and $v \in N(X(t), x) \cap B$ conditions (i) and (ii) imply conditions 1 and 2 of Theorem 2.1, respectively. □

Next, we give an application in the field of optimization theory. Let $f : X \mapsto \mathbb{R}$ be a function and let c be an arbitrary non-zero vector in $\mathbb{R}^n$. Given t, $t^- \leq t \leq t^+$, an optimal solution of the problem

$$\begin{array}{rl} \max & f(x) \\ s.t. & x \in X(t) \end{array}$$

is called *an optimum with respect to c* of f on X. Then we have the following result saying that there exists a continuum of optima with respect to c in case f is concave and smooth.

Theorem 5.9 *Let $f : X \mapsto \mathbb{R}$ be a concave smooth function and let c be a non-zero vector in $\mathbb{R}^n$. Then there exists a connected set C of optima with respect to c of f on X satisfying that $C \cap X^- \neq \emptyset$ and $C \cap X^+ \neq \emptyset$.*

Proof: For $x \in X$, let $\nabla f(x)$ be the gradient of f at x. Since f is a concave smooth function, $\nabla f(x)$ is a continuous function from X to $\mathbb{R}^n$. Let $g(x)$ be the projection of the point $x + \nabla f(x)$ on $X(t)$, where $t = c^\top x$. Then g is a continuous function from X to X satisfying $g(x) \in X(t)$ whenever $x \in X(t)$. Therefore, g satisfies the condition of Corollary 5.4. Hence, there exists a connected set of fixed points of g intersecting with both X^- and X^+. Clearly, a fixed point of g is an optimum of f with respect to c. $\square$

Finally, we establish a general existence theorem on a continuum of solutions to the non-linear variational inequality problem with respect to some non-zero vector; see also Talman and Yamamoto (2001). Given an arbitrary point-to-set mapping ϕ defined on the set X, *the variational inequality problem with respect to c*, where c is some non-zero vector in $\mathbb{R}^n$, is to find a point $x^* \in X$ and $f^* \in \phi(x^*)$ such that

$$(x^* - x)^\top f^* \geq 0, \ \forall x \in X(t^*),$$

where $t^* = c^\top x^*$. Recall that such a solution is called a parametrized stationary point of ϕ. We give a constructive proof.

Theorem 5.10 *Let ϕ be an upper semi-continuous mapping from X to the collection of non-empty convex and compact subsets of $\mathbb{R}^n$ and let c be a non-zero vector in $\mathbb{R}^n$. Then there exists a connected set C in X of solutions to the variational inequality problem for ϕ with respect to c satisfying that $X^- \cap C \neq \emptyset$ and $X^+ \cap C \neq \emptyset$.*

Proof: Take $\pi(v) = \mathbb{R}^n$ for all $v \in B$ and $A(x) = E(n)$ for all $x \in X$. Clearly, the mapping π satisfies condition 2 of Theorem 2.1. Corollary 3.16 implies that for any triangulation $\mathcal{T}$ of P there exists a piecewise linear path $\rho([0,1])$ in P of parametrized stationary points of the piecewise linear approximation f of $\bar{\phi}$ with repect to $\mathcal{T}$ connecting P^- and P^+. For $r \in \mathbb{N}$, let $\mathcal{T}^r$ be a triangulation of P with mesh size less than or equal to $\frac{1}{r}$ and let $\rho^r([0,1])$ be the corresponding piecewise linear path connecting P^- and P^+. Following the proof of Lemma 4.1 we obtain that for every convergent sequence $(q^r)_{r\in\mathbb{N}}$ with limit q^* satisfying

$q^r \in \rho^r([0,1])$ for all $r \in \mathbb{N}$, it holds that $x^* = p(q^*)$ is a parametrized stationary point of ϕ with respect to c. By taking the set Z as the set of parametrized stationary points of ϕ in X it follows from the proof of Theorem 2.1 that there exists a connected set of parametrized stationary points of ϕ in X having a non-empty intersection with both X^- and X^+. $\square$

Notice that to show that the variational inequality problem with respect to c has a continuum of solutions, no additional assumptions on ϕ are needed. Moreover, the result holds for any non-zero vector c.

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