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On the Lovász ϑ-number of almost regular graphs with application to Erdős–Rényi graphs

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Abstract

We consider k-regular graphs with loops, and study the Lovász ϑ-numbers and Schrijver ϑ'–numbers of the graphs that result when the loop edges are removed. We show that the ϑ-number dominates a recent eigenvalue upper bound on the stability number due to Godsil and Newman [C.D. Godsil and M.W. Newman. Eigenvalue bounds for independent sets. *Journal of Combinatorial Theory B*, to appear].

As an application we compute the ϑ and ϑ' numbers of certain instances of Erdős–Rényi graphs. This computation exploits the graph symmetry using the methodology introduced in [E. de Klerk, D.V. Pasechnik and A. Schrijver. Reduction of symmetric semidefinite programs using the regular *-representation. *Mathematical Programming B*, to appear].

The computed values are strictly better than the Godsil-Newman eigenvalue bounds.

Key Words: Erdős–Rényi graph, stability number, Lovász ϑ-number, Schrijver ϑ'–number, C$*$–algebra, semidefinite programming

AMS subject classification: 05C69, 90C35, 90C22

JEL code: C60

1 Introduction

In this paper we study the Lovász ϑ-number [13] and Schrijver ϑ'–number [17] for classes of almost regular graphs, i.e. graphs that become regular if a 'small' number of loops are added to the edge set.

The purpose is to study upper bounds on the stability (independence) numbers of such graphs.

Assume now that G is a k-regular graph with ℓ loops and adjacency matrix A, and let τ denote the smallest eigenvalue of A. Godsil and Newman [11] recently derived the

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following upper bound on $\alpha(G)$:

$$\alpha(G) \leq \frac{-\tau + \sqrt{\tau^2 + 4\left(\frac{k-\tau}{n}\right)\ell}}{2\left(\frac{k-\tau}{n}\right)}, \tag{1}$$

where n is the number of vertices, and $\alpha(G)$ is the stability number of G. Here, and throughout the paper, we use the convention that vertices with loops are allowed in a stable set.

For k-regular graphs without loops, i.e. if $\ell = 0$, (1) reduces to the well-known Hoffman-Delsarte eigenvalue bound; see [4] §3.3, or [3] page 115.

The Lovász ϑ-number is not defined for graphs with loops, but for the purpose of providing an upper bound on $\alpha(G)$ we simply delete the loop edges and compute the ϑ-number of the resulting graph. We will show that this ϑ-number, and therefore also the related Schrijver ϑ'–number, dominate the bound (1). This is a generalization of the well-known result that the ϑ-number dominates the Hoffman-Delsarte eigenvalue bound for k-regular graphs without loops.

In practice it is possible to compute ϑ and ϑ' for large graphs with symmetries, by using a methodology introduced in [9].

As an application we compute the ϑ and ϑ' numbers of certain instances of Erdős–Rényi graphs. The Erdős-Rényi graph $ER(q)$ is the graph whose vertices are the points of the projective plane $PG(2,q)$, with two vertices x and y adjacent if they are distinct and $x^T y = 0$. The graph $ER(q)$ has $q^2 + q + 1$ vertices and can be made $(q+1)$-regular by adding $q + 1$ loops. In the present work we restrict ourselves to q being an odd prime.

The $ER(q)$ graphs were first introduced in [2, 5] as examples of graphs with many edges but no 4-cycle. They were further studied in [16, 6, 7, 14, 11].

Godsil and Newman [11] showed that, for $ER(q)$, the eigenvalue bound (1) becomes

$$\alpha(ER(q)) \leq \frac{\sqrt{q} + \sqrt{q + 4(q+1)\frac{q+\sqrt{q}+1}{q^2+q+1}}}{2\frac{q+\sqrt{q}+1}{q^2+q+1}} = q^{3/2} - q + 2\sqrt{q} - 1/q + 3/q^2 + O\left(\frac{1}{q^3}\right). \tag{2}$$

Recently, Mubayi and Williford [14] proved that

$$\alpha(ER(q)) \geq \frac{120}{73\sqrt{73}} q^{3/2} > 0.19239 q^{3/2},$$

which shows that the upper bound (2) is tight in terms of the dependence of its leading term on q.

In this paper, we apply the approach from [9] to compute the Lovász ϑ and Schrijver ϑ'-numbers of $ER(q)$. We show that, for $q \leq 31$, odd and prime, the computed bounds are in fact *strictly* better than the eigenvalue bounds (2), although the differences are small.

Outline of the paper

The paper is organized in the following way. In Section 2 we provide basic facts on finite groups and regular $*$–representations of matrix algebras. In Section 3 we review how regular $*$-representations may be used to reduce the size of certain semidefinite programming problems, and in Section 4 we apply this methodology to reduce the sizes of the semidefinite programming problems that define ϑ and ϑ'. In this section we also show that the ϑ number dominates the eigenvalue bound (1). In Section 5 we define Erdős–Rényi graphs $ER(q)$ and give their properties, and in Section 6 we provide numerical results on the computation of $\vartheta(ER(q))$ and $\vartheta'(ER(q))$ for $q \leq 31$, odd, and prime.

Notation

We use $\mathrm{tr}(A)$ to denote the trace of a square matrix A. The space of symmetric matrices:

$$\mathcal{S}_n := \{X \in \mathbb{R}^{n\times n} : X = X^T\}$$

is endowed with the trace inner product.

For $A, B \in \mathcal{S}_n$, $A \succeq 0$ (resp. $A \succ 0$) denotes positive semidefiniteness (resp. positive definiteness), and $A \succeq B$ denotes $A - B \succeq 0$. The cone of $n \times n$ positive semidefinite matrices is denoted by

$$\mathcal{S}_n^+ := \{X \in \mathcal{S}_n : z^T X z \geq 0 \ \ \forall z \in \mathbb{R}^n\}.$$

For two matrices $A, B \in \mathcal{S}_n$, $A \geq B$, $(A > B)$ means $a_{ij} \geq b_{ij}$, $(a_{ij} > b_{ij})$ for all i, j. The vector of all ones is denoted by e and the matrix of all ones by J. We denote the Kronecker delta by δ_{ij}.

A graph with vertex set $V = \{1, \ldots, n\}$ and edge set E is denoted by $G = (V, E)$.

2 Finite groups and regular *–representations

Let $\mathcal{G}$ be a group, Z a finite set and S_Z the group of all permutations of Z. Let $\mathcal{G}$ be a finite group acting on Z, and for each $g \in \mathcal{G}$ define $\pi_g : Z \to Z$ by $\pi_g(z) = g \cdot z$. Then $\pi_g \in S_Z$, and $\phi : \mathcal{G} \to S_Z$ given by $\phi(g) := \pi_g$ is a homomorphism. Moreover $\phi_{gg'} = \phi_g \phi_{g'}$ and $\phi_{g^{-1}} = \phi_g^{-1}$ for all $g, g' \in \mathcal{G}$.

The image ϕ_g of g under ϕ can be represented by the permutation matrix $P_g \in \mathbb{R}^{|Z|\times|Z|}$,

$$(P_g)_{x,y} := \begin{cases} 1 & \text{if} \quad \phi_g(x) = y \\ 0 & \text{otherwise,} \end{cases}$$

for $x, y \in Z$. The representation ϕ is orthogonal, i.e.

$$P_{g\cdot g'} = P_g P_{g'} \quad \text{and} \quad P_{g^{-1}} = P_g^T.$$

In the sequel we will identity $\mathcal{G}$ with its representation $\phi(\mathcal{G})$.

The *orbit* of an element $z \in Z$ under the action of a group $\mathcal{G}$ is the set

$$\{\bar{x} : \bar{x} = \phi_g(z) \text{ for some } g \in \mathcal{G}\}.$$

Similarly the *orbit of a pair* $(x, y) \in Z \times Z$ under the action of a group $\mathcal{G}$ is the set

$$\{(\bar{x}, \bar{y}) : (\bar{x}, \bar{y}) = (\phi_g(x), \phi_g(y)) \ \text{ for some } g \in \mathcal{G}\}.$$

Recall that $x \in Z$ and $y \in Z$ either have the same orbits under the action of $\mathcal{G}$, or disjoint orbits.

The *centralizer ring* (or commutant) of the group $\mathcal{G}$ is defined as

$$\mathcal{A} := \left\{ X : X = \frac{1}{|\mathcal{G}|} \sum_{P\in\mathcal{G}} P^T X P, \ X \in \mathbb{R}^{|Z|\times|Z|} \right\}. \tag{3}$$

$\mathcal{A}$ is a *-algebra, i.e. $\mathcal{A}$ is a collection of matrices closed under addition, scalar and matrix multiplication and transposition. An equivalent definition of the centralizer ring is

$$\mathcal{A} = \left\{ X \in \mathbb{R}^{|Z|\times|Z|} : XP = PX \quad \forall P \in \mathcal{G} \right\}.$$

Note that from the definition of the centralizer ring (3) one can also derive orbits of elements of $Z \times Z$. Namely, for $Z = \{1, \dots, n\}$, the orbit of $(i,j) \in Z \times Z$ corresponds to the positions of the nonzero entries of

$$\frac{1}{|\mathcal{G}|} \sum_{P \in \mathcal{G}} P^T e_i e_j^T P,$$

where e_i denotes the ith standard unit vector.

The matrix *–algebra $\mathcal{A}$ has a basis of $0-1$ matrices

$$B_k := \sum_{\{i,j\} \text{ has orbit } k} \frac{1}{|\mathcal{G}|} \left(\sum_{P \in \mathcal{G}} P^T e_i e_j^T P \right) \quad (k = 1, \dots, d). \tag{4}$$

Note that these matrices represent the orbits of pairs in the sense that

$$(B_k)_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ in orbit } k; \\ 0 & \text{otherwise} \end{cases} \qquad ((i,j) \in Z \times Z,\ k = 1, \dots, d).$$

Also note that:

- $\sum_i B_i = J$;
- For each i there is an i^* (possibly $i^* = i$) with $B_i = B_{i^*}^T$.

For what follows, we need to normalize the basis B_i, $i = 1, \dots, d$:

$$D_i := \frac{1}{\sqrt{\operatorname{tr}(B_i^T B_i)}} B_i, \quad i = 1, \dots, d. \tag{5}$$

Note that

$$\operatorname{tr}(D_i^T D_j) = \delta_{ij}.$$

The *multiplication parameters* γ_{ij}^k are defined by

$$D_i D_j = \sum_k \gamma_{ij}^k D_k$$

for $i, j = 1, \dots, d$. For γ_{ij}^k $(i, j, k = 1, \dots, d)$ one has:

$$\gamma_{ij}^k = \operatorname{tr}(D_{k^*} D_i D_j) \tag{6}$$

and

$$\gamma_{ij}^k = \gamma_{j^* i^*}^{k^*} = \gamma_{k^* i}^{j^*} = \gamma_{j k^*}^{i^*}. \tag{7}$$

Now, for $k = 1, \dots, d$ we define $d \times d$ matrices L_k;

$$(L_k)_{ij} := \gamma_{ik}^j, \quad i, j = 1, \dots, d. \tag{8}$$

By using (7) one can easily show that $L_i^T = L_{i^*}$. The matrices L_k form a basis as a vector space of a *faithful* representation of $\mathcal{A}$, say $\mathcal{A}'$, that is called the regular $*$-representation of $\mathcal{A}$.

Theorem 1 (see e.g. [9]). *The linear map $\varphi : D_i \to L_i$, $i = 1, \dots, d$ defines a *–isomorphism from $\mathcal{A}$ to $\mathcal{A}'$.*

The following is a consequence of this theorem.

Corollary 2 ([9]). *Let $x \in \mathbb{R}^d$. One has*

$$\sum_{i=1}^d x_i D_i \succeq 0 \Longleftrightarrow \sum_{i=1}^d x_i L_i \succeq 0.$$

3 Exploiting symmetry in semidefinite programs

We now show how to use the ideas from the previous section to reduce the size of certain semidefinite programs. The methodology we will describe is essentially due to [9], where it was used to bound crossing numbers of complete bipartite graphs.

Assume that the following semidefinite programming problem is given

$$\min_{X \succeq 0} \{ \operatorname{tr}(A_0 X) : \operatorname{tr}(A_k X) = b_k \quad k = 1, \ldots, m \}, \tag{9}$$

where the matrices $A_i \in \mathcal{S}_n$ $(i = 0, \ldots, m)$ and the vector $b \in \mathbb{R}^n$ are given. Assume further that there is a group of permutation matrices $\mathcal{G}$ such that the associated Reynolds operator

$$R(X) := \frac{1}{|\mathcal{G}|} \sum_{P \in \mathcal{G}} P^T X P, \; X \in \mathbb{R}^{n \times n}$$

maps the feasible set of (9) into itself and leaves the objective value invariant, i.e.

$$\operatorname{tr}(A_0 R(X)) = \operatorname{tr}(A_0 X) \text{ if } X \text{ is a feasible solution of (9).}$$

Since the Reynolds operator maps the convex feasible set into itself and preserves the objective values of feasible solutions, we may restrict the optimization to feasible solutions in the centralizer ring of $\mathcal{G}$. As explained in the previous section, we may obtain a normalized basis D_i $(i = 1, \ldots, d)$ of the centralizer ring via (4) and (5), by determining the orbits of pairs under the action of $\mathcal{G}$.

In other words, we may restrict our attention to feasible solutions of (9) of the form $X = \sum_{i=1}^{d} x_i D_i$ for some $x \in \mathbb{R}^d$.

From Corollary 2 it follows that the SDP problem (9) can be formulated as

$$\min_{x \in \mathbb{R}^d} \left\{ \sum_{i=1}^{d} x_i \operatorname{tr}(A_0 D_i) : \sum_{i=1}^{d} x_i \operatorname{tr}(A_k D_i) = b_k \quad \forall k, \; \sum_{i=1}^{d} x_i L_i \succeq 0 \right\}, \tag{10}$$

where the L_i's are defined in (8).

Note that problem (10) only involves $d \times d$ data matrices (i.e. the L_i matrices) as opposed to $n \times n$ matrices (i.e. the matrices D_i). Thus we may have a considerable reduction of the size of the matrices to which we apply semidefinite programming.

If problem (9) has the additional constraint $X \geq 0$, then its reformulation is identical to (10) except for the additional requirement $x \geq 0$.

4 The maximum stable set problem, ϑ and ϑ'

Given a graph $G = (V, E)$, a subset $V' \subseteq V$ is called a stable set of G if the induced subgraph on V' contains no edges except loops. The *maximum stable set problem* is to find the stable set of maximum cardinality. The *stability number* $\alpha(G)$ is the cardinality of the largest stable set in the graph G.

The Lovász ϑ number

The Lovász ϑ number, introduced in [13],

$$\left.\begin{array}{rcl}\vartheta(G) & := & \max \operatorname{tr}(JX) \\ & & \text{s.t. } \; X_{ij}=0, \quad \{i,j\}\in E \; (i\neq j) \\ & & \qquad \operatorname{tr}(X)=1 \\ & & \qquad X\in \mathcal{S}_n^+, \end{array}\right\} \tag{11}$$

gives an upper bound on $\alpha(G)$. We now show how to compute $\vartheta(G)$ using the symmetry reduction technique described in the previous section.

Lemma 3. *Let $G=(V,E)$ be given and denote $\mathcal{G} := Aut(G)$ and $n=|V|$. If X is a feasible solution of (11), then*

$$R(X)=\frac{1}{|\mathcal{G}|}\sum_{P\in\mathcal{G}} P^T X P, \;\; X\in\mathbb{R}^{n\times n}$$

is also a feasible solution with the same objective value.

Thus we may reformulate the SDP problem (11) using the technique described in Section 3. The details are given as the following theorem.

Theorem 4. *Let $G=(V,E)$ be given and denote $\mathcal{G} := Aut(G)$. Denote the number of orbits of $V\times V$ under the action of $\mathcal{G}$ by d, and the length of orbit i by l_i ($i=1,\ldots,d$). One has*

$$\vartheta(G)=\max_{x\in\mathbb{R}^d}\sum_{i=1}^d x_i\sqrt{l_i}$$

subject to

$$\begin{array}{rcl} x_k & = & 0 \quad \textit{if orbit } k \textit{ intersects } E \;\; (k=1,\ldots,d) \\ \sum_j \sqrt{l_j}x_j & = & 1 \quad (\textit{summation over orbits of pairs } (v,v),\ v\in V) \\ \sum_{i=1}^d x_i L_i & \succeq & 0, \end{array}$$

where the $d\times d$ matrices L_i ($i=1,\ldots,d$) are constructed from the orbit matrices B_i ($i=1,\ldots,d$) via (5), (6), and (8).

The Schrijver ϑ' number

The Schrijver ϑ'–function [17] is defined as:

$$\begin{array}{rcl}\vartheta'(G) & := & \max \operatorname{tr}(JX) \\ & & \text{s.t. } \; \operatorname{tr}((A+I)X)=1 \\ & & \qquad X\geq 0 \\ & & \qquad X\in\mathcal{S}_n^+. \end{array} \tag{12}$$

Clearly one has

$$\alpha(G)\leq\vartheta'(G)\leq\vartheta(G).$$

Similar to the ϑ case, we may reformulate the problem as follows.

Theorem 5. *Let $G = (V, E)$ be given and denote $\mathcal{G} := Aut(G)$. Denote the number of orbits of $V \times V$ under the action of $\mathcal{G}$ by d, and the length of orbit i by l_i $(i = 1, \ldots, d)$. One has*

$$\left.\begin{array}{rcl} \vartheta'(G) & = & \max_{x\in\mathbb{R}^d, x\geq 0} \sum_{i=1}^d x_i\sqrt{l_i} \\ s.t. & & \\ x_k & = & 0 \quad \textit{if orbit } k \textit{ intersects } E \ \ (k = 1, \ldots, d) \\ \sum_j \sqrt{l_j} x_j & = & 1 \quad \textit{(summation over orbits of pairs } (v, v),\ v \in V\textit{)} \\ \sum_{i=1}^d x_i L_i & \succeq & 0, \end{array}\right\} \qquad (13)$$

where the $d \times d$ matrices L_i $(i = 1, \ldots, d)$ are constructed from the orbit matrices B_i $(i = 1, \ldots, d)$ via (5), (6), and (8).

Note that the only difference between the reformulations for ϑ and ϑ' is the requirement that $x \geq 0$ for ϑ'.

An eigenvalue bound and its relation to ϑ

Let $G = (V, E)$ be a k-regular graph with ℓ loops. Let A denote its adjacency matrix and $\tau < 0$ the smallest eigenvalue of A.

Godsil and Newman [11] derived the upper bound (1) on $\alpha(G)$ as follows. Let z be the incidence vector of a maximum stable set of G, and assume that this stable set contains $\bar{\ell}$ loops.

Since $A - \tau I \succeq 0$ one has:

$$\left(z - \frac{\alpha(G)}{n}e\right)^T (A - \tau I)\left(z - \frac{\alpha(G)}{n}e\right) \geq 0$$

which simplifies to

$$\left(\frac{k-\tau}{n}\right)\alpha(G)^2 + \tau\alpha(G) \leq \bar{\ell}.$$

Using $\bar{\ell} \leq \ell$, we obtain the bound (1), and we reproduce it here for convenience:

$$\alpha(G) \leq \frac{-\tau + \sqrt{\tau^2 + 4\left(\frac{k-\tau}{n}\right)\ell}}{2\left(\frac{k-\tau}{n}\right)}.$$

We show will show that $\vartheta(G)$ dominates the eigenvalue bound (1). To this end, consider the following formulation of the ϑ-number:

$$\left.\begin{array}{rcl} \vartheta(G) & = & \max e^T x \\ \text{s.t.} & & \\ X - xx^T & \succeq & 0 \\ X_{ii} & = & x_i \quad (i \in V) \\ X_{ij} & = & 0 \quad (\{i, j\} \in E,\ i \neq j). \end{array}\right\} \qquad (14)$$

Note that for any feasible solution one has $x_i \in [0, 1]$ $(i \in V)$.

Theorem 6. *Let $G = (V, E)$ be a k-regular graph with ℓ loops. Let $\vartheta(G)$ be the Lovász ϑ number of the graph obtained by removing the loop edges from E. One has*

$$\vartheta(G) \leq \frac{-\tau + \sqrt{\tau^2 + 4\left(\frac{k-\tau}{n}\right)\ell}}{2\left(\frac{k-\tau}{n}\right)}.$$

Proof. Let x, X denote an optimal solution of the ϑ formulation (14). Since

$$A - \tau I - \frac{k-\tau}{n} J \succeq 0,$$

one has

$$x^T \left(A - \tau I - \frac{k-\tau}{n} J \right) x \geq 0.$$

Using $J = ee^T$ and $e^T x = \vartheta(G)$ this becomes

$$x^T (A - \tau I) x \geq \frac{k-\tau}{n} \vartheta(G)^2.$$

We now use $X - xx^T \succeq 0$ to find

$$\begin{aligned} x^T (A - \tau I) x &= \operatorname{tr}\left((A - \tau I) xx^T \right) \\ &\leq \operatorname{tr}\left((A - \tau I) X \right) \\ &\leq \ell - \tau \vartheta(G), \end{aligned}$$

where the last inequality is due to $\operatorname{tr}(AX) \leq \ell$ (since $X_{ii} = x_i \in [0,1]$ $(i \in V)$), and $\operatorname{tr}(X) = e^T x = \vartheta(G)$.

Thus we have obtained

$$\left(\frac{k-\tau}{n} \right) \vartheta(G)^2 + \tau \vartheta(G) - \ell \leq 0,$$

and the required result follows. □

5 Erdős-Rényi Graphs

Let V be a vector space over the finite field of order q, $GF(q)$. There are $q^2 + q + 1$ 1-dimensional subspaces of V: these are the *points* of $PG(2, q)$. There are $q^2 + q + 1$ 2-dimensional subspaces of V: these are the *lines* of $PG(2, q)$. Each point may be represented by any non-zero vector in its 1-dimensional subspace (which then spans that subspace). For background on projective planes, see [12].

The Erdős-Rényi graph $ER(q)$ is the graph whose vertices are the points of $PG(2, q)$, with two vertices x and y adjacent if they are distinct and $x^T y = 0$.

Consider the graph whose vertices are the points of $PG(2, q)$, with x and y adjacent if they are distinct and $x^T My = 0$, where

$$M = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

By the classification of bilinear forms over $GF(q)$ (see [12]), this graph is isomorphic to $ER(q)$. For convenience, we will use this definition of $ER(q)$ and let $\langle x, y \rangle := x^T My$.

Most vertices of $ER(q)$ have degree $q+1$ but there are $q+1$ vertices of degree q. These are known as *absolute vertices*, and are self-orthogonal (removing the word "distinct" from the definition of $ER(q)$ would make it regular, with loops). The absolute vertices form an independent set. There are $(q^2+q)/2$ vertices that are adjacent to exactly 2 absolute vertices each; these are the *external* vertices. The remaining $(q^2-q)/2$ vertices are adjacent to no absolute vertices; these are the *internal* vertices. See [16] for more details. We will denote the absolute, external, and internal vertices by $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{M}$, respectively. The automorphism group of $ER(q)$, for q an odd prime, is shown in [16] to be $P\mathcal{O}_3(q)$. There are exactly three orbits of vertices: $\mathcal{R}$, $\mathcal{L}$, and $\mathcal{M}$.

The absolute vertices are exactly the vertices x such that $\langle x, x\rangle = 0$. Due to our choice of M, for the external vertices $\langle x, x\rangle$ is a square and for the internal vertices $\langle x, x\rangle$ is a non-square. So we may scale the external vertices so that $\langle x, x\rangle = 1$ and the internal vertices so that $\langle x, x\rangle = g$. (There is an abuse of notation here: we are using x to represent both a 1-dimensional subspace and a particular vector in that subspace.)

We will now compute the orbits of the automorphism group of $ER(q)$ on the pairs of vertices. (See also [1], where they derive the parameters of the association schemes on the external and internal vertices, which can be used to read off the orbits for $\mathcal{L}\times\mathcal{L}$ and $\mathcal{M}\times\mathcal{M}$.)

There are of course three diagonal orbits on pairs, corresponding to the three orbits on vertices:

- $\{(x, x) : x \in \mathcal{R}\}$
- $\{(x, x) : x \in \mathcal{L}\}$
- $\{(x, x) : x \in \mathcal{M}\}$

For a pair of distinct vertices (x, y), let X be the matrix whose columns are x and y, and let $A := X^T M X$. Similarly, for (x', y') we define X' and A'. Assume (x, y) and (x', y') are in the same orbit. Then $X' = mXd$ for some $m \in P\mathcal{O}_3(q)$ and some non-singular diagonal matrix d (as $P\mathcal{O}_3(q)$ acts on 1-subspaces, we may need to rescale to achieve our normalization, hence d). Now

$$X' = mXd \iff X'^T M X' = dX^T m^T M m X d \iff A' = dAd. \tag{15}$$

The diagonal elements of A are either 0, 1, or g (according to the type of x and y) and must be identical to the diagonal elements of A'. Our task is then to classify such matrices A under the equivalence suggested by (15).

If x is absolute then all pairs (x, y) where y is of fixed type and $\langle x, y\rangle \neq 0$ are in the same orbit; this can be seen from

$$\begin{pmatrix} 0 & b \\ b & c \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}.$$

Recalling that for absolute vertices adjaceny means equality, and that absolute vertices are never adjacent to internal ones, we have the following orbits on pairs of distinct vertices:

- $\{(x, y) : x \in \mathcal{R}, y \in \mathcal{R}, x \neq y\}$
- $\{(x, y) : x \in \mathcal{R}, y \in \mathcal{L}, \langle x, y\rangle = 0\}$
- $\{(x, y) : x \in \mathcal{R}, y \in \mathcal{L}, \langle x, y\rangle \neq 0\}$
- $\{(x, y) : x \in \mathcal{R}, y \in \mathcal{M}\}$

(There are of course two analogous orbits in $\mathcal{L} \times \mathcal{R}$, and one in $\mathcal{M} \times \mathcal{R}$.)

If neither vertex is absolute then the diagonal entries of d are constrained to be ± 1, and we have the following orbits on pairs of distinct vertices:

- $\{(x, y) : x \in \mathcal{L}, y \in \mathcal{L}, \langle x, y\rangle = 0\}$
- $\{(x, y) : x \in \mathcal{L}, y \in \mathcal{L}, \langle x, y\rangle = \pm g^t\}$, $t = 0, 1, 2, \ldots, \frac{q-3}{2}$
- $\{(x, y) : x \in \mathcal{M}, y \in \mathcal{M}, \langle x, y\rangle = 0\}$
- $\{(x, y) : x \in \mathcal{M}, y \in \mathcal{M}, \langle x, y\rangle = \pm g^t\}$, $t = 0, 2, \ldots, \frac{q-3}{2}$
- $\{(x, y) : x \in \mathcal{L}, y \in \mathcal{M}, \langle x, y\rangle = 0\}$
- $\{(x, y) : x \in \mathcal{L}, y \in \mathcal{M}, \langle x, y\rangle = \pm g^t\}$, $t = 0, 1, 2, \ldots, \frac{q-3}{2}$

(Similarly for orbits in $\mathcal{M} \times \mathcal{L}$.) Note that it can be shown that there are no internal vertices x, y with $\langle x, y\rangle = g$.

In total there are $2q + 11$ orbits of pairs and they form a basis for the centralizer ring $\mathcal{A}$ of $\mathrm{Aut}(ER(q))$, for q odd and prime.

6 Numerical results

In this section we give numerical results on upper bounds for the stability number of the Erdős–Rényi graph $ER(q)$. For q odd and prime, we formulate the $d = 2q + 11$ orbits B_k ($k = 1, \ldots, d$) that are of the form given in Section 5. After normalizing the matrices B_k ($k = 1, \ldots, d$), we use (6) to obtain the matrices L_k ($k = 1, \ldots, d$). Finally, we solve the SDP problems described in Section 4 to obtain $\vartheta(ER(q))$ and $\vartheta'(ER(q))$.

By the properties of ϑ, ϑ' and Theorem 6 we know that

$$\alpha(ER(q)) \le \vartheta'(ER(q)) \le \vartheta(ER(q)) \le \frac{\sqrt{q} + \sqrt{q + 4(q+1)\frac{q+\sqrt{q}+1}{q^2+q+1}}}{2\frac{q+\sqrt{q}+1}{q^2+q+1}},$$

where the last expression is the Godsil-Newman eigenvalue bound (2) for $ER(q)$.

Note that, for given q, the Schrijver ϑ'-function in the form (12) is an SDP problem with a matrix variable of order $q^2 + q + 1$ and $O(q^4)$ sign constraints. For $q > 17$, say, solving such an SDP problem is difficult. However, using the regular $*$-representation, we reduce this to obtain problem (13) that involves matrices of order $2q + 11$ only. Thus it is possible to obtain $\vartheta'(ER(q))$ for the values of q listed in the table by interior-point methods in couple of seconds on a standard pc.

In Table 1 we present our numerical results. All computations were done using the semidefinite programming software SeDuMi [18] and Matlab 6.5. In the first column we give the order q of the projective plane which defines the Erdős–Rényi graph; the second column lists known stability numbers (due to J. Williford, private communication); in the third column we give the computed values for the Schrijver ϑ'– number, and in the fourth column the the values of the Lovász theta number for $ER(q)$. In the last column we give the eigenvalue bound (2) from [11].

Note that the $\vartheta(ER(q))$ bounds are strictly better the eigenvalue bounds (2), but the differences between the bounds are small. In six cases the bound $\lfloor\vartheta'(ER(q))\rfloor$ improves on the bound from (2) (rounded down), but in all these cases the difference is only 1. Also note that $\lfloor\vartheta(ER(q))\rfloor$ gives the same bound as $\lfloor\vartheta'(ER(q))\rfloor$ in all cases except $q = 29$.

q	α(ER(q))	$\vartheta'(ER(q))$	$\vartheta(ER(q))$	(2)
3	5	5.00	5.00	5.56
5	10	10.07	10.09	10.56
7	15	15.74	15.82	16.73
11	29	31.09	31.29	32.05
13	38	40.51	40.52	41.03
17	n.a.	60.22	60.42	61.29
19	n.a.	71.30	71.49	72.49
23	n.a.	96.2400	96.2408	96.86
29	n.a.	136.98	137.07	137.91
31	n.a.	151.70	151.95	152.71

Table 1: Bounds for the stability number of the graph $ER(q)$.

7 Conclusion

In this paper we have studied the Lovász ϑ-number [13] and Schrijver ϑ'–number for certain classes of graphs. We have showed how the semidefinite programming problems used to compute these numbers for a given graph G are determined solely by the orbits of pairs of vertices under the action of $\mathrm{Aut}(G)$. Thus one may reduce the order of the matrices involved in the computation from the number of vertices to the number of orbits of pairs. This is an application of a technique introduced in [9], where it was used to bound crossing numbers of complete bipartite graphs.

In the second instance we showed that the ϑ-number dominates a recent eigenvalue bound from [11] on the independence number of almost regular graphs. This result is an extension of the well-known result that the ϑ-number dominates the Hoffman-Delsarte eigenvalue bound on the stability number of a regular graph without loops.

Finally, we have illustrated these results by computing the ϑ and ϑ'-numbers of the Erdős–Rényi graph $ER(q)$ for $q \leq 31$, odd, and prime. The computation of $\vartheta'(ER(31))$, for example, would not be possible without using the techniques described here.

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