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Abstract

This paper shows that some classes of relaxed sequencing games, which arise from the class of sequencing games as introduced in Curiel, Pederzoli, Tijs (1989), are balanced.
Keywords: Sequencing situations, sequencing games, balancedness

1 Introduction

Sequencing games, introduced by Curiel, Pederzoli, Tijs (1989), are cooperative combinatorial optimization games that arise from one-machine sequencing situations. They introduced and characterized the Equal Gain Splitting (EGS) rule that generates a core element for the corresponding sequencing game. Moreover, they showed that sequencing games are convex. Hamers, Suijs, Borm, Tijs (1996) introduced and characterized the split core, a set solution which is a generalization of the EGS-rule and yields a re¯nement of the core. Curiel, Potters, Rajendra Prasad, Tijs, Veltman (1993,1994) showed that balancedness is preserved if the cost function is generalized.

A new direction in this game theoretical research is established by studying sequencing games that arise from one-machine sequencing situations in which restrictions on the jobs are imposed. Hamers, Borm, Tijs (1995) impose ready times on the jobs. In this case the corresponding sequencing games are balanced, but not necessarily convex. However, for a special subclass convexity is established. Similar results are obtained in Borm, Fiestras-Janeiro, Hamers, S¶anchez, Voorneveld (2002) and Hamers, Klijn, van Velzen (2002). The ¯rst consider sequencing situation in which due dates play a role and the second consider sequencing situations in which precedence constraints are involved.

Another direction in this area has been found in extending the number of machines. Hamers, Klijn, Suijs (1999) consider sequencing situations with m parallel and identical machines. These games are balanced if only two machines are present. If three or more machines are present balancedness is proven for a special class. Calleja, Borm, Hamers, Klijn, Slikker (2001) prove balancedness for a special class of sequencing games that arise from two-machine sequencing situations in which a di®erent type of cost criterion is considered. Van den Nouweland, Krabbenborg, Potters (1992) obtained some balancedness results for sequencing games that arise from multiple machine sequencing situations with a dominant machine.

In all previous mentioned sequencing games the value of a coalition is obtained by taking an admissible rearrangement that maximizes the cost savings of this coalition. These admissible rearrangements have the same feature that two players can only exchange their position in a queue if all players in between these two players in that queue are also member of this coalition. In Curiel, Potters, Rajendra Prasad, Tijs, Veltman (1993) it is argued that this set of admissible rearrangements is too restrictive. In fact, they propose to investigate sequencing games in which the set of admissible rearrangements is relaxed, i.e. rearrangements in which players of a coalition can switch positions in a queue, even if not necessarily all players in between these two players are member of this coalition. This paper tackles this issue by relaxing the set of admissible rearrangements. This has resulted in the investigation of weak-relaxed, relaxed and rigid sequencing games. Hence, for all these games coalitions have different sets of admissible rearrangements they can use to optimize their cost savings than in the classical approach of sequencing games.

The organization of this paper is as follows. Section 2 recalls sequencing situations and games as introduced in Curiel, Pederzoli, Tijs (1989). Section 3 discusses weak-relaxed and relaxed sequencing games. For the class of weak-relaxed sequencing games a core element is provided and some balancedness results with respect to relaxed sequencing games are discussed. Finally, Section 4 shows that the class of rigid sequencing games is balanced.

2 Sequencing situations and games

In this section we describe the class of sequencing situations and the corresponding games as introduced in Curiel et al. (1989). Before that we will recall some notions from cooperative game theory.

A cooperative game is a pair $(N;v)$ where N is a finite (player-)set and v, the characteristic function, is a map $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. The map v assigns to each subset $S \subset N$, called a coalition, a real number $v(S)$ called the worth of S. The core of a game $(N;v)$ is the set

$$C(v) = \{x \in \mathbb{R}^N | x(S) \geq v(S) \text{ for every } S \subset N; x(N) = v(N)\};$$

where $x(S) = \sum_{i \in S} x_i$. Intuitively the core is the set of payoff vectors for which no coalition has an incentive to split off from the grand coalition. The core can be an empty set. If the core is nonempty, the game is called balanced.

A cooperative game $(N;v)$ is called superadditive if

$$v(S) + v(T) \leq v(S \cup T) \text{ for all } S;T \subset N \text{ with } S \cap T = \emptyset;$$

and convex if for all $S;T \subset N$ it holds

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T):$$

Note, that convex games are balanced (Shapley (1971)).

Now we will describe the class of sequencing games as introduced in Curiel et al. (1989). In a sequencing situation there are n agents waiting in front of a machine. Each agent has a job that has to be processed on that machine. The set of agents will be denoted by $N = \{1;\ldots;n\}$. A processing order will be denoted by a bijection $\sigma : \{1;\ldots;n\} \to N$ which describes the position of each agent in the queue. Formally, $\sigma(i) = j$ means that agent j is in the i-th position in the queue. It is assumed that there is an initial processing order σ_0. The set of all processing orders is denoted by $\Pi(N)$. For every agent $i \in N$ the positive processing time equals p_i. Further, it is assumed that for every agent $i \in N$ the cost function c_i is described by a function $c_i : [0;\infty) \to \mathbb{R}$ defined by $c_i(t) = \alpha_i t$, with $\alpha_i > 0$ the cost per time unit. The completion time of agent i with respect to the processing order σ is defined by $C_{\sigma;i} := \sum_{j:\sigma^{-1}(j) \leq \sigma^{-1}(i)} p_j$. The above described sequencing situation will be denoted by a quadruple $(N;\sigma_0;\alpha;p)$, with $N = \{1;\ldots;n\}$ the set of agents, $\sigma_0 : \{1;\ldots;n\} \to N$ the initial order, $\alpha = (\alpha_i)_{i \in N} \in \mathbb{R}^N_+$ the cost vector and $p = (p_i)_{i \in N} \in \mathbb{R}^N_+$ the processing times. For convenience, we assume in this paper that σ_0 is the identity bijection, i.e. $\sigma_0(i) = i$ for every $i \in N$.

Next we focus on the cost resulting from a processing order. The total cost of all agents with respect to the processing order σ is given by $c_\sigma(N) = \sum_{i \in N} \alpha_i C_{\sigma,i}$. Clearly, there must be a processing order for which the cost is minimal. Smith (1956) showed that the total cost of the group is minimized if and only if the agents are lined up in decreasing order of urgency indices. The urgency index u_i of agent i is equal to $\frac{\alpha_i}{p_i}$. Formally, for a sequencing situation $(N;\sigma_0;\alpha;p)$ we have that $c_\pi(N) = \min_{\sigma \in \Pi(N)} c_\sigma(N)$ if and only if $u_{\pi(1)} \geq u_{\pi(2)} \geq \cdots \geq u_{\pi(n)}$.

Now we describe sequencing games as introduced in Curiel et al. (1989). Let $(N;\sigma_0;\alpha;p)$ be a sequencing situation and let π be an optimal processing order of N. The maximal cost savings for coalition N are equal to $c_{\sigma_0}(N) - c_\pi(N)$. Now we want to determine the maximal cost savings for each coalition S. For this we have to agree which rearrangements of S are admissible with respect to the initial order σ_0. Curiel et al. (1989) called a rearrangement for coalition S admissible with respect to the initial order if it is described by a bijection $\sigma : \{1,\ldots,n\} \to N$ such that each player outside S remains in his initial position and and no jumps take place over players outside S. Formally, these two conditions can be summarised in the following equation:

$$(i;\sigma) = (i;\sigma_0) \text{ for all } i \in N \backslash S \tag{1}$$

where $(i;\tau) = \{j \in N : \tau^{-1}(j) < \tau^{-1}(i)\}$ is the set of predecessors of i with respect to τ, excluding i. We denote the set of these admissible rearrangements for coalition S by $A(S)$. The worth of a coalition S will now be defined as the maximum cost savings it can accomplish by means of an admissible rearrangement, i.e.

$$v(S) := \max_{\sigma \in A(S)} \Big(\sum_{i \in S} \alpha_i C_{\sigma_0,i} - \sum_{i \in S} \alpha_i C_{\sigma,i}\Big).$$

Curiel et al. (1989) showed that sequencing games are convex games. As a consequence these games are balanced.

3 Weak-relaxed sequencing games

In this section we consider weak-relaxed and relaxed sequencing games. We show that weak-relaxed sequencing games are balanced.

In sequencing games as introduced in Curiel et al. (1989), referred to as classical sequencing games, the value of a coalition is obtained by taking an admissible rearrangement that maximizes the cost savings of this coalition. In such an admissible rearrangement, two players can only exchange their position in the queue if all players in between these two players are also member of this coalition. In weak-relaxed sequencing games we weaken this requirement. We assume that one specific player j can switch with any player in the coalition provided that the players outside this coalition do not suffer from this switch. So player j is allowed to switch with a player in front of him in the queue if this player has a larger processing time, and with a player behind him in the queue if this player has a smaller processing time. Because there is one player with a special property, we denote the weak-relaxed sequencing situation by $(N;j;\sigma_0;\alpha;p)$, where $j \in N$ is the player with the special property and $N;\sigma_0;\alpha$ and p are the same as in Section 2.

Let $(N;j;\sigma_0;\alpha;p)$ be a weak-relaxed sequencing situation. To define the set of admissible orders $WR^j(S)$ with respect to the initial order we need to distinguish between two sets of coalitons: coalitions that include player j and coalitions that do not include player j. For coalition $S \subset N$ that does not include j, i.e. $j \notin S$, the set of admissible orders is equal to the set of admissible rearrangements in the classical sequencing games, i.e. the set of admissible rearrangements is $A(S)$.

For coalition $S \subset N$ that does include player j, i.e. $j \in S$, a reordering σ is called admissible if it leaves the positions of the players in $N \setminus S$ fixed, at most one jump takes places (of player j with another player in S), and the completion time of the players outside S does not increase. Formally these demands can be expressed as

$$C_{\sigma_0;k} \geq C_{\sigma;k} \text{ for all } k \in N \setminus S$$

and for an $m \in S$

$$(k;\sigma) \cap (N \setminus S) = (k;\sigma_0) \cap (N \setminus S) \text{ for all } k \in S \setminus \{j;m\}:$$

From a weak-relaxed sequencing situation $(N;j;\sigma_0;\alpha;p)$ we derive a weak-relaxed sequencing game $(N;w)$ by

$$w(S) := \max_{\sigma \in WR^j(S)} \Big(\sum_{i \in S} \alpha_i C_{\sigma_0;i} - \sum_{i \in S} \alpha_i C_{\sigma;i} \Big):$$

Note that, because for all $S \subset N$ we have $A(S) \subset WR^j(S)$, it holds that $v(S) \leq w(S)$ for all $S \subset N$. Moreover, as a consequence of $A(N) = WR^j(N)$, we have that $v(N) = w(N)$.

Example 1 Let $(N;j;\sigma_0;\alpha;p)$ be a weak-relaxed sequencing situation with $N = \{1;2;3\}$, $j = 3$, $\alpha = (2;3;5)$ and $p = (2;1;1)$. The corresponding weak-relaxed sequencing game $(N;w)$ is displayed in Table 1. We explain $w(13)$. Because $p_1 \geq p_3$ we have that $WR^3(13) = \{123;321\}$. Therefore $w(13) = \max(0;\alpha_3(p_1+p_2) - \alpha_1(p_2+p_3)) = \max(0;11) = 11$. Note that in the classical sequencing game that arises from $(N;\sigma_0;\alpha;p)$ we have $v(13) = 0$.

S	1	2	3	12	13	23	123
w(S)	0	0	0	4	11	2	14

Table 1: Weak-relaxed sequencing game

Similar to classical sequencing games it is easy to show that weak-relaxed sequencing games are superadditive. Contrary to the classical sequencing games, weak-relaxed sequencing games do not belong to the class of convex games.

Example 2 Consider $(N;w)$ as in Example 1. Then $w(1) = 0$, $w(12) = 4$, $w(13) = 11$ and $w(N) = 14$. This implies that $w(13) - w(1) = 11 > w(N) - w(12) = 10$. Hence $(N;w)$ is not convex.

The next theorem shows that weak-relaxed sequencing games have a non-empty core. From Example 1 we see that $w(13) \neq w(1) + w(3)$ which implies that weak-relaxed sequencing games are not σ_0-component additive games (cf. Potters, Reijnierse (1995)), in contrary to classical sequencing games. Hence, balancedness of weak-relaxed sequencing games cannot be established using the balancedness of σ_0-component additive games.
We will show that weak-relaxed sequencing games are permutationally convex games. For the notion of permutationally convexity we first need to introduce permutations. A permutation π, denoted by $\pi = \pi(1)\ldots\pi(n)$, is a bijection $\pi : \{1;\ldots;n\} \to N$. In this paper a permutation should be interpreted as an order of the player set. Let $[l;\pi]$ be the set of predecessors of $l \in N$ including l, i.e. $[l;\pi] = \{j \in N : \pi^{-1}(j) \leq \pi^{-1}(l)\}$. A game is called permutationally convex if there exists a permutation π such that for all $i;k \in \{1;\ldots;n\}$ with $\pi^{-1}(i) < \pi^{-1}(k)$ and $S \subset N \setminus [k;\pi]$ the following inequalities are satisfied:

$$v([i;\pi] \cup S) - v([i;\pi]) \leq v([k;\pi] \cup S) - v([k;\pi]) \tag{2}$$

$$v(S) \leq v([k;\pi] \cup S) - v([k;\pi]). \tag{3}$$

Granot, Huberman (1982) showed that permutationally convex games are balanced by showing that the marginal vector $m^{\pi}(v) := (v([1;\pi]) - v((1;\pi)), \ldots, v([n;\pi]) - v((n;\pi)))$ is in the core.

Theorem 1 Weak-relaxed sequencing games are balanced.

Proof: Let $(N; j; \sigma_0; \alpha; p)$ be a weak-relaxed sequencing situation and let $(N; w)$ be the corresponding weak-relaxed sequencing game. Let $(N; \sigma_0; \alpha; p)$ be a classical sequencing situation and let $(N; v)$ be the corresponding classical sequencing game.

Now take $\pi = j-1 \ldots 1\ j+1 \ldots n\ j$. We will show that (2) and (3) are satisfied, which implies that $(N; v)$ is permutationally convex.

Let $i, k \in N$ such that $\pi^{-1}(i) < \pi^{-1}(k)$, i.e. i is a predecessor of k with respect to π. Weak-relaxed sequencing games are superadditive games and hence (3) is satisfied. Therefore it is sufficient to show that (2) holds.

Suppose that $k = j$. Then $[k;\pi] = N$, henceforth $S = \emptyset$, which proves (2). So we may assume that $k \neq j$. This implies that $j \notin [i;\pi] \subset [k;\pi]$. Hence, $w([i;\pi]) = v([i;\pi])$ and $w([k;\pi]) = v([k;\pi])$.

Now let $S \subset N \backslash [k;\pi]$ and let $\sigma_{opt}(S \cup [i;\pi]) \in WR(S \cup [i;\pi])$ denote the optimal reordering of coalition $S \cup [i;\pi]$. We distinguish between two cases.

Case 1: $\sigma_{opt}(S \cup [i;\pi]) \in A(S \cup [i;\pi])$. This implies that $w(S \cup [i;\pi]) = v(S \cup [i;\pi])$.

By definition,

$$w(S \cup [k;\pi]) \geq v(S \cup [k;\pi]).$$

So,

$$w(S \cup [k;\pi]) - w(S \cup [i;\pi]) \geq v(S \cup [k;\pi]) - v(S \cup [i;\pi])$$
$$\geq v([k;\pi]) - v([i;\pi]) = w([k;\pi]) - w([i;\pi]);$$

where the second inequality follows from the convexity of the classical sequencing games.

Case 2: $\sigma_{opt}(S \cup [i;\pi]) \notin A(S \cup [i;\pi])$. In this case player j switches with a player, say player m, from a different component. Because $\{j\} \cup [k;\pi]$ is connected with respect to the initial order σ_0, we have that $m \notin [k;\pi]$. We will provide the proof for the case that $j > m$, the proof for $j < m$ runs similar and is therefore omitted. The optimal order of $S \cup [i;\pi]$ can be attained by first switching players j and m and then putting each component in decreasing order of urgency indices. Therefore the total cost savings of $S \cup [i;\pi]$ can be decomposed into two parts: first the cost savings obtained by only switching j and m, and, second, the cost savings by putting the players in each component in decreasing order of urgency indices. First, the cost savings obtained by switching j and m, denoted by S^i_{jm}, equal

$$S^i_{jm} = \alpha_m(p_j + \ldots + p_{m-1}) - \alpha_j(p_{j+1} + \ldots + p_m) + \sum_{h \in [i;\pi]: j<h<m} (p_j - p_m)\alpha_h.$$

Second, the cost savings that can be obtained by putting the players in each component $S \cup [i;\pi]$ in decreasing order of urgency indices can be expressed using classical sequencing games. Because after the switch of players j and m two initial components have changed these cost savings are equal to the cost savings of the classical sequencing game that arise from the sequencing situation $(N;\sigma;\alpha;p)$ with σ such that $\sigma(i) = i$ for all $i \in \{1,\ldots,n\}\backslash\{j,m\}$, $\sigma(j) = m$ and $\sigma(m) = j$. Let $(N; v_\sigma)$ be the corresponding classical sequencing game. Now the cost savings obtained by putting the players in decreasing order of urgency indices equal $v_\sigma(S \cup [i;\pi])$. Therefore

$$w(S \cup [i;\pi]) = S^i_{jm} + v_\sigma(S \cup [i;\pi]). \tag{4}$$

Now we look at the cost savings coalition $S \cup [k;\pi]$ can obtain. For coalition $S \cup [k;\pi]$ we rearrange the initial order σ_0, even if j and m are in the same component, as follows. First we switch players j and m and, second, we put each component in decreasing order of urgency indices. This is certainly an admissible rearrangement, but it is not necessarily optimal.

The cost savings of the switch of j and m, denoted by S^k_{jm}, is equal to

$$S^k_{jm} = \alpha_m(p_j + \ldots + p_{m-1}) - \alpha_j(p_{j+1} + \ldots + p_m) + \sum_{h \in [k;\pi]: j<h<m} (p_j - p_m)\alpha_h.$$

After the switch of j and m, coalition $S \cup [k;\pi]$ can obtain extra cost savings by putting each components in decreasing order of urgency indices. By expressing these extra cost savings in terms of the classical sequencing game $(N; v_\sigma)$ we obtain,

$$w(S \cup [k;\pi]) \geq S^k_{jm} + v_\sigma(S \cup [k;\pi]). \tag{5}$$

Because $p_j \geq p_m$ and $[i;\pi] \subset [k;\pi]$ it holds that $S^k_{jm} \geq S^i_{jm}$. Therefore

$$w(S \cup [k;\pi]) \geq S^i_{jm} + v_\sigma(S \cup [k;\pi])$$

Now we obtain

$$\begin{aligned} w(S \cup [k;\pi]) - w(S \cup [i;\pi]) &\geq v_\sigma(S \cup [k;\pi]) - v_\sigma(S \cup [i;\pi]) \\ &\geq v_\sigma([k;\pi]) - v_\sigma([i;\pi]) = w([k;\pi]) - w([i;\pi]). \end{aligned}$$

The first inequality holds by (4) and (5), the second inequality follows from the convexity of classical sequencing games and the equality follows from $j, m \notin [k;\pi]$. □

As a consequence, the marginal vector $m^\pi(w)$ is a core-element, where $\pi = j-1 \cdots 1\ j+1 \cdots n\ j$.

Example 3 Let $(N; w)$ be a weak-relaxed sequencing game for which $N = \{1,\ldots,7\}$ and $j = 3$. Then $m^\pi(w) \in C(w)$ with $\pi = 2145673$.

In the final part of this section we discuss relaxed sequencing games, which are games that arise from sequencing situations where every player is allowed to jump over other players as long as the interests of the players outside the coalition are not hurt. Let $(N;\sigma_0;\alpha;p)$ be a sequencing situation. An order σ is called relaxed admissible for coalition S with respect to the initial order if it leaves the position of the players in $N\backslash S$ fixed, and the completion time of players in $N\backslash S$ does not increase. Formally, the set of admissible rearrangements of a coalition S satisfies the following two properties and will be denoted by $R(S)$,

$$\sigma(j) = j \text{ for all } j \in N\backslash S$$

and

$$C_{\sigma,j} \leq C_{\sigma_0,j} \text{ for all } j \in N \backslash S.$$

Then a relaxed sequencing game (N, v) that arises from a sequencing situation (N, σ_0, α, p) is defined by

$$z(S) := \max_{\sigma \in R(S)} \Big(\sum_{i \in S} \alpha_i C_{\sigma_0,i} - \sum_{i \in S} \alpha_i C_{\sigma,i} \Big),$$

for all $S \subseteq N$. Note that because $A(S) \subseteq WR^j(S) \subseteq R(S)$ for every $S \subseteq N$, we have that $v(S) \leq w(S) \leq z(S)$ for each $S \subseteq N$. Moreover, because $A(N) = WR^j(N) = R(N)$, it holds that $v(N) = w(N) = z(N)$.

Example 4 Let (N, σ_0, α, p) be a sequencing situation with $N = \{1, 2, 3, 4\}$, $\alpha = (2, 1, 4, 3)$ and $p = (1, 1, 1, 1)$. The corresponding relaxed sequencing game is printed in Table 2.

S	i	12	13	14	23	24	34	123	124	134	234	N
$z(S)$	0	0	4	3	3	4	0	5	5	5	5	8

Table 2: Relaxed sequencing game.

Observe that the coalitions $\{1, 3\}$ and $\{2, 4\}$ have a positive value, a situation that can not occur in weak relaxed sequencing games.

Because a 3-player relaxed sequencing game is a weak-relaxed sequencing game with $j = 3$, these games are balanced. In Hamers (1988) it is shown that 4-player relaxed sequencing games are balanced by checking all balancedness conditions. For $n \geq 5$, no results on the balancedness of relaxed sequencing games are known. Contrary to weak-relaxed sequencing games, the class of relaxed sequencing games is not inbedded in the class of permutationally convex games.

Example 5 Consider Example 4. Let $x \in C(v)$. Because $z(13) + z(24) = 4 + 4 = 8 = z(N)$ we have that $x_1 + x_3 = 4$ and $x_2 + x_4 = 4$. Now $z(123) = 5$ implies that $x_1 + x_2 + x_3 \geq 5$ and thus that $x_2 \geq 1$. Similarly it can be shown that $x_i \geq 1$ for all $i \in \{1, 3, 4\}$. As $z(i) = 0$ for all $i \in N$ we can conclude that x is not a marginal vector. Therefore no marginal vector is in the core, and hence this game is not permutationally convex.

4 Rigid sequencing games

In this section we introduce rigid sequencing games and we prove that these games are balanced. Rigid sequencing situations are situations in which there are fixed time slots, i.e. the machine is programmed to start processing a new job on fixed times. Therefore, in rigid sequencing situations only jobs that have equal processing time can be switched.

Let (N, σ_0, α, p) be a sequencing situation. For coalition $S \subseteq N$ a reordering σ is called rigid admissible with respect to the initial order if it leaves $N \backslash S$ fixed and if switching takes place only between players with equal processing time. Formally,

$$|P(\sigma_0, j)| = |P(\sigma, j)| \text{ for all } j \in N \backslash S$$

and

$$p_i = p_j \text{ for all } i, j \in N \text{ with } i = \sigma(j).$$

By applying the Smith-rule to players with equal processing time, we can derive the optimal processing order for each coalition $S \subset N$, i.e. an order is optimal if players with equal processing time stand in decreasing order of urgency indices.

Let $RIG(S)$ denote the set of rigid admissible rearrangements of coalition $S \subset N$. Then a rigid sequencing game (N, r) that arises from a sequencing $(N, \mathrm{sigma}_0, \alpha, p)$ is defined by

$$r(S) := \max_{\sigma \in RIG(S)} \Big(\sum_{i \in S} \alpha_i C_{\sigma_0,i} - \sum_{i \in S} \alpha_i C_{\sigma,i} \Big).$$

The following example illustrates a rigid sequencing game and shows that not necessarily $RIG(S) \subset A(S)$ or $A(S) \subset RIG(S)$ for every $S \subset N$.

Example 6 Let (N, σ_0, α, p) be a sequencing situation with $N = \{1, 2, 3\}$, $\alpha = (2, 3, 5)$ and $p = (2, 1, 2)$. Then the corresponding rigid sequencing game (N, r) is displayed in the following table.

S	1	2	3	12	13	23	123
$r(S)$	0	0	0	0	9	0	9.

Table 3: Rigid sequencing game.

Because $p_1 = p_3 \neq p_2$ it holds that $r(12) = r(23) = 0$ and $r(13) = \max(0, \alpha_3(p_1+p_2) - \alpha_1(p_2+p_3)) = \max(0, 9) = 9$. Further, observe that $A(\{1, 2\}) = \{123, 213\}$ whereas $RIG(\{1, 2\}) = \{1, 2, 3\}$, and $A(\{1, 3\}) = \{123\}$ whereas $RIG(\{1, 3\}) = \{123, 321\}$.

The next theorem tells us that the rigid sequencing games have a nonempty core by showing that rigid sequencing games are permutation games. A cooperative game (N, v) is called a permutation game if there is an $n \times n$-matrix A such that

$$v(S) = \max_{\sigma \in \Pi_S} \Big(\sum_{i \in S} a_{ii} - \sum_{i \in S} a_{i\sigma(i)} \Big)$$

for all $S \subset N$, $S \neq \emptyset$ and $v(\emptyset) = 0$, where Π_S is the set of all permutations of S and a_{ij} denote the entries of A. In Tijs, Parthasarathy, Potters, Rajendra Prasad (1984) it is shown that permutation games are balanced.

Theorem 2 Rigid sequencing games are balanced.

Proof: We show that rigid sequencing games are permutation games. Let (N, σ_0, α, p) be a sequencing situation and let (N, r) be the corresponding rigid sequencing game. Let a be such that

$$a_{ij} = \begin{cases} \alpha_i C_{\sigma,j} & \text{if } p_i = p_j \\ \infty & \text{else} \end{cases}$$

For coalition $S \subset N$ a reordering is rigid admissible if and only if it is a permutation of players of S with equal processing time. Therefore $r(S) = \max_{\sigma \in RIG(S)} (\sum_{i \in S} \alpha_i C_{\sigma,i} - \sum_{i \in S} \alpha_i C_{\sigma,j}) = \max_{\sigma \in \Pi_S} (\sum_{i \in S} a_{ii} - \sum_{i \in S} a_{i\sigma(i)})$, where the last equality follows from $\alpha_{ij} = \infty$ if $p_i \neq p_j$. $\square$

The following example shows that the class of rigid sequencing game is a proper subset of the class of permutation games.

Example 7 Let

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and let $(N; z)$ be the corresponding permutation game. Then $N = f1; 2; 3g$, $z(i) = 0$ for i 2 $f1; 2; 3g$, $z(12) = z(13) = z(23) = 2$ and $z(N) = 3$. Suppose this game is a rigid sequencing game $(N; r)$. Because all two-player coalitions have values strictly larger than 0, it follows that $p_1 = p_2 = p_3$. Now $z(12) = r(12) = g_{12} = 2$ and $v(23) = r(23) = g_{23} = 2$. But this implies that $r(N) = g_{12} + g_{13} + g_{23}$ ¸ $g_{12} + g_{23} = 4 > 3 = z(N)$. Therefore we may conclude that the class of rigid sequencing games is a proper subset of the class of permutation games.

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