LOWER TAIL DEPENDENCE FOR ARCHIMEDEAN COPULAS: CHARACTERIZATIONS AND PITFALLS

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Lower tail dependence for Archimedean copulas: characterizations and pitfalls

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Abstract

Tail dependence copulas provide a natural perspective from which one can study the dependence in the tail of a multivariate distribution. For Archimedean copulas with continuously differentiable generators, regular variation of the generator near the origin is known to be closely connected to convergence of the corresponding lower tail dependence copulas to the Clayton copula. In this paper, these characterizations are refined and extended to the case of generators which are not necessarily continuously differentiable. Moreover, a counterexample is constructed showing that even if the generator of a strict Archimedean copula is continuously differentiable and slowly varying at the origin, then the lower tail dependence copulas do not need to converge to the independent copula.

JEL: C14, C16

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1 Introduction

In financial and actuarial risk management, appropriate models for dependence between risks are of obvious importance (e.g. Bäuerle and Müller, 1987;

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Frees and Valdez, 1998; Klugman and Parsa, 1999; Dhaene et al., 2005). Copulas form a widely accepted tool for building such dependence models. A versatile subclass of copulas is the one of Archimedean copulas, introduced in Kimberling (1974) and studied intensively since (e.g. Genest and MacKay, 1986; Genest and Rivest, 1993; Müller and Scarsini, 2004). The need for accurate modelling of extremal events then requires a better understanding of the behavior of these and other copulas in the tails.

In Juri and Wüthrich (2002), tail dependence for bivariate Archimedean copulas is described using the concept of lower tail dependence copulas. The lower tail dependence copula of a copula C at level $0 < u < 1$ is defined as the copula of the conditional distribution of a random pair (U, V) with distribution function C when conditioned to be contained in the square $[0, u]^2$.

The central topic in Juri and Wüthrich (2002) is the asymptotic behavior of the lower tail dependence copula of a strict Archimedean copula as the threshold u decreases to zero. The main result is that, if the generator is continuously differentiable, then regular variation of the generator near zero is equivalent to convergence of the lower tail dependence copula to a Clayton copula, the parameter of the latter being determined by the index of regular variation of the generator. The key role of the Clayton copula was also recognized later in Charpentier and Juri (2004) or Bassan and Spizzichino (2005).

Our aim, then, is twofold: First, we extend the results in Juri and Wüthrich (2002) to the case of generators that are not necessarily continuously differentiable. Here we rely on results in Charpentier and Segers (2006), extending known characterizations for convergence of Archimedean copulas (Nelsen, 1999, Theorems 4.4.7 and 4.4.8) by removing redundant smoothness assumptions. Second, we correct the statement in Theorem 3.5 in Juri and Wüthrich (2002) that slow variation of the generator implies convergence of the lower tail dependence copula to the independent one. Indeed, we construct a counterexample contradicting the previous claim.

The outline of the paper is as follows: after some preliminaries in section 2, the main result on convergence of lower tail dependence copulas is stated and proven in section 3. A counterexample is constructed in section 4, followed by a technical discussion in section 5 on the relation between the various conditions imposed on the behaviour of the generator near the origin.

2 Preliminaries

A function $C : [0, 1]^2 \to [0, 1]$ is called a *bivariate copula* if it is the restriction to $[0, 1]^2$ of a bivariate distribution function whose marginals are given by the

uniform distribution on the interval $[0, 1]$. A function $\psi : [0, 1] \to [0, \infty]$ is called a *strict generator* if it is decreasing, convex, $\psi(0) = \infty$ and $\psi(1) = 0$. The inverse function of a strict generator ψ is denoted by ψ^{-1}. A function $C : [0, 1]^2 \to [0, 1]$ is called a *strict Archimedean copula* if there exists a strict generator ψ such that

$$C(u, v) = \psi^{-1}\{\psi(u) + \psi(v)\}, \qquad (u, v) \in [0, 1]^2.$$

Note that the generator is unique up to a multiplicative constant. A strict Archimedean copula is a copula. See the survey monograph by Nelsen (1999) and the references therein for more details.

Let C be a copula and let (U, V) be a random pair with joint distribution function C. Let $0 < u < 1$ be such that $C(u, u) > 0$. The *lower tail dependence copula relative to C at level u* is defined as the copula, C_u, of the joint distribution of (U, V) conditionally on the event $\{U \le u, V \le u\}$. Formally,

$$C_u(x, y) = \frac{C(x', y')}{C(u, u)}$$

where $0 \le x' \le u$ and $0 \le y' \le u$ are the solutions to the equations

$$\frac{C(x', u)}{C(u, u)} = x \text{ and } \frac{C(u, y')}{C(u, u)} = y;$$

see Juri and Wüthrich (2002, Definition 3.1) or Juri and Wüthrich (2003, Definition 2.2). *Upper tail dependence copulas* are defined in a similar way (Juri and Wüthrich, 2003, Definition 2.1). Moreover, the definition can be extended by allowing the thresholds for the two margins to be different, that is, by conditioning on the event $\{U \le u, V \le v\}$, where $(u, v) \in (0, 1]^2$ are such that $C(u, v) > 0$, see Charpentier and Juri (2004, Definition 2.5). In this note, we will be interested only in the diagonal.

If C is a strict bivariate Archimedean copula with generator ψ, then the lower tail dependence copula relative to C at level u is given by the strict Archimedean copula with generator ψ_u defined by

$$\psi_u(t) = \psi(tv) - \psi(v), \qquad 0 \le t \le 1, \tag{1}$$

where $v = v(u) = \psi^{-1}\{2\psi(u)\}$ (Juri and Wüthrich, 2002, Proposition 3.2). Since $v(u) \to 0$ as $u \downarrow 0$, the asymptotic behaviour of the lower tail dependence copula C_u as $u \downarrow 0$ depends on the asymptotic behaviour of ψ near the origin.

A positive, measurable function f defined in a right-neighbourhood of zero is said to be *regularly varying at zero of index $\tau \in \mathbb{R}$* if

$$\lim_{u \downarrow 0} \frac{f(ux)}{f(u)} = x^\tau, \qquad 0 < x < \infty;$$

notation: $f \in \mathcal{R}_\tau$. If $\tau = 0$, then the limit is equal to one for all $0 < x < \infty$; in this case, f is said to be *slowly varying at zero*. A limiting case is obtained when $\tau = -\infty$: f is said to be *rapidly varying at zero of index* $-\infty$, notation $f \in \mathcal{R}_{-\infty}$, if

$$
\lim_{u \downarrow 0} \frac{f(ux)}{f(u)} = \begin{cases} 0 & \text{if } 1 < x < \infty, \\ 1 & \text{if } x = 1, \\ \infty & \text{if } 0 < x < 1. \end{cases}
$$

Classically, regular (and rapid) variation are considered at infinity rather than at zero. However, it is typically straightforward to translate results from regular variation at infinity to regular variation at zero by considering the function $y \mapsto f(1/y)$ (see for instance Bingham et al., 1987, p. 18).

The *Clayton copula* with parameter $\alpha \in [0, \infty)$ is the Archimedean copula with strict generator given by

$$
\psi(x; \alpha) = \int_x^1 t^{-\alpha - 1} \mathrm{d}t = \begin{cases} \dfrac{x^{-\alpha} - 1}{\alpha} & \text{if } 0 < \alpha < \infty, \\ -\log(x) & \text{if } \alpha = 0 \end{cases}
$$

for $0 < x \leq 1$; the corresponding copula is

$$
C(x, y; \alpha) = \begin{cases} (x^{-\alpha} + y^{-\alpha} - 1)^{-1/\alpha} & \text{if } 0 < \alpha < \infty, \\ xy & \text{if } \alpha = 0 \end{cases}
$$

for $(x, y) \in (0, 1]^2$. Note that $\lim_{\alpha \downarrow 0} \psi(t; \alpha) = \psi(t; 0)$ and $\lim_{\alpha \downarrow 0} C(x, y; \alpha) = C(x, y; 0)$. The comonotone copula, which is itself not an Archimedean copula, arises as the limit of the Clayton copula as $\alpha \to \infty$, that is,

$$
C(x, y; \infty) = \lim_{\alpha \to \infty} C(x, y; \alpha) = \min(x, y)
$$

for $(x, y) \in [0, 1]^2$. The Clayton copula has the special property that at every level $0 < u < 1$, its lower tail dependence copula is again a Clayton copula and with the same parameter; see also Charpentier and Juri (2004, Proposition 4.15).

3 Main result

Our main result, Theorem 1, can be seen as an extension of Theorems 3.3, 3.5 and 3.6 of Juri and Wüthrich (2002). The asymptotic behavior of lower tail dependence copulas for general symmetric bivariate copulas is studied in Juri

and Wüthrich (2003), and for nonsymmetric bivariate copulas in Charpentier and Juri (2004).

Theorem 1 *Let C be a strict Archimedean copula with generator ψ, whose right-hand derivative is denoted by ψ'. Let $0 \leq \alpha \leq \infty$. Consider the following four statements:*

(i) $\lim_{u \downarrow 0} C_u(x, y) = C(x, y; \alpha)$ *for all* $(x, y) \in [0, 1]^2$;
(ii) $-\psi' \in \mathcal{R}_{-\alpha-1}$.
(iii) $\psi \in \mathcal{R}_{-\alpha}$.
(iv) $\lim_{u \downarrow 0} u\psi'(u)/\psi(u) = -\alpha$.

If $\alpha = 0$ (tail independence),

$$(i) \Longleftrightarrow (ii) \Longrightarrow (iii) \Longleftrightarrow (iv),$$

and if $\alpha \in (0, \infty]$ (tail dependence),

$$(i) \Longleftrightarrow (ii) \Longleftrightarrow (iii) \Longleftrightarrow (iv).$$

As we will see in Theorem 2, if $\alpha = 0$, then *(iii)* does not imply *(i)* or *(ii)*, in contradiction to Lemma 3.4 and Theorem 3.5 of Juri and Wüthrich (2002).

Proof. Recall from (1) that the lower tail dependence copula of C at $0 < u < 1$ is the Archimedean copula with generator $\psi_u(x) = \psi(xv) - \psi(v)$ for $0 \leq x \leq 1$, where $v \equiv v(u) = \psi^{-1}\{2\psi(u)\}$. Note that $v(u)$ is continuous in u and decreases to 0 as u decreases to zero. The (right-hand) derivative of ψ_u at $0 < y < 1$ is equal to $\psi'_u(y) = v\psi'(vy)$.

• **The case $\alpha \in (0, \infty)$ (tail dependence)**

(i) implies (ii). By Charpentier and Segers (2006, Proposition 2) [extending Nelsen (1999, Theorem 4.4.7) and Genest and MacKay (1986, Proposition 4.2) to the case of generators which are not twice continuously differentiable], for $0 < x \leq 1$,

$$\lim_{u \downarrow 0} \frac{\psi_u(x)}{\psi'_u(1/2)} = \frac{\psi(x; \alpha)}{\psi'(1/2; \alpha)} = -2^\alpha \psi(x; \alpha).$$

Define $g(v) = -\psi'_u(1/2) = -v\psi'(v/2)$. By the previous display, for $0 < x \leq 1$,

$$\lim_{v \downarrow 0} \frac{\psi(vx) - \psi(v)}{g(v)} = -2^\alpha \psi(x; \alpha).$$

For $0 < x \leq 1$, we get

$$\frac{g(vx)}{g(v)} = \left(\frac{\psi(vx^2) - \psi(v)}{g(v)} - \frac{\psi(vx) - \psi(v)}{g(v)} \right) \Big/ \frac{\psi(vx^2) - \psi(vx)}{g(vx)}$$

$$\to \{\psi(x^2; \alpha) - \psi(x; \alpha)\}/\psi(x; \alpha) = x^{-\alpha}, \qquad \text{as } v \downarrow 0.$$

Hence, $g \in \mathcal{R}_{-\alpha}$, and thus $-\psi' \in \mathcal{R}_{-\alpha-1}$.

(ii) implies (i). If $-\psi' \in \mathcal{R}_{-\alpha-1}$, then for all $0 < x < 1$ and $0 < y < 1$,

$$\frac{\psi_u(x)}{\psi_u'(y)} = \frac{\psi(vx) - \psi(v)}{v\psi'(vy)} = -\int_{vx}^v \frac{\psi'(t)}{v\psi'(vy)} \mathrm{d}t = -\int_x^1 \frac{\psi'(vt)}{\psi'(vy)} \mathrm{d}t$$

$$\to -\int_x^1 \left(\frac{t}{y} \right)^{-\alpha-1} \mathrm{d}t = \frac{\psi(x; \alpha)}{\psi'(y; \alpha)}, \qquad u \downarrow 0.$$

By Charpentier and Segers (2006, Proposition 2), extending Nelsen (1999, Theorem 4.4.7), we find that (i) must hold.

(iii) implies (iv). This follows from the Monotone Density Theorem (Bingham et al., 1987, Theorem 1.7.2) applied to the function $x \mapsto \psi(1/x)$.

(iv) implies (iii). This follows from the Representation Theorem for regularly varying functions (Bingham et al., 1987, equation (1.5.2)).

So far, we have established the equivalences *(i)* $\Longleftrightarrow$ *(ii)* and *(iii)* $\Longleftrightarrow$ *(iv)*.

(ii) implies (iii). This follows from Karamata's Theorem (Bingham et al., 1987, Proposition 1.5.8) applied to the function $x \mapsto \psi(1/x)$.

(iii) and (iv) imply (ii). This is immediate, since $-\psi'(x) \sim \alpha x^{-1} \psi(x)$ as $x \downarrow 0$ and $\psi \in \mathcal{R}_{-\alpha}$.

- **The case $\alpha = 0$ (tail independence)**

The proofs of all the implications, except for the last one, also hold when $\alpha = 0$.

- **The case $\alpha = \infty$ (tail comonotonicity)**

By Charpentier and Segers (2006, Proposition 3), extending Nelsen (1999, Theorem 4.4.8) and Genest and MacKay (1986, Proposition 4.3), *(i)* is equivalent to

$$\lim_{u \downarrow 0} \frac{\psi_u(x)}{\psi_u'(x)} = 0, \qquad 0 < x \le 1.$$

Combine the above three displays to find that *(i)* is equivalent to

$$\lim_{v \downarrow 0} \frac{\psi(vx) - \psi(v)}{v\psi'(vx)} = 0, \qquad 0 < x \le 1. \tag{2}$$

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We show first the circle of implications *(i)* $\Longrightarrow$ *(ii)* $\Longrightarrow$ *(iv)* $\Longrightarrow$ *(i)* and then the equivalence *(iii)* $\Longleftrightarrow$ *(iv)*.

(i) implies (ii). Since ψ is decreasing and convex,

$$0 \le (x-1)v\psi'(v) \le \psi(vx) - \psi(v), \qquad 0 < x \le 1; 0 < v \le 1.$$

Since *(i)* is equivalent to (2), the above inequality implies

$$\lim_{v \downarrow 0} \frac{\psi'(v)}{\psi'(vx)} = 0, \qquad 0 < x < 1.$$

Hence $\psi' \in \mathcal{R}_{-\infty}$.

(ii) implies (iv). Let $1 < x < \infty$. There exists $0 < u_0 \le 1/x$ such that

$$\frac{\psi'(ux)}{\psi'(u)} \le \frac{1}{2x}, \qquad 0 < u \le u_0.$$

Let $0 < u \le u_0$ and let $k = 0, 1, 2, \dots$ be such that $ux^k < u_0 \le ux^{k+1}$. Since ψ is decreasing and convex,

$$
\begin{aligned}
\psi(u) &= \sum_{j=0}^{k} \{\psi(ux^j) - \psi(ux^{j+1})\} + \psi(ux^{k+1}) \\
&\le \sum_{j=0}^{k} ux^j(1-x)\psi'(ux^j) + \psi(u_0) \\
&\le u\psi'(u)(1-x)\sum_{j=0}^{k} x^j \frac{1}{(2x)^j} + \psi(u_0) \\
&\le 2(1-x)u\psi'(u) + \psi(u_0).
\end{aligned}
$$

Since ψ is strict, there exists $0 < u_1 < u_0$ such that $\psi(u) \ge 2\psi(u_0)$ for all $0 < u \le u_1$. Hence, by the previous display,

$$\psi(u) \le 4(1-x)u\psi'(u), \qquad 0 < u \le u_1.$$

Let u decrease to zero to find

$$\limsup_{u \downarrow 0} \frac{\psi(u)}{-u\psi'(u)} \le 4(x-1).$$

Since x was an arbitrary element in $(1, \infty)$, we arrive at *(iv)*.

(iv) implies (i). Since *(i)* is equivalent to (2), it is sufficient to show that *(iv)* implies (2). Let $0 < v \le 1$ and $0 < x < 1$. We have

$$\left| \frac{\psi(vx) - \psi(v)}{v\psi'(vx)} \right| \le \frac{\psi(vx)}{v|\psi'(vx)|} \le \frac{\psi(vx)}{vx|\psi'(vx)|}.$$

By (iv), the right-hand side of this equation tends to zero as $v \downarrow 0$, whence (2), as required.

(iii) implies (iv). Let $0 < u < 1$ and $1 < x < 1/u$. Since ψ is convex,

$$\psi(u) - \psi(ux) \leq (1 - x)u\psi'(u).$$

By *(iii)*, $\lim_{u \downarrow 0} \psi(ux)/\psi(u) = 0$ for every $1 < x < \infty$. Divide both sides of the inequality in the previous display by $\psi(u)$ and let u decrease to zero to find

$$\liminf_{u \downarrow 0} \frac{-u\psi'(u)}{\psi(u)} \geq \frac{1}{x - 1}, \qquad 1 < x < \infty.$$

The right-hand side in the previous display becomes arbitrarily large as $x \downarrow 1$, whence *(iv)*.

(iv) implies (iii). Let $0 < x < 1$. Since ψ is convex, we have for $0 < u \leq 1$,

$$\psi(ux) - \psi(u) \geq (x - 1)u\psi'(u),$$

whence

$$\frac{\psi(ux)}{\psi(u)} \geq (x - 1)\frac{u\psi'(u)}{\psi(u)} + 1.$$

By *(iv)*, the right-hand side side of this inequality tends to infinity as $u \downarrow 0$, yielding *(iii)*. □

4 Counterexample

We claim in Theorem 1 that for general $\alpha \in [0, \infty]$, statements *(i)* and *(ii)* imply statements *(iii)* and *(iv)*. If $\alpha > 0$, the converse is also true. However, if $\alpha = 0$, then the converse does not hold, as shown by the following counterexample, contradicting Juri and Wüthrich (2002, Theorem 3.5).

Theorem 2 *There exists a strict Archimedean copula C whose generator ψ is continuously differentiable and slowly varying at the origin, but such that the lower tail dependence copula of C at level u does not converge to the independence copula as $u \downarrow 0$.*

Proof. Let $f : (0, 1] \to \mathbb{R}$ be the piece-wise linear function with knots

$$f(2^{-k}) = 2^k, \qquad k = 0, 1, 2, \ldots.$$

That is, f is the linear interpolation of the function $(0, 1] \ni x \mapsto x^{-1}$ at the points $\{2^{-k} \mid k = 0, 1, 2, \ldots\}$. Define the function $\psi : [0, 1] \to [0, \infty]$ by

$$\psi(s) = \int_s^1 f(x)\mathrm{d}x, \qquad s \in [0, 1].$$

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By construction, the function ψ is continuously differentiable with derivative $\psi' = -f$. Since f is decreasing, ψ' is increasing, whence ψ is convex. Hence, ψ is a strict generator.

As $s^{-1} \le f(s) \le 2s^{-1}$ for all $s \in (0,1]$, we have $\psi(s) \ge \log(1/s)$ and thus

$$0 \le \frac{sf(s)}{\psi(s)} \le \frac{2}{\log(1/s)} \to 0, \qquad \text{as } s \downarrow 0.$$

Hence, as ψ is convex, for every $1 < x < \infty$,

$$0 \le 1 - \frac{\psi(sx)}{\psi(s)} \le \frac{s(1-x)f(s)}{\psi(s)} \to 0, \qquad \text{as } s \downarrow 0.$$

Therefore, ψ is slowly varying at the origin.

Let C be the Archimedean copula with generator ψ and let C_u be the tail dependence copula relative to C at level $0 < u < 1$. We will show that $C_{2^{-k}} = C$ for every positive integer k. Hence, C_u cannot converge to the independence copula as $u \downarrow 0$.

By the definition of the function f,

$$\psi(2^{-k-1}) - \psi(2^{-k}) = \int_{2^{-k-1}}^{2^{-k}} f(x)\mathrm{d}x = \frac{3}{4}$$

for all nonnegative integer k. Since also $\psi(1) = 0$, we get $\psi(2^{-k}) = \frac{3}{4}k$ and thus $\psi^{-1}\{2\psi(2^{-k})\} = 2^{-2k}$ for all nonnegative integer k. By (1), the tail dependence copula of C at level $u = 2^{-k}$ is therefore Archimedean with generator

$$\psi_{2^{-k}}(t) = \psi(2^{-2k}t) - \psi(2^{-2k}) = \int_{2^{-2k}t}^{2^{-2k}} f(x)\mathrm{d}x = \int_t^1 2^{-2k}f(2^{-2k}x)\mathrm{d}x$$

for $t \in [0,1]$. The function $(0,1] \ni x \mapsto f_k(x) = 2^{-2k}f(2^{-2k}x)$ is piece-wise linear with knots $f_k(2^{-j}) = 2^j$ for all nonnegative integer j. Hence, f_k must coincide with f. But then, $\psi_{2^{-k}}$ coincides with ψ, and thus $C_{2^{-k}}$ coincides with C for all nonnegative integer k, as required $\qquad\square$

5   Discussion: Asymptotic independence

The problem with Theorem 3.5 in Juri and Wüthrich (2002) comes from the auxiliary Lemma 3.4 in the same paper. In this Lemma, it is claimed that if ψ is a strict generator, differentiable and slowly varying at the origin, then there

exists a positive function g on $(0, 1)$ such that

$$\lim_{u \downarrow 0} \frac{\psi(ux) - \psi(u)}{g(u)} = -\log(x) \tag{3}$$

for every $0 < x < \infty$. However, the generator ψ appearing in the proof of Theorem 2 satisfies $\psi(ux) - \psi(u) = \psi(x)$ for every $u = 2^{-k}$ with $k = 0, 1, 2, \ldots$, contradicting the claim.

The condition (3) states that the function ψ belongs to the de Haan class Π with auxiliary function g, notation $\psi \in \Pi_g$ (e.g. Bingham et al., 1987, chapter 3). [Here, we conveniently shift from asymptotics at infinity to asymptotics at zero by considering the function $y \mapsto \psi(1/y)$ for $y \geq 1$.] By the Monotone Density Theorem (Bingham et al., 1987, Theorem 3.6.8), equation (3) is equivalent to

$$-\psi' \in \mathcal{R}_{-1} \tag{4}$$

and in this case, $g(s) \sim -s\psi'(s)$ as $s \downarrow 0$. Moreover, by Karamata's theorem (Bingham et al., 1987, Proposition 1.5.9a), (4) implies $\psi \in \mathcal{R}_0$. The converse is not true however, as demonstrated by our counterexample.

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