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GRAPHS COSPECTRAL WITH KNESER GRAPHS

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Graphs cospectral with Kneser graphs

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Abstract

We construct graphs that are cospectral but nonisomorphic with Kneser graphs $K(n, k)$, when $n = 3k - 1$, $k > 2$ and for infinitely many other pairs (n, k) . We also prove that for $3 \leq k \leq n - 3$ the Modulo-2 Kneser graph $K_2(n, k)$ is not determined by the spectrum.

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1 Introduction

Given a simple graph G , the spectrum of G is the multi-set of eigenvalues of the adjacency matrix of G . Two graphs with the same spectrum are called *cospectral*. A graph G is said to be *determined by the spectrum* if every graph cospectral with G is isomorphic to G . Two non-isomorphic cospectral graphs G and G' are called *cospectral mates*.

An important research topic in the theory of graph spectra is to find out which graphs are determined by their spectra. Especially for graphs with a high degree of regularity, like distance-regular graphs, the problem has received much attention. See [2] and [3] for a survey and recent developments. Here we consider some graphs in the Johnson association scheme $J(n, k)$. In other words, the vertex set $V(n, k)$ consists of all k -subsets of $\{1, \dots, n\}$, and adjacency only depends on the intersection size of the corresponding k -subsets. For a subset S of $K = \{0, \dots, k - 1\}$, we denote by $J_S(n, k)$ the graph with vertex set $V(n, k)$, where two vertices are adjacent if the intersection size of the corresponding subsets is in S . The graph $J_0(n, k)$ is better known as the Kneser graph $K(n, k)$, and $J_{k-1}(n, k)$ is the Johnson graph. If S consists of all even numbers in K , we

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call the graph $J_S(n, k)$ the Modulo-2 Kneser graph, and denote it by $K_2(n, k)$. Note that $J_S(n, k)$ is isomorphic with $J_{S+n-2k}(n, n-k)$, and with the complement of $J_{K \setminus S}(n, k)$. In particular, $K_2(n, k)$ is isomorphic with $K_2(n, n-k)$ if n is even, and isomorphic with the complement of $K_2(n, n-k)$ if n is odd.

Hoffman and Chang (see for example [8]) have shown that $J_1(n, 2)$ is determined by its spectrum if and only if $n \neq 8$. A regular graph is determined by its spectrum if and only if its complement is, therefore also $K(n, 2)$ is determined by its spectrum if $n \neq 8$. Other Kneser graphs which are known to be determined by the spectrum are the trivial cases $k = 1$ and $k \geq n/2$, and the Odd graphs $K(2k+1, k)$ (see [7]). In [4], it is proved that the Johnson graphs $J_{k-1}(n, k)$ are not determined by the spectrum if $k > 2$. The odd graphs $K(2k+1, k)$, and the Johnson graphs $J_{k-1}(n, k)$ are distance-regular. These graphs and their complements are graphs in the Johnson scheme for which the answer to the characterization problem is known. However, for almost all other graphs in the Johnson scheme the problem has been unsolved. In this note we give an answer for Kneser graphs $K(n, k)$, when $n = 3k - 1$, and for infinitely many other values of n and k by constructing cospectral mates. In addition we find cospectral mates for all Modulo-2 Kneser graphs $K_2(n, k)$ with $3 \leq k \leq n - 3$.

Our main tool for constructing a cospectral mate is the following result due to Godsil and McKay [6] (see also [2]):

Proposition 1.1 *Let G be a graph and let H be an induced regular subgraph of G of even order h (say). Assume that each vertex outside H is adjacent to $h, h/2$ or 0 vertices of H . Make a new graph G' as follows. For each vertex v outside H with $h/2$ neighbors in H , delete the $h/2$ edges between v and H , and join v instead to the $h/2$ other vertices in H . Then G and G' have the same spectrum.*

The operation that changes G into G' is called *Godsil-McKay switching*. The vertex set of H will be called a *(Godsil-McKay) switching set*.

2 Kneser graphs

For a positive integer $\ell \leq k - 1$ we define:

$$V_\ell = \{v \in V(n, k) : \{1, \dots, k - \ell\} \subset v\}.$$

It is clear that the set V_ℓ induces a coclique (independent vertex set) in $K(n, k)$. The following result states for which values of n, k , and ℓ the set V_ℓ is a switching set.

Theorem 2.1 *Let n, k and ℓ be positive integers such that $\ell < k < n/2$. Then V_ℓ is a Godsil-McKay switching set in $K(n, k)$ if n, k , and ℓ satisfy the following equation*

$$\binom{n - k + \ell}{\ell} = 2 \binom{n - 2k + \ell}{\ell}. \quad (1)$$

Moreover, if $\ell < k - 1$, then the graph $K'(n, k)$ obtained by switching is nonisomorphic with $K(n, k)$.

Proof. Clearly V_ℓ has order $h = \binom{n-k+\ell}{\ell}$. Let $x \in V(n, k)$ be a vertex of $K(n, v)$, which is not in V_ℓ . Suppose x is disjoint from $\{1, \dots, k - \ell\}$. Then x has $\binom{n-2k-\ell}{\ell} = h/2$ neighbors in S_ℓ . If x has nonempty intersection with $\{1, \dots, k - \ell\}$, then x is nonadjacent to all vertices of S_ℓ . Therefore V_ℓ is a switching set, and $K'(n, k)$ is cospectral with $K(n, k)$ by Proposition 1.1.

To see that these graphs are nonisomorphic, first observe that Equation 1 implies that

$$\frac{n - k + i}{n - 2k + i} \leq 2 \text{ for } i = 1, \dots, \ell.$$

In particular $n - k + 1 \leq 2(n - 2k + 1)$, which yields $n \geq 3k - 1$. This implies that for the considered values of n and k , the Kneser graph $K(n, k)$ has diameter 2. Consider the vertices $v = \{1, \dots, k\}$ and $u = \{1, \dots, k - \ell - 1, k - \ell + 1, \dots, k + 1\}$. Then $v \in V_\ell$ and $u \notin V_\ell$. Since $\ell < k - 1$, u is adjacent to no vertex of V_ℓ , so u and v are nonadjacent in $K(n, k)$ and in $K'(n, k)$. We claim that there is no vertex adjacent to both u and v . Suppose not, and let x be a common neighbor of u and v . Then $x \notin V_\ell$ (since V_ℓ is a coclique). If $\ell \in x$, x is adjacent to no vertex of V_ℓ which will remain so after switching, contradiction. If $\ell \notin x$, then $x \subset \{k + 2, \dots, n\}$ because x is adjacent to u . But then x is adjacent to v before switching, but becomes nonadjacent after switching. This proves our claim. Therefore $K'(n, k)$ has diameter at least 3, and hence is nonisomorphic to $K(n, k)$. \square

If $\ell = 1$ then we easily have that Equation 1 has a solution whenever $n = 3k - 1$. Note that for $K(5, 2)$ (the Petersen graph), the set V_1 is a Godsil-McKay switching set, but $\ell = k - 1$, and $K'(5, 2)$ is isomorphic with $K(5, 2)$.

If $\ell = 2$ we need integral solutions of the equation: $2n = 6k - 3 + \sqrt{8k^2 + 1}$. So we want values of $k > 1$ for which $8k^2 + 1$ is a square. There are infinitely many such values, being the solutions of the second order recurrence relation: $k_t = 6k_{t-1} - k_{t-2}$ with $k_0 = 0$ and $k_1 = 1$. An explicit formula is $k = \lfloor (3 + 2\sqrt{2})^i / 4\sqrt{2} \rfloor$, for integer $i \geq 2$. The smallest solutions with $k > 1$ are $(n, k) = (25, 6)$, $(153, 35)$, $(899, 204)$ (see [11]).

For $\ell \geq 3$ we don't know of any solution. We checked by computer $\ell = 3$ and $n \leq 10000$. After that, Blokhuis and De Weger [1] proved that there is no solution with $\ell = 3$. It was conjectured by Erdős [5] that for given $\ell \geq 3$ there are only finitely many solutions to Equation 1.

We can conclude that the Kneser graph $K(n, k)$ is not determined by its spectrum if $n = 8$, $k = 2$, if $n = 3k - 1$, $k \geq 3$, and if

$$n = \frac{6k - 3 + \sqrt{8k^2 + 1}}{2}, \quad k = \left\lfloor \frac{(3 + 2\sqrt{2})^i}{4\sqrt{2}} \right\rfloor, \quad i = 2, 3, \dots$$

We mentioned that for $k \leq 2$, $(n, k) \neq (8, 2)$ and for $k \geq \lfloor n/2 \rfloor$ the Kneser graphs are determined by their spectrum. For all other values we don't know the answer. The smallest open case is $(n, k) = (9, 3)$. We looked by computer

for switching sets of order 4 and 6 in $K(9, 3)$ and $K(10, 3)$. There were no such switching sets. However we found switching sets for the graph $J_{\{0,2\}}(9, 3) = K_2(9, 3)$. One is the affine plane of order 3. Another one, which generalizes to arbitrary n and $k \geq 3$ will be discussed in the next section.

3 Modulo-2 Kneser graphs

In this section we use Godsil-McKay switching in the Modulo-2 Kneser graphs $K_2(n, k)$, to construct cospectral mates. Assume $3 \leq k \leq n - 3$, define $S = \{1, \dots, 6\}$, and $R = \{7, \dots, k + 3\}$. Then the following subsets of $\{1, \dots, n\}$ are vertices of $K_2(n, k)$.

$$\begin{aligned} v_1 &= \{1, 2, 3\} \cup R, \\ v_2 &= \{1, 5, 6\} \cup R, \\ v_3 &= \{2, 4, 6\} \cup R, \\ v_4 &= \{3, 4, 5\} \cup R, \\ v_5 &= \{4, 5, 6\} \cup R. \end{aligned}$$

Proposition 3.1 *The set $V = \{v_1, v_2, v_3, v_4\}$ is a switching set of $K_2(n, k)$.*

Proof. Any two vertices from V intersect in $k - 2$ elements, therefore the subgraph of $K_2(n, k)$ induced by V is a clique or a coclique, and therefore regular. Consider an arbitrary vertex $x \in V(n, k) \setminus V$. Every element from $\{1, \dots, k + 3\}$ occurs an even number of times in a subset of V , therefore $\sum_{v \in V} |v \cap x|$ is even. So x cannot have odd intersection with an odd number of vertices from V . Hence x has 0, 2 or 4 neighbors in V , and therefore V is a switching set. \square

Let G' be the graph obtained from $G = K_2(n, k)$ by switching with respect to V . The hard part of this section is to show that G and G' are nonisomorphic. To achieve this we need a couple of lemmas and definitions. For two vertices x and y of a graph G , the number of common neighbors of x and y will be denoted by $\lambda_G(x, y)$. The *common neighbor pattern* of a vertex x of G is the multi-set of all possible values of $\lambda_G(x, y)$, where y runs through the vertex set of G .

Lemma 3.2 *If G and G' are isomorphic, then $\lambda_G(v_1, v_5) = \lambda_{G'}(v_1, v_5)$.*

Proof. Clearly all vertices in G have the same common neighbor pattern. If G and G' are isomorphic then also all vertices of G' have this pattern. In particular, v_5 has the same common neighbor pattern before and after switching. From the switching operation it easily follows that $\lambda_G(v_5, v_i) = \lambda_{G'}(v_5, v_i)$ for $i \geq 5$. Thus

$$\begin{aligned} \{\lambda_G(v_5, v_1), \lambda_G(v_5, v_2), \lambda_G(v_5, v_3), \lambda_G(v_5, v_4)\} = \\ \{\lambda_{G'}(v_5, v_1), \lambda_{G'}(v_5, v_2), \lambda_{G'}(v_5, v_3), \lambda_{G'}(v_5, v_4)\}. \end{aligned}$$

Moreover, the permutation $(1, 2, 3)(4, 5, 6)$ of $\{1, \dots, n\}$ induces an automorphism of G that fixes v_1 and v_5 , and cyclicly shifts (v_2, v_3, v_4) . Therefore

$\lambda_G(v_5, v_2) = \lambda_G(v_5, v_3) = \lambda_G(v_5, v_4)$. This automorphism remains an automorphism after switching, and thus $\lambda_{G'}(v_5, v_2) = \lambda_{G'}(v_5, v_3) = \lambda_{G'}(v_5, v_4)$. Therefore $\lambda_G(v_5, v_1) = \lambda_{G'}(v_5, v_1)$. \square

Lemma 3.3 *If $\lambda_G(v_1, v_5) = \lambda_{G'}(v_1, v_5)$, then*

$$(n, k) = (7, 3), (7, 4), (10, 3), (10, 7), (8, 4), (12, 4), \text{ or } (12, 8).$$

Proof. First we compute

$$\Delta = \lambda_G(v_1, v_5) - \lambda_{G'}(v_1, v_5).$$

If a vertex x of G is adjacent to all or no vertices of V , it will remain so after switching, so x does not contribute to Δ . In particular, if $|x \cap S| = 0$ or 6 , x does not contribute to Δ .

There are $6\binom{n-6}{k-1}$ vertices x with $|x \cap S| = 1$, half of which is adjacent to v_5 and not adjacent to v_1 , whilst none is adjacent to both. Therefore these vertices contribute with $-3\binom{n-6}{k-1}$ to Δ . Similarly, the $6\binom{n-6}{k-5}$ vertices x with $|x \cap S| = 5$, contribute to Δ with $-3\binom{n-6}{k-5}$.

There are $15\binom{n-6}{k-2}$ vertices x with $|x \cap S| = 2$. Out of these the ones whose intersection with S is $\{1, 4\}$, $\{2, 5\}$, or $\{3, 6\}$ are adjacent to all or no vertices of V . So $12\binom{n-6}{k-2}$ are adjacent to two vertices of V , and exactly half of these are adjacent to v_5 and v_1 , and none is adjacent to v_5 and v_1 after switching. So in this case the contribution to Δ is $6\binom{n-6}{k-2}$. Similarly, vertices x with $|x \cap S| = 4$ contribute with $6\binom{n-6}{k-4}$.

If $x \cap S$ is one of the following $\{1, 2, 3\}$, $\{4, 5, 6\}$, $\{1, 5, 6\}$, $\{2, 3, 4\}$, $\{2, 4, 6\}$, $\{1, 3, 5\}$, $\{3, 4, 5\}$, $\{1, 2, 6\}$, then x is adjacent to all or no vertices of V . Of the remaining $12\binom{n-6}{k-3}$ vertices x with $|x \cap S| = 3$, half is adjacent to v_5 , but none is adjacent to both v_5 and v_1 . So in this case the contribution to Δ is $-6\binom{n-6}{k-3}$. Thus we have

$$\Delta = -3\binom{n-6}{k-1} + 6\binom{n-6}{k-2} - 6\binom{n-6}{k-3} + 6\binom{n-6}{k-4} - 3\binom{n-6}{k-5}.$$

Assume that $\Delta = 0$. By use of straightforward computations we get:

$$\begin{aligned} & (n-k-1)(n-k-2)(n-k-3)(n-k-4) \\ & -2(n-k-1)(n-k-2)(n-k-3)(k-1) \\ & +2(n-k-1)(n-k-2)(k-1)(k-2) \\ & -2(n-k-1)(k-1)(k-2)(k-3) \\ & +(k-1)(k-2)(k-3)(k-4) = 0. \end{aligned}$$

Defining $x = n - 2k$ leads to

$$x^4 + x^2(n^2 - 15n + 40) - n^3 + 14n^2 - 56n + 64 = 0,$$

a quadratic equation in x^2 . We are only interested in integral solution, in which case the discriminant

$$(n^2 - 15n + 40)^2 + 4(n^3 - 14n^2 + 56n - 64) = (n^2 - 13n + 40)^2 + 64n - 256$$

is the square of an integer. Since this number is even, it must be at least $(n^2 - 13n + 40)^2$, therefore $64n - 256 \geq 2(2n^2 - 26n + 82)$, hence $n \leq 24$. For all integer values of n and k with $n \leq 24$ and $3 \leq k \leq n - 3$ we computed Δ , and found that $\Delta = 0$ only if (n, k) is one of the seven mentioned cases. \square

Of the remaining seven cases, $K_2(7, 3)$ is the complement of $K_2(7, 4)$, and $K_2(10, 3)$ and $K_2(12, 4)$ are isomorphic to $K_2(10, 7)$ and $K_2(12, 8)$, respectively. So only the four cases $(n, k) = (7, 3)$, $(10, 3)$, $(8, 4)$ and $(12, 4)$ need to be examined. For these four graphs we have computed the spectra of the neighbor graphs of v_5 before and after switching. In all four cases switching changed the spectrum, and therefore the graphs are nonisomorphic. Thus we have:

Theorem 3.4 *For $3 \leq k \leq n - 3$, the Modulo-2 Kneser graph $K_2(n, k)$ is not determined by its spectrum.*

The graphs $K_2(7, 3)$, $K_2(7, 4)$, $K_2(10, 3)$, and $K_2(12, 4)$ are strongly regular. A graph G' is cospectral with a strongly regular graph G if and only if G' is strongly regular with the same parameters as G . All strongly regular graphs with the same parameters as $K_2(7, 3)$ have been generated by McKay and Spence [10]. This gives exactly 3853 cospectral mates for $K_2(7, 3)$ (and for $K_2(7, 4)$), one of these is the cospectral mate constructed above. Mathon and Street [9] have constructed several strongly regular graphs with the parameters of $K_2(10, 3)$. We don't know if our cospectral mate is among these. However, as far as we know, up till now only one strongly regular graph with the parameters of $K_2(12, 4)$ has been constructed. So the cospectral mate of $K_2(12, 4)$ seems to be a new strongly regular graph with parameters $(495, 238, 109, 119)$.

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