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Externalities and Compensation: Primeval Games and Solutions[1]

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Abstract

The classical literature (Pigou (1920), Coase (1960), Arrow (1970)) and the relatively recent studies (cf. Varian (1994)) associate the externality problem with efficiency. This paper focuses explicitly on the compensation problem in the context of externalities. To capture the features of inter-individual externalities, this paper constructs a new game-theoretic framework: primeval games. These games are used to design normative compensation rules for the underlying compensation problems: the marginalistic rule, the concession rule, and the primeval rule. Characterizations of the marginalistic rule and the concession rule are provided and specific properties of the primeval rule are studied.

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1 Introduction

This paper focuses on the issue of externality and the associated compensation problem. Externalities arise whenever an (economic) agent undertakes an action that has an effect on another agent. When the effect turns out to be a cost imposed on the other agent(s), it is called a negative externality. When agents benefit from an activity in which they are not directly involved, the effect is called a positive externality. An associated fundamental question in real life is how to compensate the losses incurred by the negative externalities.

Pigou (1920) suggests a solution that involves intervention by a regulator who imposes a (Pigovian) tax. An alternative solution, known as the Coase theorem (Coase (1960)), involves negotiation between the agents. Coase claims that if transactions costs are zero and property rights are well defined, agents should be able to negotiate their way to an efficient outcome. A third class of solutions, associated with Arrow (1970), involves setting up a market for the externality. If a firm produces pollution that harms another firm, then a competitive market for the right to pollute may allow for an efficient outcome. In this framework, Varian (1994) designs the so-called compensation mechanisms for internalizing externalities which encourage the firms to correctly reveal the costs they impose on the other.

In fact, all solutions and approaches above try and solve the inefficiency problems arising from externalities, whereas they cannot be viewed as normative answers in terms of fairness. In particular, the theories cannot answer a basic question like how much a household should be compensated by a polluting firm. Therefore, we are still in search of basic normative solutions which might serve as benchmarks to determine adequate compensations in environments that are featured by externalities.

Solving an externality-incurred compensation problem boils down to recommending rules or solutions for profit/cost sharing problems with externalities. A first model to solve this problem was developed by Thrall and Lucas (1963) by the concept of *partition function form games*: a partition function assigns a value to each pair consisting of a coalition and a coalition structure which includes that coalition. Solution concepts for such games can be found in Myerson (1977), Bolger (1986), Feldman (1994), Potter (2000), Pham Do and Norde (2002), Maskin (2003), Macho-Stadler, Pérez-Castrillo, and Wettstein (2004), and Ju (2004a).

However, one may observe that the framework of partition function form games does not model the externalities among individuals but restrict to specific coalitional effects. The reason is simple: Partition function form games as well as cooperative games with transferable utility (TU games) in characteristic function form always assume all the players in the player set N are present even if they do not form a coalition. Consider a partition function form game and a player i in this game. What we know about the values with respect to i has the following three cases only: complete breakdown, i.e. all the players in this game do not cooperate with each other; partial cooperation, i.e. i participates in some coalition or i stands alone while some other players cooperate; complete cooperation, i.e. all the players form a grand coalition. In fact, the externalities among individual players (inter-individual externalities) are "internalized" or "incorporated" from the very beginning because there is no explicit distinction between the case when only one player is in the game and the case when all appear.

The task attempted in this paper is essentially twofold. First, it takes players' initial situations (no other players, in an absolute stand-alone sense) into account and constructs a new class of games, primeval games, which model the externalities among individual players. Second, it discusses several compensation rules which can actually serve as specific benchmarks to solve the compensation issue related to externality problems.

Primeval games have a flavor of TU games but are more like partition function form games in structure. Two basic differences with respect to the classical cooperative games are that primeval games do not consider cooperation, and primeval games take into account all situations in which only a subgroup of players is present. In this way, all possible externalities among players are modelled.

We introduce three compensation rules for primeval games: the marginalistic rule, a modification of the Shapley value for TU games (Shapley (1953)), the concession rule, which is in the same spirit as the consensus value for TU games (Ju, Borm and Ruys (2004)), and a more context-specific compensation rule, the primeval rule. The first two solution concepts are axiomatically characterized. Properties of the primeval rule are discussed.

In addition to this section introducing the paper briefly, the remaining part has the following structure. The next section presents a small example that motivates the approach and the model. In section 3, we lay out the general model: primeval games. Section 4 defines three solution concepts for primeval games. Section 5 introduces unanimity games for the class of primeval games, which facilitates the characterizations of the marginalistic rule and the concession rule that are provided in the next section. Specific properties of the primeval rule are studied and a comparison with the marginalistic rule and the concession rule is provided for specific types of players in the final section.

2 An example: a village with three households

Consider three households, a, b, and c, living in the same village; or more generally, three (economic) agents in a certain interactive environment.

If all three households simultaneously live in the village, the utilities of a, b, and c are given by 8, 2 and 2, respectively.

It is quite common that one household may generate positive or negative externalities to the others. For instance, a's utility may not only dependent on a himself, but may also depend on the activities of b and c. That is, the realization of 8 is the outcome of every household's activities in the current structure. It could be higher or lower if some other household were absent or would stop any possible activities that may generate externalities to a.

Therefore, it is necessary and interesting to go "back" to see the "primeval" situations of the current structure.

In the case that only household b lives in the village, or equivalently, in the case that b comes into this village first while a and c are not present the utility for b would be 3 instead of 2. This case is described by the second column in the following table. E.g. the fifth column represents the fact that when both a and c live in the village while b is not present, a and c's utilities would be 5 and 1, respectively. All the other possible cases are provided as well.

(a)	(b)	(c)	(a,b)	(a,c)	(b,c)	(a,b,c)
(5)	(3)	(2)	$(8,2)$	$(5,1)$	$(3,0)$	$(8,2,2)$

Here, from the externality point of view, one can easily see that a benefits while b loses. Natural questions are: Should b be compensated? If so, by whom and how much?

3 The model: primeval games

To capture all the possibilities of inter-individual externalities and further discuss the associated compensation problem, we now construct the formal model of primeval games.

Let $N = \{1, 2, ..., n\}$ be the finite set of players. A subset S of N, in order to be distinguished from the usual concept of *coalition* in cooperative games, is called a *group of individuals* (in short, a group S). Here, the term of group should be understood as a neutral concept, which has nothing to do with cooperation or anything else, but simply means a set of individual players in N.

A pair (i, S) that consists of a player i and a group S of N to which i belongs is called an *embedded player* in S. Let $\mathcal{E}(N)$ denote the set of embedded players, i.e.

$$\mathcal{E}(N) = \left\{(i, S) \in N \times 2^N | i \in S\right\}.$$

Definition 3.1 *A mapping*

$$u : \mathcal{E}(N) \longrightarrow \mathbb{R}$$

that assigns a real value $u(i, S)$ *to each embedded player* (i, S) *is an* individual-group function. *The ordered pair* (N, u) *is called a* primeval game[1]. *The set of primeval games with player set* N *is denoted by* PRI^N.

The value $u(i, S)$ represents the payoff, or utility, of player i, given that all players in S are present while all players in $N \backslash S$ are absent. For a given group S and an individual-group function u, let $\bar{u}(S)$ denote the vector $(u(i, S))_{i \in S}$. We call $\bar{u}(N)$ the *status quo* of a primeval game u, and $u(i, \{i\})$ the absolute stand-alone payoff, or the *Rubinson Crusoe payoff* (in short, R-C payoff) of player i in game u.

Definition 3.2 *A (compensation) rule on* PRI^N *is a function* f, *which associates with each primeval game* (N, u) *in* PRI^N *a vector* $f(N, u) = (f_i(N, u))_{i \in N} \in \mathbb{R}^N$ *of individual payoffs.*

4 Compensation rules

This section introduces several compensation rules for primeval games. Since it is assumed that for any primeval game every player has the same right to enter it, there is no predetermined ordering of players. However, we need to take orders into account because they help to clearify the relationship among players with respect to externalities. Therefore, we consider all different orderings of players when determining compensations in the context of externality.

4.1 The marginalistic rule

People generally believe that one should not do harm to the others, and otherwise, one must provide compensation. Analogously, if a player's activities impose a positive effect on the others, then he has the right to ask them to pay for that. Meanwhile we might adopt a practical principle known as *first come, first served.* That is, the player who comes into a game first should be well protected: Any later entrant must compensate him if she causes loss on him while he need not worry about any possible negative effects he could impose on the later entrants, i.e., he has the right to assume no responsibility for his

[1]Since a primeval game models inter-individual externalities and aims to solve the associated compensation problem, an alternative name would be *individual externality-compensation game.*

behavior, irrespective of what consequence it might cause on the others. Along the same line of reasoning, the second entrant only cares about the first player but does not have any responsibility for his successors whereas all his successors should take care of the first two entrants' payoffs. More specifically, given an ordering of players, the early entrants should be well protected such that the losses due to negative externalities that possibly arise later are compensated. Also, the gains from positive externalities should be transferred to whom they are produced by. Those effects can be well captured by the so-called marginal values. Thus, the corresponding rule is in fact a completely marginal treatment of externalities.

The formal definition is provided as follows. For a primeval game $u \in PRI^N$, let $\Pi(N)$ be the set of all bijections $\sigma : \{1, 2, ..., |N|\} \longrightarrow N$. For a given $\sigma \in \Pi(N)$ and $k \in \{1, 2, ..., |N|\}$ we define $S_k^\sigma = \{\sigma(1), \sigma(2), ..., \sigma(k)\}$ and $S_0^\sigma = \emptyset$.

We construct the marginal vector $m^\sigma(u)$, which corresponds to the situation where the players enter the game one by one in the order $\sigma(1), \sigma(2), ..., \sigma(|N|)$ and where each player $\sigma(k)$ is given the marginal value he creates by entering. Formally, it is the vector in $\mathbb{R}^N$ defined by

$$m^\sigma_{\sigma(k)}(u) = \begin{cases} u(\sigma(1), \{\sigma(1)\}) & \text{if } k = 1 \\ \\ u(\sigma(k), S_k^\sigma) + \sum_{j=1}^{k-1} \left(u(\sigma(j), S_k^\sigma) - u(\sigma(j), S_{k-1}^\sigma) \right) & \text{if } k \in \{2, ..., |N|\}. \end{cases}$$

Therefore, player $\sigma(k)$ might be involved in four kinds of compensating behavior or circumstances: compensating the incumbents if he produces negative externalities on them, being compensated from the incumbents if they benefit from his showing up (i.e., he produces positive externalities on the incumbents), being compensated by the later entrants if they impose negative externalities on him; paying compensation to the later entrants if they generate positive externalities on him.

Here, one can readily check that for a primeval game $u \in PRI^N$ and an order $\sigma \in \Pi(N)$,

$$\sum_{k=1}^{t} m^\sigma_{\sigma(k)}(u) = \sum_{k=1}^{t} u(\sigma(k), S_t^\sigma)$$

for all $t \in \{1, ..., |N|\}$.

Furthermore, since no predetermined ordering of players exists, we take all possible permutations into consideration. Thus, the *marginalistic rule* $\Phi(u)$ is defined as the average of the marginal vectors, i.e.,

$$\Phi(u) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m^\sigma(u).$$

Note that the marginalitic rule for primeval games is in the same spirit of the Shapley value for TU games.

Example 4.1 *Consider the three-household village example.*

With $\sigma : \{1,2,3\} \longrightarrow N$ *defined by* $\sigma(1) = b$, $\sigma(2) = a$ *and* $\sigma(3) = c$, *which is shortly denoted by* $\sigma = (b\ a\ c)$, *we get*

$$\begin{aligned} m_b^\sigma(u)(= m_{\sigma(1)}^\sigma(u)) = &\quad u(b,\{b\}) = 3, \\ m_a^\sigma(u)(= m_{\sigma(2)}^\sigma(u)) = &\quad u(a,\{a,b\}) + u(b,\{a,b\}) - u(b,\{b\}) = 8 + 2 - 3 = 7, \\ m_c^\sigma(u)(= m_{\sigma(3)}^\sigma(u)) = &\quad u(c,\{a,b,c\}) + u(b,\{a,b,c\}) + u(a,\{a,b,c\}) \\ - &\quad u(b,\{a,b\}) - u(a,\{a,b\}) \\ = &\quad 2 + 2 + 8 - 2 - 8 = 2. \end{aligned}$$

Similarly, all marginal vectors are given by

σ	$m_a^\sigma(u)$	$m_b^\sigma(u)$	$m_c^\sigma(u)$
$(a\ b\ c)$	5	5	2
$(a\ c\ b)$	5	6	1
$(b\ a\ c)$	7	3	2
$(b\ c\ a)$	9	3	0
$(c\ a\ b)$	4	6	2
$(c\ b\ a)$	9	1	2

Then, we get $\Phi(u) = \left(6\frac{1}{2}, 4, 1\frac{1}{2}\right)$. *Thus, to compensate for externalities,* a *needs to pay* $1\frac{1}{2}$ *to* b, *and* c *will pay* $\frac{1}{2}$ *to* b.

4.2 The concession rule

One might oppose the “first come, first served” idea and rather prefer an equal responsibility based rule: From the bilateral point of view, both parties (the incumbents and the entrant) should be equally responsible for an externality due to the showing up of the new entrant. Take the village example. Suppose player b comes first and player a follows. In this case we observe that b is negatively affected. One may argue that not only a but also b should account for the loss of 1 because it is the outcome of the joint effect between a's activities and b's feelings. An alternative argument could be that the households have the rights of living in the village and equally enjoy the rights to produce externalities, irrespective of the timing about entering the village. Then, a 50-50 rule seems suitable. This point of view is reflected in the definition of the concession rule below.

In order to define the concession rule for primeval games, we construct the concession vector $C^\sigma(u)$, which corresponds to the situation where players enter the game u one by one in an order $\sigma \in \Pi(N)$ and where every new entrant, say $\sigma(k)$, first obtains the payoff when entering, $u(\sigma(k), S_k^\sigma)$, and then equally shares with every incumbent her surplus/loss incurred by the corresponding positive/negative externality imposed by him, and also equally shares his surplus/loss with all his successors. The word of *concession* is used here because players concede to each other and make a compromise on assuming responsibilities of the externalities.

We first define player $\sigma(k)$'s *concession payoff for the externalities on previous players* as

$$\mathcal{P}_{\sigma(k)}^{\sigma}(u) = \sum_{j=1}^{k-1} \frac{u(\sigma(j), S_k^\sigma) - u(\sigma(j), S_{k-1}^\sigma)}{2}$$

and his *concession payoff from the subsequent externalities* as

$$\mathcal{S}_{\sigma(k)}^{\sigma}(u) = \sum_{l=k+1}^{|N|} \frac{u(\sigma(k), S_l^\sigma) - u(\sigma(k), S_{l-1}^\sigma)}{2}.$$

Apparently, when a player enters the game u in the very first place, he has no concession payoff for the externalities on previous players. Therefore $\mathcal{P}_{\sigma(1)}^{\sigma}(u) = 0$. Correspondingly, when a player enters a game in the very last place, there is no subsequent externality for him. Hence, $\mathcal{S}_{\sigma(|N|)}^{\sigma}(u) = 0$.

Moreover, the concession payoff from the subsequent externalities for player $\sigma(k)$ can be simplified as

$$\mathcal{S}_{\sigma(k)}^{\sigma}(u) = \frac{u(\sigma(k), N) - u(\sigma(k), S_k^\sigma)}{2}$$

for all $k = \{1, ..., |N| - 1\}$.

Now, formally, the concession vector is the vector in $\mathbb{R}^N$ defined by

$$C_{\sigma(k)}^{\sigma}(u) = \begin{cases} u(\sigma(1), \{\sigma(1)\}) + \mathcal{S}_{\sigma(1)}^{\sigma}(u) & \text{if } k = 1 \\ \\ u(\sigma(k), S_k^\sigma) + \mathcal{P}_{\sigma(k)}^{\sigma}(u) + \mathcal{S}_{\sigma(k)}^{\sigma}(u) & \text{if } k = \{2, ..., |N| - 1\} \\ \\ u(\sigma(|N|), N) + \mathcal{P}_{\sigma(|N|)}^{\sigma}(u) & \text{if } k = |N|. \end{cases}$$

And more explicitly,

$$C_{\sigma(k)}^{\sigma}(u) = \begin{cases} \frac{u(\sigma(1), N) + u(\sigma(1), \{\sigma(1)\})}{2} & \text{if } k = 1 \\ \\ \mathcal{P}_{\sigma(k)}^{\sigma}(u) + \frac{u(\sigma(k), N) + u(\sigma(k), S_k^\sigma)}{2} & \text{if } k = \{2, ..., |N| - 1\} \\ \\ u(\sigma(|N|), N) + \mathcal{P}_{\sigma(|N|)}^{\sigma}(u) & \text{if } k = |N|. \end{cases}$$

We want to note that for a primeval game $u \in PRI^N$ and an order $\sigma \in \Pi(N)$,

$$\sum_{k=1}^{|N|} C^{\sigma}_{\sigma(k)}(u) = \sum_{k=1}^{|N|} u(\sigma(k), N),$$

but generally,

$$\sum_{k=1}^{t} C^{\sigma}_{\sigma(k)}(u) \neq \sum_{k=1}^{t} u(\sigma(k), S^{\sigma}_t)$$

for $t \in \{1, ..., |N| - 1\}$.

The *concession rule* $\mathcal{C}(u)$ is defined as the average of the concession vectors, i.e.,

$$\mathcal{C}(u) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} C^{\sigma}(u).$$

Note that the concession rule for primeval games is in the same spirit as the consensus value for TU games (cf. Ju, Borm and Ruys (2004)).

Example 4.2 *Consider the three-household village example.*

With $\sigma = (b\ a\ c)$, which is a shorthand notation as in Example 4.1, we get

$$\begin{aligned}
C^{\sigma}_b(u) &= \frac{u(b,\{b,a,c\}) + u(b,\{b\})}{2} = \frac{2+3}{2} = 2\frac{1}{2}, \\
C^{\sigma}_a(u) &= \frac{u(b,\{b,a\}) - u(b,\{b\})}{2} + \frac{u(a,\{b,a,c\}) + u(a,\{b,a\})}{2} \\
&= \frac{2-3}{2} + \frac{8+8}{2} = 7\frac{1}{2}, \\
C^{\sigma}_c(u) &= u(c,\{b,a,c\}) + \frac{(u(b,\{b,a,c\}) - u(b,\{b,a\})) + (u(a,\{b,a,c\}) - u(a,\{b,a\}))}{2} \\
&= 2.
\end{aligned}$$

Similarly, all concession vectors are given by

σ	$C^{\sigma}_a(u)$	$C^{\sigma}_b(u)$	$C^{\sigma}_c(u)$
$(a\ b\ c)$	$6\frac{1}{2}$	$3\frac{1}{2}$	2
$(a\ c\ b)$	$6\frac{1}{2}$	4	$1\frac{1}{2}$
$(b\ a\ c)$	$7\frac{1}{2}$	$2\frac{1}{2}$	2
$(b\ c\ a)$	$8\frac{1}{2}$	$2\frac{1}{2}$	1
$(c\ a\ b)$	6	4	2
$(c\ b\ a)$	$8\frac{1}{2}$	$1\frac{1}{2}$	2

Then, we get $\mathcal{C}(u) = \left(7\frac{1}{4}, 3, 1\frac{3}{4}\right)$. Thus, to compensate for externalities, a needs to pay $\frac{3}{4}$ to b, and c will pay $\frac{1}{4}$ to b. Compared to the outcome of the marginalistic rule, both a and c give less compensation to b.

Theorem 4.3 *The outcome prescribed by the concession rule turns out to be the average of the status quo payoff vector and the outcome of the marginalitic rule. For any game $u \in PRI^N$, we have*

$$\mathcal{C}_i(u) = \frac{1}{2}u(i,N) + \frac{1}{2}\Phi_i(u)$$

for all $i \in N$.

Proof. Given a game $u \in PRI^N$ and $\sigma \in \Pi(N)$, let $i = \sigma(k)$, where $k \in \{1, 2, ...|N|\}$.

By definition, we know for $k \in \{2, ..., |N| - 1\}$

$$\begin{aligned} C_i^\sigma(u) = \quad & C_{\sigma(k)}^\sigma(u) \\ = \quad & \frac{1}{2}\left(u(i,N) + u(i,S_k^\sigma)\right) + \frac{1}{2}\sum_{j=1}^{k-1}\left(u(\sigma(j),S_k^\sigma) - u(\sigma(j),S_{k-1}^\sigma)\right) \\ = \quad & \frac{1}{2}(u(i,N) + \frac{1}{2}\left(u(i,S_k^\sigma) + \sum_{j=1}^{k-1}\left(u(\sigma(j),S_k^\sigma) - u(\sigma(j),S_{k-1}^\sigma)\right)\right) \\ = \quad & \frac{1}{2}u(i,N) + \frac{1}{2}m_{\sigma(k)}^\sigma(u). \end{aligned}$$

Moreover, this equality is obvious for the cases that $k = 1$ or $|N|$. Hence, $\mathcal{C}_i(u) = \frac{1}{2}u(i,N) + \frac{1}{2}\Phi_i(u)$. ■

4.3 The primeval rule

We now propose an alternative rule, the basic idea of which is that the losses due to negative externalities should be compensated whereas the benefits from the positive externalities are enjoyed for free. This somehow is a general and natural attitude when people face externalities in reality. Thus, the rule based on this idea might be easy to be accepted and implemented in practice.

The corresponding rule could be described as the *chargeable negative externalities and free positive externalities rule*. For shorthand we call it the *primeval rule.*

For a primeval game $u \in PRI^N$ and an ordering $\sigma \in \Pi(N)$ and $k \in \{1, 2, ..., |N|\}$, we construct the *primeval vector* $B^\sigma(u)$, which corresponds to the situation where the players enter the game one by one in the order $\sigma(1), \sigma(2), ..., \sigma(|N|)$ and where each player $\sigma(k)$ compensates the losses of his predecessors but enjoys positive externalities from his successors freely.

We now define player $\sigma(k)$'s *loss for compensating negative externalities* as

$$L_{\sigma(k)}^\sigma(u) = \sum_{j=1}^{k-1} \max\left\{u(\sigma(j),S_{k-1}^\sigma) - u(\sigma(j),S_k^\sigma), 0\right\}$$

and his *gain from subsequent positive externalities* as

$$G^{\sigma}_{\sigma(k)}(u) = \sum_{l=k+1}^{|N|} \max\left\{u(\sigma(k), S^{\sigma}_{l}) - u(\sigma(k), S^{\sigma}_{l-1}), 0\right\}.$$

Apparently, when a player enters the game u in the very first place, he assumes no responsibility for the others. Therefore, $L^{\sigma}_{\sigma(1)}(u) = 0$. Similarly, when a player enters a game in the very last place, he cannot enjoy any subsequent positive externalities. Hence, $G^{\sigma}_{\sigma(|N|)}(u) = 0$.

Formally, the primeval vector $B^{\sigma}(u)$ is the vector in $\mathbb{R}^N$ defined by

$$B^{\sigma}_{\sigma(k)}(u) = \begin{cases} u(\sigma(1), \{\sigma(1)\}) + G^{\sigma}_{\sigma(1)}(u) & \text{if } k = 1 \\ \\ u(\sigma(k), S^{\sigma}_{k}) - L^{\sigma}_{\sigma(k)}(u) + G^{\sigma}_{\sigma(k)}(u) & \text{if } k \in \{2, ..., |N| - 1\} \\ \\ u(\sigma(|N|), N) - L^{\sigma}_{\sigma(|N|)}(u) & \text{if } k = |N|. \end{cases}$$

Similar to the concession rule, here one can check that for a primeval game $u \in PRI^N$ and an order $\sigma \in \Pi(N)$,

$$\sum_{k=1}^{|N|} B^{\sigma}_{\sigma(k)}(u) = \sum_{k=1}^{|N|} u(\sigma(k), N),$$

but generally,

$$\sum_{k=1}^{t} B^{\sigma}_{\sigma(k)}(u) \neq \sum_{k=1}^{t} u(\sigma(k), S^{\sigma}_{t})$$

for $t \in \{1, ..., |N| - 1\}$.

The *primeval rule* $\zeta(u)$ is defined as the average of the primeval vectors, i.e.,

$$\zeta(u) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} B^{\sigma}(u).$$

Example 4.4 *Consider the three-household village example.*

With $\sigma = (b\ a\ c)$, we get

$$\begin{aligned} B^{\sigma}_{b}(u) &= u(b, \{b\}) = 3, \\ B^{\sigma}_{a}(u) &= u(a, \{b, a\}) - (u(b, \{b\}) - u(b, \{b, a\}) = 8 - (3 - 2) = 7, \\ B^{\sigma}_{c}(u) &= u(c, \{b, a, c\}) = 2. \end{aligned}$$

Similarly, all primeval vectors are given by

σ	$B_a^\sigma(u)$	$B_b^\sigma(u)$	$B_c^\sigma(u)$
$(a\ b\ c)$	8	2	2
$(a\ c\ b)$	8	2	2
$(b\ a\ c)$	7	3	2
$(b\ c\ a)$	7	3	2
$(c\ a\ b)$	7	2	3
$(c\ b\ a)$	7	1	4

Then, we get $\zeta(u) = \left(7\frac{1}{3}, 2\frac{1}{6}, 2\frac{1}{2}\right)$. *Thus, to compensate for externalities, a needs to pay* $\frac{1}{6}$ *to b and* $\frac{1}{2}$ *to c. Note that in this case c even becomes a compensation receiver instead of a provider like in the previous two cases. This is due to the underlying assumption that all positive externalities are for free.*

5 Unanimity games

This section introduces unanimity games for the class of primeval games as a generalization of unanimity games for the class of TU games. Every primeval game can be written in a unique way as a linear combination of unanimity games.

Recall that the unanimity games $\{(N, u_T) | T \in 2^N \backslash \{\emptyset\}\}$ form a basis for the class of all TU games with player set N. Below we will define *unanimity games* for primeval games.

Definition 5.1 *Let* $(j,T) \in \mathcal{E}(N)$ *be an embedded player. The* unanimity game $w_{(j,T)}$, *corresponding to* (j,T), *is given by*

$$w_{(j,T)}(i,S) = \begin{cases} 1, & \text{if } j = i \text{ and } T \subset S \\ \\ 0, & \text{otherwise} \end{cases}$$

for every $(i,S) \in \mathcal{E}(N)$.

Example 5.2 *Consider the three-household village game and denote the player set with* $N = \{1,2,3\}$ *instead of* $\{a,b,c\}$. *The following table gives the values of* $w_{(j,T)}(i,S)$ *for all embedded players* (j,T) *and* (i,S).

For saving space, we use the following notations. $\tau_1 = (1,\{1\})$, $\tau_2 = (2,\{2\})$, $\tau_3 = (3,\{3\})$, $\tau_4 = (1,\{1,2\})$, $\tau_5 = (2,\{1,2\})$, $\tau_6 = (1,\{1,3\})$, $\tau_7 = (3,\{1,3\})$, $\tau_8 = (2,\{2,3\})$, $\tau_9 = (3,\{2,3\})$, $\tau_{10} = (1,\{1,2,3\})$, $\tau_{11} = (2,\{1,2,3\})$, $\tau_{12} = (3,\{1,2,3\})$.

$(i,S)\backslash(j,T)$	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	τ_8	τ_9	τ_{10}	τ_{11}	τ_{12}
τ_1	1	0	0	0	0	0	0	0	0	0	0	0
τ_2	0	1	0	0	0	0	0	0	0	0	0	0
τ_3	0	0	1	0	0	0	0	0	0	0	0	0
τ_4	1	0	0	1	0	0	0	0	0	0	0	0
τ_5	0	1	0	0	1	0	0	0	0	0	0	0
τ_6	1	0	0	0	0	1	0	0	0	0	0	0
τ_7	0	0	1	0	0	0	1	0	0	0	0	0
τ_8	0	1	0	0	0	0	0	1	0	0	0	0
τ_9	0	0	1	0	0	0	0	0	1	0	0	0
τ_{10}	1	0	0	1	0	1	0	0	0	1	0	0
τ_{11}	0	1	0	0	1	0	0	1	0	0	1	0
τ_{12}	0	0	1	0	0	0	1	0	1	0	0	1

One can prove, similar to the case of TU games, that the unanimity games form a basis for the class of primeval games (cf. Ju (2004b, p.100-101, Lemma 5.5.3)). This means that if (N,u) is a primeval game, then there exist uniquely determined real numbers $d_{(j,T)}$, $(j,T)\in\mathcal{E}(N)$, such that $u=\sum_{(j,T)\in\mathcal{E}(N)} d_{(j,T)}w_{(j,T)}$.

The following example shows the linear expansion of a primeval game (N,u) with respect to the unanimity games $w_{(j,T)}$.

Example 5.3 *Consider the primeval game (N,u) in the three-household village example. The decomposition of u is given by*

$$\begin{aligned} u = & \quad 5w_{(1,\{1\})}+3w_{(2,\{2\})}+2w_{(3,\{3\})}+3w_{(1,\{1,2\})}-w_{(2,\{1,2\})}-w_{(3,\{1,3\})}-2w_{(3,\{2,3\})} \\ + & \quad 3w_{(3,\{1,2,3\})}. \end{aligned}$$

6 Properties and characterizations

This section discusses possible properties of a compensation rule for primeval games. We then provide characterizations using those properties.

As the status quo of a primeval game always exists (in fact, it is the only situation that happens in reality), we require the efficiency (or balanced-budget) property for a compensation rule: the sum of all the players' values according to the rule equals the sum of their status quo payoffs.

- Property 1 *(Efficiency)*: $\sum_{i\in N} f_i(u) = \sum_{i\in N} u(i, N)$ for all $u \in PRI^N$.

A second property is *symmetry*. For a primeval game $u \in PRI^N$, we say that two players $i, j \in N$ are *symmetric* if for all $S \subset N\backslash\{i, j\}$,

$$u(i, S \cup \{i\}) + \sum_{k\in S} u(k, S \cup \{i\}) = u(j, S \cup \{j\}) + \sum_{k\in S} u(k, S \cup \{j\}).$$

It implies that in terms of total payoffs, the showing up of i has the same effect as that of j for any group of players without i and j.

- Property 2 *(Symmetry)*: $f_i(u) = f_j(u)$ for all $u \in PRI^N$, and for all symmetric players i, j in (N, u).

We now turn to a third property, which focuses on the externality side of a primeval game.

Given a game $u \in PRI^N$, a player $i \in N$ is called an *immune player* if $u(i, S) = u(i, \{i\})$ for all $S \subset N$ and $i \in S$. Thus, an immune player is a player who is not affected by the presence of the others.

Given a game $u \in PRI^N$, a player $i \in N$ is called an *uninfluential player* if $u(j, S \cup \{i\}) = u(j, S)$ for all $S \subset N\backslash\{i\}$ and $j \in S$. Thus, an uninfluential player is a player who never affects another player.

Given a game $u \in PRI^N$, a player $i \in N$ is called a *neutral player* if it is both an immune player and an uninfluential player in (N, u).

- Property 3 *(The neutral player property)*: $f_i(u) = u(i, \{i\})$, for all $u \in PRI^N$ and for any neutral player i in (N, u).

Given a game $u \in PRI^N$, a player $i \in N$ is called a *dummy* if

$$\sum_{j\in S} u(j, S \cup \{i\}) + u(i, S \cup \{i\}) = \sum_{j\in S} u(j, S) + u(i, \{i\})$$

for all $S \subset N\backslash\{i\}$.

- Property 4 *(The dummy property)*: $f_i(u) = u(i, \{i\})$, for all $u \in PRI^N$ and for any dummy player i in (N, u).

We now introduce the following property.

- Property 5 *(Additivity)*: $f(u_1 + u_2) = f(u_1) + f(u_2)$ for all $u_1, u_2 \in PRI^N$, where $u_1 + u_2$ is defined by $(u_1 + u_2)(i, S) = u_1(i, S) + u_2(i, S)$ for every $(i, S) \in \mathcal{E}(N)$.

Theorem 6.1 *The marginalistic rule satisfies efficiency, symmetry, the neutral player property, the dummy property and additivity.*

Proof.
(i) Efficiency: Clearly, by construction, $m^\sigma(u)$ is efficient for all $\sigma \in \Pi(N)$.
(ii) Symmetry: Let i_1, i_2 be two symmetric players in $u \in PRI^N$. Consider $\sigma \in \Pi(N)$, and without loss of generality, $\sigma(k) = i_1$, $\sigma(h) = i_2$, where $i_1, i_2 \in N$. Let $\bar{\sigma} \in \Pi(N)$ be the permutation which is obtained from σ by interchanging the positions of i_1 and i_2, i.e.

$$\bar{\sigma}(w) = \begin{cases} \sigma(w) & \text{if } w \neq k, h \\ i_1 & \text{if } w = h \\ i_2 & \text{if } w = k. \end{cases}$$

As $\sigma \mapsto \bar{\sigma}$ is bijective, it suffices to prove that $m^\sigma_{i_1}(u) = m^{\bar{\sigma}}_{i_2}(u)$.
Case 1: $1 < k < h$.
By definition, we know

$$\begin{aligned} m^\sigma_{i_1}(u) = m^\sigma_{\sigma(k)}(u) &= \sum_{l=1}^{k} u(\sigma(l), S^\sigma_k) - \sum_{j=1}^{k-1} u(\sigma(j), S^\sigma_{k-1}) \\ m^{\bar{\sigma}}_{i_2}(u) = m^{\bar{\sigma}}_{\bar{\sigma}(k)}(u) &= \sum_{l=1}^{k} u(\bar{\sigma}(l), S^{\bar{\sigma}}_k) - \sum_{j=1}^{k-1} u(\bar{\sigma}(j), S^{\bar{\sigma}}_{k-1}). \end{aligned}$$

Obviously, $u(\sigma(j), S^\sigma_{k-1}) = u(\bar{\sigma}(j), S^{\bar{\sigma}}_{k-1})$ for all $j \in \{1, ..., k-1\}$. Moreover, by symmetry, $\sum_{l=1}^{k} u(\sigma(l), S^\sigma_k) = \sum_{l=1}^{k} u(\bar{\sigma}(l), S^{\bar{\sigma}}_k)$. Therefore, $m^\sigma_{i_1}(u) = m^{\bar{\sigma}}_{i_2}(u)$.
Case 2: $1 < h < k$. The proof is analogous to the above.
Case 3: $1 = k < h$. Apparently,

$$m^\sigma_{i_1}(u) = m^\sigma_{\sigma(1)}(u) = u(i_1, \{i_1\}) = u(i_2, \{i_2\}) = m^{\bar{\sigma}}_{\bar{\sigma}(1)}(u) = m^{\bar{\sigma}}_{i_2}(u).$$

Case 4: $1 = h < k$. Analogously, the proof is easy to be established.
As a consequence, $m^\sigma_{i_1}(u) = m^{\bar{\sigma}}_{i_2}(u)$.
(iii) The neutral player property: If player i is a neutral player in (N, u) then $m^\sigma_i(u) = u(i, \{i\})$ for any $\sigma \in \Pi(N)$.
(iv) The dummy property: Obvious.
(v) Additivity: It follows from the fact that $m^\sigma_{\sigma(k)}(u_1 + u_2) = m^\sigma_{\sigma(k)}(u_1) + m^\sigma_{\sigma(k)}(u_2)$ for all

$u_1, u_2 \in PRI^N$ and for all $k \in \{1, 2, ..., |N|\}$. ■

As the following example shows, the concession rule and the primeval rule do not satisfy the dummy property.

Example 6.2 *Here, the three-household village game is manipulated into a new primeval games* (N, u_1) *with* $N = \{a, b, c\}$ *such that player* c *is a dummy in the game* u_1*.*

	(a)	(b)	(c)	(a,b)	(a,c)	(b,c)	(a,b,c)
u_1	(5)	(3)	(2)	$(8,2)$	$(3,4)$	$(4,1)$	$(6,0,6)$

The solutions for the above game are given as follows.

$\Phi(u_1)$	$=$	$(6,$	$4,$	$2)$
$\mathcal{C}(u_1)$	$=$	$(6,$	$2,$	$4)$
$\zeta(u_1)$	$=$	$(5\frac{1}{2},$	$2,$	$4\frac{1}{2})$

Theorem 6.3 *There is a unique compensation rule on* PRI^N *satisfying efficiency, symmetry, the dummy property and additivity. This rule is the marginalistic rule.*

Proof. From Theorem 6.1, it follows that the marginalistic rule Φ satisfies efficiency, symmetry, the dummy property and additivity.

Conversely, suppose that a compensation rule f satisfies these four properties. We have to show that $f = \Phi$. Let u be a primeval game on N. Then,

$$u = \sum_{(j,T)\in\mathcal{E}(N)} d_{(j,T)} w_{(j,T)}$$

where $d_{(j,T)}$ is uniquely determined.

By the additivity property,

$$f(u) = \sum_{(j,T)\in\mathcal{E}(N)} f(d_{(j,T)} w_{(j,T)}) \text{ and } \Phi(u) = \sum_{(j,T)\in\mathcal{E}(N)} \Phi(d_{(j,T)} w_{(j,T)}).$$

Thus, it suffices to show that for all $(j,T) \in \mathcal{E}(N)$ and $d_{(j,T)} \in \mathbb{R}$ we have $f(d_{(j,T)} w_{(j,T)}) = \Phi(d_{(j,T)} w_{(j,T)})$.

Let $(j,T) \in \mathcal{E}(N)$ and $d_{(j,T)} \in \mathbb{R}$. For any $i \notin T$, one readily verifies that i is a dummy player of game $(N, d_{(j,T)} w_{(j,T)})$. Therefore, by the dummy property,

$$f_i(d_{(j,T)} w_{(j,T)}) = \Phi_i(d_{(j,T)} w_{(j,T)}) = 0 \text{ for all } i \notin T. \quad (1)$$

Then, for players $i, k \in T$, we can easily see that i and k are symmetric player in $(N, d_{(j,T)}w_{(j,T)})$. By symmetry,

$$f_i(d_{(j,T)}w_{(j,T)}) = f_k(d_{(j,T)}w_{(j,T)}) \text{ for all } i, k \in T; \quad (2)$$

and similarly,

$$\Phi_i(d_{(j,T)}w_{(j,T)}) = \Phi_k(d_{(j,T)}w_{(j,T)}) \text{ for all } i, k \in T. \quad (3)$$

Therefore, efficiency and (1)-(3) imply that

$$f_i(d_{(j,T)}w_{(j,T)}) = \Phi_i(d_{(j,T)}w_{(j,T)}) = \frac{1}{|T|}d_{(j,T)} \text{ for all } i \in T.$$

■

Consider the dummy property which takes a marginal contribution perspective and assigns a dummy player his R-C payoff. As we know, without taking compensation into account, a dummy player i will get his status quo payoff in game u, i.e., $u(i, N)$. As $u(i, \{i\})$ and $u(i, N)$ represent two polar opinions, one may argue that taking the average could be a fair compromise.

- Property 6 *(The quasi dummy property)*: $f_i(u) = \frac{u(i,\{i\})+u(i,N)}{2}$, for all $u \in PRI^N$ and for any dummy player i in (N, u).

Now we introduce the property of *adjusted symmetry*. Similar to the quasi dummy property, one may have the following argument. On the one hand, when considering the same effect on total payoffs that symmetric players have, they may require the same value in a game. On the other hand, since symmetric players can have different R-C payoffs or status quo payoffs, their values should reflect such differences. An immediate and easy way to deal with this problem is to adjust the values by their status quo payoffs.

- Property 7 *(Adjusted symmetry)*: There is an $\alpha(u) \in \mathbb{R}$ such that

$$f_i(u) = \frac{\alpha(u) + u(i, N)}{2} \text{ and } f_j(u) = \frac{\alpha(u) + u(j, N)}{2}$$

  for all $u \in PRI^N$, and for all symmetric players i, j in u, where $\alpha(u)$ is called the standard value for symmetric players in u.

Theorem 6.4 *The concession rule satisfies efficiency, adjusted symmetry, the neutral player property, the quasi dummy property and additivity.*

Proof.
(i) Efficiency: Clearly, by construction, $C^\sigma(u)$ is efficient for all $\sigma \in \Pi(N)$.
(ii) Adjusted symmetry: By Theorem 4.3, the proof is readily established.
(iii) The neutral player property: If player i is a neutral player in (N, u) then $C_i^\sigma(u) = u(i, \{i\})$ for any $\sigma \in \Pi(N)$.
(iv) The quasi dummy property: Given a game $u \in PRI^N$ and $\sigma \in \Pi(N)$, let player i be a dummy player in u and $i = \sigma(k)$. By definition, one can readily check that for all $k \in \{2, ..., |N|\}$,

$$\begin{aligned}\mathcal{P}^\sigma_{\sigma(k)}(u) &= \frac{1}{2}\sum_{j=1}^{k-1}\left(u(\sigma(j), S^\sigma_k) - u(\sigma(j), S^\sigma_{k-1})\right) \\ &= \frac{1}{2}\left(u(i, \{i\}) - u(i, S^\sigma_k)\right).\end{aligned}$$

Then, by the definition of the concession vector, we know

$$C^\sigma_{\sigma(k)}(u) = \frac{u(i, \{i\}) + u(i, N)}{2}$$

for all $k \in \{1, 2, ..., |N|\}$.
Hence, what remains is obvious.
(v) Additivity: It is immediate, by definition, to see that $C^\sigma_{\sigma(k)}(u_1 + u_2) = C^\sigma_{\sigma(k)}(u_1) + C^\sigma_{\sigma(k)}(u_2)$ for all $u_1, u_2 \in PRI^N$ and for all $k \in \{1, 2, ..., |N|\}$. ■

Theorem 6.5 *There is a unique compensation rule on PRI^N satisfying efficiency, adjusted symmetry, the quasi dummy property and additivity. This rule is the concession rule.*

Proof. From Theorem 6.4, it follows that the concession rule $\mathcal{C}$ satisfies efficiency, adjusted symmetry, the quasi dummy property and additivity.

Conversely, suppose a compensation rule f satisfies these four properties. We have to show that $f = \mathcal{C}$. Let u be a primeval game on N. Then,

$$u = \sum_{(j,T)\in\mathcal{E}(N)} d_{(j,T)} w_{(j,T)}$$

where $d_{(j,T)}$ is uniquely determined.

By the additivity property,

$$f(u) = \sum_{(j,T)\in\mathcal{E}(N)} f(d_{(j,T)} w_{(j,T)}) \text{ and } \mathcal{C}(u) = \sum_{(j,T)\in\mathcal{E}(N)} \mathcal{C}(d_{(j,T)} w_{(j,T)}).$$

Thus, it suffices to show that for all $(j,T) \in \mathcal{E}(N)$ and $d_{(j,T)} \in \mathbb{R}$ we have $f(d_{(j,T)}w_{(j,T)}) = \mathcal{C}(d_{(j,T)}w_{(j,T)})$.

Let $(j,T) \in \mathcal{E}(N)$ and $d_{(j,T)} \in \mathbb{R}$. For any $i \notin T$, one readily verifies that i is a dummy player of game $(N, d_{(j,T)}w_{(j,T)})$. Therefore, by the quasi dummy property,

$$f_i(d_{(j,T)}w_{(j,T)}) = \mathcal{C}_i(d_{(j,T)}w_{(j,T)}) = 0 \text{ for all } i \notin T. \tag{4}$$

Moreover, we know that all players in group T are symmetric players in $(N, d_{(j,T)}w_{(j,T)})$. By adjusted symmetry,

$$f_i(d_{(j,T)}w_{(j,T)}) = \frac{\alpha_f}{2} \text{ for all } i \in T\backslash\{j\} \text{ and some } \alpha_f \in \mathbb{R}, \tag{5}$$

and

$$\mathcal{C}_i(d_{(j,T)}w_{(j,T)}) = \frac{\alpha_{\mathcal{C}}}{2} \text{ for all } i \in T\backslash\{j\} \text{ and some } \alpha_{\mathcal{C}} \in \mathbb{R}. \tag{6}$$

And for player j, by adjusted symmetry as well, we have

$$f_j(d_{(j,T)}w_{(j,T)}) = \frac{\alpha_f + d_{(j,T)}}{2} \text{ and } \mathcal{C}_j(d_{(j,T)}w_{(j,T)}) = \frac{\alpha_{\mathcal{C}} + d_{(j,T)}}{2}. \tag{7}$$

Thus, efficiency and (4)-(7) imply that

$$\alpha_f = \alpha_{\mathcal{C}} = \frac{1}{|T|}d_{(j,T)}.$$

■

Before introducing the next property, we first define completely symmetric players. Given a primeval game $u \in PRI^N$, we say that two players $i,j \in N$ are *completely symmetric* if for all $S \subset N\backslash\{i,j\}$,

$$u(i, S \cup \{i\}) = u(j, S \cup \{j\}) \text{ and } u(i, S \cup \{j\} \cup \{i\}) = u(j, S \cup \{j\} \cup \{i\})$$

and for all $k \in S$

$$u(k, S \cup \{i\}) = u(k, S \cup \{j\}).$$

It is natural to require that two complete symmetric players get the same value in a primeval game as their emergences generate the same influence to other players while getting the same influence from the emergences of the others.

- Property 8 *(Complete symmetry)*: $f_i(u) = f_j(u)$ for all $u \in PRI^N$, and for all completely symmetric players $i,j \in N$.

Obviously, from the stronger versions of symmetry considered before, it readily follows that the marginalisitic rule and the concession rule satisfy complete symmetry.

Now we discuss another property which pays more attention to the compensation aspect and therefore seems important in the context of primeval games.

Given a game $u \in PRI^N$, a player $i \in N$ is called a *harmful player* if $u(j, S \cup \{i\}) \leq u(j, S)$ for all $S \subset N\backslash\{i\}$ and $j \in S$. Thus, a harmful player is a player who never generates positive externalities to other players.

Given a game $u \in PRI^N$, a player $i \in N$ is called a *harmless player* if $u(j, S \cup \{i\}) \geq u(j, S)$ for all $S \subset N\backslash\{i\}$ and $j \in S$. Thus, a harmless player is a player who never produces negative externalities to others.

Given a game $u \in PRI^N$, a player $i \in N$ is called an *immune-harmful player* if it is both an immune player and a harmful player in u; or is called an *immune-harmless player* if it is both an immune player and a harmless player in u.

- Property 9 *(The immune-harmless player property)*: $f_i(u) = u(i, \{i\})$, for all $u \in PRI^N$ and for any immune-harmless player i in (N, u).

Theorem 6.6 *The primeval rule satisfies efficiency, complete symmetry, the neutral player property and the immune-harmless player property.*

Proof.
(i) Efficiency: Clearly, by construction, $B^{\sigma}(u)$ is efficient for all $\sigma \in \Pi(N)$.
(ii) Complete symmetry: Let i_1, i_2 be two completely symmetric players in $u \in PRI^N$. Consider $\sigma \in \Pi(N)$, and without loss of generality, $\sigma(k) = i_1$, $\sigma(h) = i_2$, where $i_1, i_2 \in N$. Let $\bar{\sigma} \in \Pi(N)$ be the permutation which is obtained from σ by interchanging the positions of i_1 and i_2, i.e.

$$\bar{\sigma}(w) = \begin{cases} \sigma(w) & \text{if } w \neq k, h \\ i_1 & \text{if } w = h \\ i_2 & \text{if } w = k. \end{cases}$$

As $\sigma \mapsto \bar{\sigma}$ is bijective, it suffices to prove that $B^{\sigma}_{i_1}(u) = B^{\bar{\sigma}}_{i_2}(u)$.
Case 1: $1 < k < h$.
By definition, we know

$$\begin{aligned} B^{\sigma}_{i_1}(u) = B^{\sigma}_{\sigma(k)}(u) &= u(\sigma(k), S^{\sigma}_k) - L^{\sigma}_{\sigma(k)}(u) + G^{\sigma}_{\sigma(k)}(u) \\ B^{\bar{\sigma}}_{i_2}(u) = B^{\bar{\sigma}}_{\bar{\sigma}(k)}(u) &= u(\bar{\sigma}(k), S^{\bar{\sigma}}_k) - L^{\bar{\sigma}}_{\bar{\sigma}(k)}(u) + G^{\bar{\sigma}}_{\bar{\sigma}(k)}(u). \end{aligned}$$

Obviously, $u(\sigma(k), S^{\sigma}_k) = u(\bar{\sigma}(k), S^{\bar{\sigma}}_k)$. Moreover, since i_1, i_2 are completely symmetric players, $L^{\sigma}_{\sigma(k)}(u) = L^{\bar{\sigma}}_{\bar{\sigma}(k)}(u)$ and $G^{\sigma}_{\sigma(k)}(u) = G^{\bar{\sigma}}_{\bar{\sigma}(k)}(u)$. Therefore, $B^{\sigma}_{i_1}(u) = B^{\bar{\sigma}}_{i_2}(u)$.

Case 2: $1 < h < k$. The proof is analogous to the above.
Case 3: $1 = k < h$. Apparently,

$$B^{\sigma}_{i_1}(u) = u(\sigma(1), \{\sigma(1)\}) + G^{\sigma}_{\sigma(1)}(u) = u(\bar{\sigma}(1), \{\bar{\sigma}(1)\}) + G^{\bar{\sigma}}_{\bar{\sigma}(1)}(u) = B^{\bar{\sigma}}_{\bar{\sigma}(1)}(u) = B^{\bar{\sigma}}_{i_2}(u).$$

Case 4: $1 = h < k$. Analogously, the proof is easy to be established.
As a consequence, $B^{\sigma}_{i_1}(u) = B^{\bar{\sigma}}_{i_2}(u)$.
(iii) The neutral player property: If player i is a neutral player in (N, u) then $B^{\sigma}_i(u) = u(i, \{i\})$ for any $\sigma \in \Pi(N)$.
(iv) The immune-harmless player property: Obvious. ■

The following example shows that the primeval rule does not satisfy symmetry and additivity.

Example 6.7 *Consider the following two primeval games.*

	(a)	(b)	(c)	(a,b)	(a,c)	(b,c)	(a,b,c)
u_1	(1)	(1)	(5)	$(2,3)$	$(2,4)$	$(0,6)$	$(3,4,2)$
u_2	(6)	(4)	(7)	$(10,5)$	$(7,6)$	$(3,6)$	$(11,6,4)$

In game u_1, a and b are symmetric players. However, $\zeta(u_1) = \left(1\frac{1}{2}, 3\frac{1}{2}, 4\right)$.

The game u_2 is obtained by adding u_1 to the three-household village game. The primeval rule yields that $\zeta(u_2) = \left(10\frac{1}{6}, 5\frac{1}{3}, 5\frac{1}{2}\right)$, which is not equal to the sum of the outcomes of the primeval rule applied in game u_1 and the three-household village game.

By investigating the gains of specific types of players under different compensation rules, we can see the relationships and differences among those rules.

We first consider the following corollary which discusses the gains of an uninfluential player according to the primeval rule and the marginalistic rule. The result is consistent with our intuition: As an uninfluential player, he need not compensate the others while he could benefit from the positive externalities from the others. So, for an uninfluential player, the outcome of the primeval rule is always no less than that of the marginalistic rule for a primeval game.

Corollary 6.8 *For any game $u \in PRI^N$ and any uninfluential player $i \in N$, it holds that*

$$\zeta_i(u) \geq \Phi_i(u).$$

Proof. Given a game $u \in PRI^N$ and let $i \in N$ be an uninfluential player. Given $\sigma \in \Pi(N)$, let $i = \sigma(k)$, it suffice to show $B_i^\sigma(u) \geq m_i^\sigma(u)$. As we know

$$B_i^\sigma(u) = B_{\sigma(k)}^\sigma(u) = \begin{cases} u(i, \{i\}) + G_{\sigma(1)}^\sigma(u) & \text{if } k = 1 \\ \\ u(i, S_k^\sigma) + G_{\sigma(k)}^\sigma(u) & \text{if } k \in \{2, ..., |N| - 1\} \\ \\ u(i, N) & \text{if } k = |N| \end{cases}$$

and

$$m_i^\sigma(u) = m_{\sigma(k)}^\sigma(u) = u(i, S_k^\sigma).$$

So, $B_i^\sigma(u) \geq m_i^\sigma(u)$. ■

We would like to note that there is no general relationship between the concession rule and the other two rules with respect to uninfluential players.

For an immune-harmful player, since he cannot get any positive externalities but needs to compensate the others as he always does harm to them, the outcome of the primeval rule is equivalent to that of the marginalistic rule. An immune-harmless player may be expected to obtain his R-C payoff: He need not compensate the others because he does not do anything harmful. Meanwhile, he need not be compensated because nobody affects him. The primeval rule is consistent with this idea while the marginalistic rule and the concession rule may give extra payoff to such a player as they take a different perspective such that the positive externalities are not for free.

Corollary 6.9 *For any game $u \in PRI^N$, we have*
(a) $\Phi_i(u) = \zeta_i(u) \leq \mathcal{C}_i(u) \leq u(i, \{i\})$ for any immune-harmful player $i \in N$; and
(b) $\Phi_i(u) \geq \mathcal{C}_i(u) \geq \zeta_i(u) = u(i, \{i\})$ for any immune-harmless player $i \in N$.

Proof.
(a) Given $\sigma \in \Pi(N)$ and let $i = \sigma(k)$ for $k \in \{1, 2, ..., |N|\}$. First, in order to prove $\Phi_i(u) = \zeta_i(u)$, it suffices to show $m_i^\sigma(u) = B_i^\sigma(u)$. Apparently, when $k = 1$, $m_i^\sigma(u) =$

$B_i^\sigma(u) = u(i,\{i\})$. When $k \in \{2,...,|N|\}$, we get

$$\begin{aligned} m_i^\sigma(u) &= \sum_{l=1}^{k} u(\sigma(l), S_k^\sigma) - \sum_{j=1}^{k-1} u(\sigma(j), S_{k-1}^\sigma) \\ &= u(i, S_k^\sigma) + \sum_{j=1}^{k-1} u(\sigma(j), S_k^\sigma) - \sum_{j=1}^{k-1} u(\sigma(j), S_{k-1}^\sigma) \\ &= u(i, S_k^\sigma) - \sum_{j=1}^{k-1} \big(u(\sigma(j), S_{k-1}^\sigma) - u(\sigma(j), S_k^\sigma)\big) \\ &= B_i^\sigma(u). \end{aligned}$$

Moreover, since $\sum_{j=1}^{k-1}\big(u(\sigma(j), S_{k-1}^\sigma) - u(\sigma(j), S_k^\sigma)\big) \geq 0$, we know $m_i^\sigma(u) = B_i^\sigma(u) \leq u(i,\{i\})$ for all $k \in \{2,...,|N|\}$. Then, $\Phi_i(u) = \zeta_i(u) \leq u(i,\{i\})$. By Theorem 4.3, we have

$$\begin{aligned} \mathcal{C}_i^\sigma(u) &= \frac{1}{2}u(i,N) + \frac{1}{2}\Phi_i(u) \\ &= \frac{1}{2}u(i,\{i\}) + \frac{1}{2}\Phi_i(u) \\ &\geq \Phi_i(u). \end{aligned}$$

(b) By definition and analogous to part (a), the proof is easy to be established. ■

Corollary 6.10 *For any game $u \in PRI^N$ and any harmless player $i \in N$ with $u(i,N) \geq u(i,\{i\})$, it holds that*

$$\zeta_i(u) \geq u(i,\{i\}).$$

Proof. For a primeval game $u \in PRI^N$, let i be a harmless player in u. For an ordering $\sigma \in \Pi(N)$, let $i = \sigma(k)$, $k \in \{1,...,|N|\}$. By definition and since $u(i,N) \geq u(i,\{i\})$, we know $G_i^\sigma(u) \geq 0$ if $k=1$; $L_i^\sigma(u) = 0$ for all $k \in \{2,...,|N|\}$; and

$$u(i, S_k^\sigma) + G_i^\sigma(u) \geq u(i,N) \geq u(i,\{i\})$$

for all $k \in \{2,...,|N|-1\}$. Hence, $B_i^\sigma(u) \geq u(i,\{i\})$. ■

Note that Corollary 6.10 can be understood as the property of *individual rationality* for harmless players: If a player's presence never does harm to others and his status quo payoff is greater than his R-C payoff, he should get at least his R-C payoff.

As the following example shows, the marginalistic rule and the concession rule do not satisfy this property.

Example 6.11 *Consider the following game u with three players, a, b and c.*

(a)	(b)	(c)	(a,b)	(a,c)	(b,c)	(a,b,c)
(3)	(1)	(5)	$(0,1)$	$(0,5)$	$(0,6)$	$(3,1,6)$

Here a is a harmless player. According to the marginalistic rule, $\Phi_a(u) = 2\frac{1}{3}$; and according to the concession rule, $\mathcal{C}_a(u) = 2\frac{2}{3}$. Both are less than a's R-C payoff of 3. However, the primeval rule yields that $\zeta_a(u) = 4$.

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