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MATRICES AND GRAPHS

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Matrices and Graphs

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Abstract

The present article is designed to be a contribution to the chapter 'Combinatorial Matrix Theory and Graphs' of the Handbook of Linear Algebra, to be published by CRC Press. The format of the handbook is to give just definitions, theorems, and examples; no proofs. In the five sections given below, we present the most important notions and facts about matrices related to (undirected) graphs.

1. Graphs.
2. The adjacency matrix and its eigenvalues.
3. Other matrix representations.
4. Graph parameters.
5. Association schemes.

Keywords: Graphs, Matrices; JEL code: C0.

1 Graphs

Definitions:

A ***graph*** $G = (V, E)$ consists of a finite set $V = \{v_1, \ldots, v_n\}$ of ***vertices*** and a finite multiset E of ***edges***, where each edge is a pair $\{v_i, v_j\}$ of vertices (not necessarily distinct). If $v_i = v_j$ the edge is called a ***loop***. A vertex v_i of an edge is called an ***endpoint*** of the edge.

A ***simple graph*** is a graph with no loops where each edge has multiplicity at most one.

Two graphs (V, E) and (V', E') are ***isomorphic*** whenever there exist bijections $\phi : V \to V'$ and $\psi : E \to E'$, such that $v \in V$ is an endpoint of $e \in E$ if and only if $\phi(v)$ is an endpoint of $\psi(e)$.

A ***walk*** of ***length*** ℓ in a graph is an alternating sequence $(v_{i_0}, e_{i_1}, v_{i_1}, e_{i_2}, \ldots, e_{i_\ell}, v_{i_\ell})$ of vertices and edges (not necessarily distinct), such that $v_{i_{j-1}}$ and v_{i_j} are endpoints of e_{i_j} for $j = 1, \ldots, \ell$.

A ***path*** of ***length*** ℓ in a graph is a walk of length ℓ with all vertices distinct.

A ***cycle*** of ***length*** ℓ in a graph is a walk $(v_{i_0}, e_{i_1}, v_{i_1}, e_{i_2}, \ldots, e_{i_\ell}, v_{i_\ell})$ with $v_{i_0} = v_{i_\ell}$, $\ell \neq 0$, and $v_{i_1}, \ldots, v_{i_\ell}$ all distinct.

A graph (V, E) is ***connected*** if $V \neq \emptyset$ and there exists a walk between any two distinct vertices of V.

The ***distance*** between two vertices v_i and v_j of a graph is the length of a shortest path between v_i and v_j. (The distance is infinite if there is no path between v_i and v_j.)

The ***diameter*** of a connected graph G is the largest distance that occurs between two vertices of G.

A ***tree*** is a connected graph with no cycles.

A graph (V', E') is a ***subgraph*** of a graph (V, E) if $V' \subset V$ and $E' \subset E$. If E' contains all edges from E with endpoints in V', (V', E') is an ***induced*** subgraph of (V, E).

A ***spanning tree*** of a connected graph (V, E) is a subgraph (V', E') with $V' = V$, which is a tree.

A ***connected component*** of a graph (V, E) is an induced subgraph

(V', E'), which is connected and such that there exists no edge in E with one endpoint in V' and one outside V'. A connected component with one vertex and no edge is called an ***isolated vertex***.

Two vertices u and v are ***adjacent*** if there exists an edge with endpoints u and v. A vertex adjacent to v is called a ***neighbor*** of v. The ***degree*** or ***valency*** of a vertex v is the number of neighbors of v.

A graph (V, E) is ***bipartite*** if there is at least one edge, no loops, and the vertex set V admits a partition into two parts, such that no edge of E has both endpoints in one part.

A simple graph (V, E) is ***complete*** if E consists of all unordered pairs from V. The (isomorphism class of the) complete graph on n vertices is denoted by K_n.

A graph (V, E) is ***empty*** if $E = \emptyset$. If also $V = \emptyset$, it is called the ***null graph***.

A bipartite simple graph (V, E) with parts V_1 and V_2 is ***complete bipartite*** if E consists of all ordered pairs from V with one vertex in V_1 and one in V_2. The (isomorphism class of the) complete bipartite graph is denoted by K_{n_1,n_2}, where $n_1 = |V_1|$ and $n_2 = |V_2|$.

The (isomorphism class of the) simple graph that consists only of vertices and edges of a path of length ℓ is called ***the path of length*** ℓ, and denoted by $P_{\ell+1}$.

The (isomorphism class of the) simple graph that consists only of vertices and edges of a cycle of length ℓ is called ***the cycle of length*** ℓ, and denoted by C_ℓ.

The ***complement*** of a simple graph $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$, where $\overline{E}$ consists of all unordered pairs from V that are not in E.

A graph G is ***regular*** (or k***-regular***), if every vertex of a graph G has the same degree (equal to k).

A graph G is ***walk-regular*** if for every vertex v the number of walks from v to v of length ℓ, only depends on ℓ (not on v).

A simple graph G is ***strongly regular*** with parameters (n, k, λ, μ), whenever G has n vertices and:
• G is k-regular with $1 \le k \le n - 2$,

• every two adjacent vertices of G have exactly λ common neighbors,
• every two distinct nonadjacent vertices of G have exactly μ common neighbors.

Let G be a simple graph. The ***line graph*** $L(G)$ of G has the edges of G as vertices, and vertices of $L(G)$ are adjacent, if the corresponding edges of G have an endpoint in common.

The ***cocktail party graph*** $CP(a)$ is the graph obtained by deleting a disjoint edges from the complete graph K_{2a}. (Note that $CP(0)$ is the null graph.)

Let G be a simple graph with vertex set $\{v_1, \ldots, v_n\}$, and let $a_1, \ldots, a_n$ be nonnegative integers. The ***generalized line graph*** $L(G; a_1, \ldots, a_n)$ consist of disjoint copies of the line graph $L(G)$ and the n cocktail party graphs $CP(a_1), \ldots, CP(a_n)$ together with all edges joining a vertex $\{v_i, v_j\}$ of $L(G)$ with each vertex of $CP(a_i)$ and $CP(a_j)$.

The ***(strong) product*** $G \cdot G'$ of two simple graphs $G = (V, E)$ and $G' = (V', E')$, is the graph with vertex set $V \times V'$, where two distinct vertices are adjacent whenever in both coordinate places the vertices are adjacent or equal. The strong product of ℓ copies of a graph G is denoted by G^ℓ.

An ***imbedding*** of a graph in $\mathbb{R}^n$, consists of a representation of the vertices by distinct points in $\mathbb{R}^n$, and a representation of the edges by curve segments between the endpoints, such that a curve segment only intersects another segment or itself in an endpoint. (A curve segment between $\mathbf{x}$ and $\mathbf{y}$ is the range of a continuous map ϕ from $[0, 1]$ to $\mathbb{R}^n$ with $\phi(0) = \mathbf{x}$ and $\phi(1) = \mathbf{y}$.)

A graph is ***planar*** if it admits an imbedding in $\mathbb{R}^2$.

A graph is ***outerplanar*** if it admits an imbedding in $\mathbb{R}^2$, such that the vertices are represented by points on the unit circle, and the representations of the edges are contained in the unit disc.

A graph G is ***linklessly imbeddable***, if it admits an imbedding in $\mathbb{R}^3$, such that no two disjoint circuits of G are linked. (Two disjoint Jordan curves in $\mathbb{R}^3$ are linked if there is no topological 2-sphere in $\mathbb{R}^3$ separating them.)

Contraction of an edge e of a graph (V, E) is the operation that merges the endpoints of e in V, and deletes e from E.

A ***minor*** of a graph G is any graph that can be obtained from G by a sequence of edge deletions and contractions.

Facts:

If no reference is given, the fact is trivial or a classical result that can be found in almost every introduction to graph theory, such as [13].

1. A graph G is bipartite if and only if G has at least one edge and no cycles of odd length.

2. A tree with n vertices has $n-1$ edges.

3. [7, p.8] A regular generalized line graph is a line graph or a cocktail party graph.

4. (Whitney) The line graphs of two connected non-isomorphic graphs G and G' are non-isomorphic, unless $\{G, G'\} = \{K_3, K_{1,3}\}$.

5. [10, p.81] A strongly regular graph is walk-regular.

6. A walk-regular graph is regular.

7. The complement of a strongly regular graph with parameter set (n, k, λ, μ) is strongly regular with parameter set $(n, n-k-1, n-2k+\mu-2, n-2k+\lambda)$.

8. Every graph can be imbedded in $\mathbb{R}^3$.

9. [17] (Robertson, Seymour) For every graph property $\mathcal{P}$ that is closed under taking minors, there exists a finite list of graphs such that a graph G has property $\mathcal{P}$ if and only if no graph from the list is a minor of G.

10. The graph properties: planar, outerplanar, and linklessly imbeddable are closed under taking minors.

11. (Kuratowski, Wagner) A graph G is planar if and only if no minor of G is isomorphic to K_5 or $K_{3,3}$.

Examples:

1. The complete graph K_n is walk-regular and regular of degree $n-1$.

2. The complete bipartite graph $K_{k,k}$ is regular of degree k, walk-regular and strongly regular with parameters $(2k, k, 0, k)$.

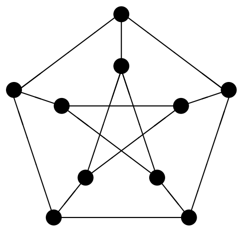

Figure 1: The Petersen graph

3. For $n \geq 4$, the line graphs $L(K_n)$ and $L(K_{k,k})$ and their complements are strongly regular. The complement of $L(K_5)$ is the Petersen graph (See Figure 1).

4. Examples of outerplanar graphs are all trees, C_n, and $\overline{P_5}$.

5. Examples of graphs that are planar, but not outerplanar are: K_4, $CP(3)$, $\overline{C_6}$ and $K_{2,n-2}$ for $n \geq 5$.

6. Examples of graphs that are not planar, but linklessly imbeddable are: K_5, and $K_{3,n-3}$ for $n \geq 6$.

7. The Petersen graph, and K_n for $n \geq 6$ are not linklessly imbeddable.

2 The adjacency matrix and its eigenvalues

Definitions:

The ***adjacency matrix*** A_G of a graph G with vertex set $\{v_1, \ldots, v_n\}$ is the symmetric $n \times n$ matrix, whose (i,j)th entry $A_G[i,j]$ is equal to the number of edges between v_i and v_j.

The ***eigenvalues*** of a graph G are the eigenvalues of its adjacency matrix.

The ***spectrum*** of a graph G is the multiset of eigenvalues with their multiplicities.

Two graphs are ***cospectral*** whenever they have the same spectrum.

A graph G is ***determined by its spectrum*** if every graph cospectral with G is isomorphic to G.

A ***Hoffman polynomial*** of a graph G is a polynomial $p(x)$ of minimum degree such that $p(A_G) = J$.

The ***main angles*** of a graph G are the cosines of the angles between the eigenspaces of A_G and the all-ones vector $\mathbf{1}$.

Facts:

If no reference is given, the fact is trivial or a standard result in algebraic graph theory that can be found in the classical books [2] and [5].

1. If A_G is the adjacency matrix of a simple graph G, then $\overline{A}_G = J - I - A_G$ is the adjacency matrix of the complement of G.

2. If A_G and $A_{G'}$ are adjacency matrices of simple graphs G and G', respectively, then $((A_G + I) \otimes (A_{G'} + I)) - I$ is the adjacency matrix of the strong product $G \cdot G'$.

3. Isomorphic graphs are cospectral.

4. Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$ and adjacency matrix A_G. The number of walks of length ℓ from v_i to v_j equals $A_G^\ell[i, j]$.

5. The eigenvalues of a graph are real numbers.

6. The adjacency matrix of a graph is diagonalizable.

7. If $\lambda_1 \geq \ldots \geq \lambda_n$ are the eigenvalues of a graph G, then $|\lambda_i| \leq \lambda_1$. If $\lambda_1 = \lambda_2$, then G is disconnected. If $\lambda_1 = -\lambda_n$ and G is not empty, then at least one connected component of G is bipartite.

8. [5, p.87] If $\lambda_1 \geq \ldots \geq \lambda_n$ ($n \geq 2$) are the eigenvalues of a graph G, then G is bipartite if and only if $\lambda_i = -\lambda_{n+1-i}$ for $i = 1, \ldots, n$.

9. If G is a simple k-regular graph, then the largest eigenvalue of G equals k, and the multiplicity of k equals the number of connected components of G.

10. [5, p.94] If $\lambda_1 \geq \ldots \geq \lambda_n$ are the eigenvalues of a simple graph G with n vertices and m edges, then $\sum_i \lambda_i^2 = 2m \leq n\lambda_1$. Equality holds if and only if G is regular.

11. [5, p.95] A graph G has a Hoffman polynomial if and only if G is regular and connected.

12. [6, p.99] Suppose $P(x)$ is the characteristic polynomial of a graph G with n vertices, r distinct eigenvalues $\nu_1, \ldots, \nu_r$ and main angles $\beta_1, \ldots, \beta_r$,

then the complement of G has characteristic polynomial

$$\overline{P}(x) = (-1)^n P(-x-1)(1 - n\sum_{i=1}^{r} \beta_i^2/(x+1+\nu_i)).$$

13. [5, p.103], [10, p.179] A connected simple regular graph is strongly regular if and only if it has just three distinct eigenvalues. The eigenvalues ($\nu_1 > \nu_2 > \nu_3$) and parameters (n, k, λ, μ) are related by $\nu_1 = k$, $\nu_2\nu_3 = \mu - k$ and $\nu_2 + \nu_3 = \lambda - \mu$.

14. [4, p.150], [10, p.180] The multiplicities of the three eigenvalues of a connected strongly regular with parameters (n, k, λ, μ) are 1 and

$$\frac{1}{2}\left(n - 1 \pm \frac{(n-1)(\mu-\lambda) - 2k}{\sqrt{(\mu-\lambda)^2 + 4(k-\mu)}}\right).$$

15. [11, p.190] A simple regular graph with at most four distinct eigenvalues is walk-regular.

16. [6, p.79] Cospectral walk-regular graphs have the same main angles.

17. [18] Almost all trees are cospectral with another tree.

18. [9] The number of graphs on n vertices, not determined by the spectrum is asymptotically bounded from below by $n^3 g_{n-1}(\frac{1}{24} - o(1))$, where g_{n-1} denotes the number of non-isomorphic graphs on $n-1$ vertices.

19. [9] The complete graph, the cycle, the path, the regular complete bipartite graph and their complements are determined by their spectrum.

20. [9] Suppose G is a regular connected simple graph on n vertices, which is determined by its spectrum. Then also the complement $\overline{G}$ of G is determined by its spectrum, and if $n+1$ is not a square, also the line graph $L(G)$ of G is determined by its spectrum.

21. [7, p.7] A generalized line graph has smallest eigenvalue at least -2.

22. [7, p.85] A connected graph with more than 36 vertices and smallest eigenvalue at least -2 is a generalized line graph.

23. [7, p.90] There are precisely 187 connected regular graphs with smallest eigenvalue at least -2, that are not a line graph or a cocktail party graph. Each of these graphs has smallest eigenvalue equal to -2, at most 28 vertices, and degree at most 16.

Examples:

1. Figure 2 gives a pair of nonisomorphic bipartite graphs with their adjacency matrices. Both matrices have spectrum $\{2,\ 0^3,\ -2\}$ (exponents indicate multiplicities), so the graphs are cospectral.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Figure 2: Two cospectral graphs with their adjacency matrices

2. The main angles of the two graphs of Figure 2 (with the given ordering of the eigenvalues) are $2/\sqrt{5},\ 1/\sqrt{5},\ 0$ and $3/\sqrt{10},\ 0, 1/\sqrt{10}$, respectively.

3. The spectrum of K_{n_1,n_2} is $\{\sqrt{n_1 n_2},\ 0^{n-2},\ -\sqrt{n_1 n_2}\}$.

4. By Fact 14, the multiplicities of the eigenvalues of any strongly regular graph with parameters $(n, k, 1, 1)$ would be nonintegral, so no such graph can exist (this result is known as the Friendship theorem).

5. The Petersen graph has spectrum $\{3,\ 1^5,\ -2^4\}$ and Hoffman polynomial $(x-1)(x+2)$. It is one of the 187 connected regular graphs with least eigenvalue -2 which is neither a line graph or a cocktail party graph.

6. The eigenvalues of the path P_n are $2\cos\frac{i\pi}{n+1}$ $(i = 1, \ldots, n)$.

7. The eigenvalues of the cycle C_n are $2\cos\frac{2i\pi}{n}$ $(i = 1, \ldots, n)$.

3 Other matrix representations

Definitions:

Let G be a simple graph with adjacency matrix A_G. Suppose D is the diagonal matrix with the degrees of G on the diagonal (with the same vertex ordering as in A_G). Then $L_G = D - A_G$ is the ***Laplacian matrix*** of G (often abbreviated to the ***Laplacian***, and also known as ***admittance matrix***), and the matrix $|L_G| = D + A_G$ is (sometimes) called the ***signless Laplacian matrix***.

The ***Laplacian eigenvalues*** of a simple graph G are the eigenvalues of the Laplacian matrix L_G.

If $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$ are the Laplacian eigenvalues of G, then μ_2 is called the ***algebraic connectivity*** of G. (See Section 6 of the present Chapter.)

Let G be simple graph with vertex set $\{v_1, \ldots, v_n\}$. A symmetric real matrix M is called a ***generalized Laplacian*** of G, whenever $M[i,j] < 0$ if v_i and v_j are adjacent, and $M[i,j] = 0$ if v_i and v_j are nonadjacent and distinct (nothing is required for the diagonal entries of M).

Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{e_1, \ldots, e_m\}$. The ***incidence matrix*** of G is the $n \times m$ matrix N_G defined by $N_G[i,j] = 1$ if vertex v_i is an endpoint of edge e_j and $N_G[i,j] = 0$ otherwise.

If A_G is the adjacency matrix of a simple graph G, then $S_G = J - I - 2A_G$ is the ***Seidel matrix*** of G.

Let I' be a diagonal matrix with ± 1 on the diagonal. Then the graph G' with Seidel matrix $S_{G'} = I' S_G I'$ is ***switching equivalent*** to G. The graph operation that changes G into G' is called ***Seidel switching***.

Facts:

If no reference is given, the fact is trivial or a classical result that can be found in [4].

1. Let G be a simple graph. The Laplacian matrix L_G and the signless Laplacian $|L_G|$ are positive semi-definite.

2. The nullity of L_G is equal to the number of connected components of G.

3. The nullity of $|L_G|$ is equal to the number of connected components of G, which are bipartite or consist of a single vertex.

4. [9] If and only if G is bipartite or empty, the Laplacian and the signless

Laplacian of G have the same spectrum.

5. (Matrix-tree theorem) Let G be a simple graph with Laplacian matrix L_G, and let c_G denote the number of spanning trees of G, then $\mathrm{adj}(L_G) = c_G J$.

6. Suppose N_G is the incidence matrix of a simple graph G, then $N_G N_G^T = |L_G|$, and $N_G^T N_G - 2I = A_{L(G)}$.

7. Suppose N_G is the incidence matrix of a simple graph G. Let N'_G be any matrix obtained from N_G by changing in each column one 1 into a -1. Then $N'_G N'^T_G = L_G$.

8. If $\mu_1 \leq \ldots \leq \mu_n$ are the Laplacian eigenvalues of a simple graph G, and $\overline{\mu}_1 \leq \ldots \leq \overline{\mu}_n$ are the Laplacian eigenvalues of $\overline{G}$, then $\mu_1 = \overline{\mu}_1 = 0$ and $\overline{\mu}_i = n - \mu_{n+2-i}$ for $i = 2, \ldots, n$.

9. [9] If $\mu_1 \leq \ldots \leq \mu_n$ are the Laplacian eigenvalues of a simple graph G with n vertices and m edges, then $\sum_i \mu_i = 2m \leq \sqrt{n \sum_i \mu_i(\mu_i - 1)}$ with equality if and only if G is regular.

10. [8] A connected simple graph G has at most three distinct Laplacian eigenvalues if and only if there exist integers μ and $\overline{\mu}$, such that any two distinct nonadjacent vertices have exactly μ common neighbors, and any two adjacent vertices have exactly $\overline{\mu}$ common nonneighbors.

11. If G is k-regular and $\mathbf{v} \not\in \mathrm{span}\{\mathbf{1}\}$, then the following are equivalent:
- λ is an eigenvalue of A_G with eigenvector $\mathbf{v}$,
- $k - \lambda$ is an eigenvalue of L_G, with eigenvector $\mathbf{v}$,
- $k + \lambda$ is an eigenvalue of $|L_G|$, with eigenvector $\mathbf{v}$,
- $-1 - 2\lambda$ is an eigenvalue of S_G with eigenvector $\mathbf{v}$.

12. [9] Consider a simple graph G with n vertices and m edges. Let $\nu_1 \leq \ldots \leq \nu_n$ be the eigenvalues of $|L_G|$, the signless Laplacian of G. Let $\lambda_1 \geq \ldots \geq \lambda_m$ be the eigenvalues of $L(G)$, the line graph of G. Then $\lambda_i = \nu_{n-i+1} - 2$ if $1 \leq i \leq \min\{m, n\}$, and $\lambda_i = -2$ if $\min\{m, n\} < i \leq m$.

13. [5, p.184] The Seidel matrices of switching equivalent graphs have the same spectrum.

Examples:

1. The Laplacian eigenvalues of the Petersen graph are $\{0,\ 2^5,\ 5^4\}$.

2. The two graphs of Figure 3 are nonisomorphic, but the Laplacian matrices have the same spectrum. Both Laplacian matrices have $12J$ as adjugate, so both have 12 spanning trees. They are not cospectral with respect to the adjacency matrix, because one is bipartite and the other one is not.

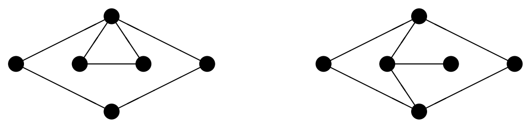

Figure 3: Graphs with cospectral Laplacian matrices

3. Figure 4 gives two graphs with cospectral signless Laplacian matrices. They are not cospectral with respect to the adjacency matrix, because one is bipartite and the other one is not. They also don't have cospectral Laplacian matrices, because the numbers of components differ.

Figure 4: Graphs with cospectral signless Laplacian matrices

4. The eigenvalues of the Laplacian and the signless Laplacian matrix of the path P_n are $2 + 2\cos\frac{i\pi}{n}$ $(i = 1, \ldots, n)$.

5. The complete bipartite graph K_{n_1,n_2} is Seidel switching equivalent to the empty graph on $n = n_1 + n_2$ vertices. The Seidel matrices have the same spectrum, being $\{n-1, -1^{n-1}\}$.

4 Graph parameters

Definitions:

A subgraph G' on n' vertices of a simple graph G is a ***clique*** if G' is isomorphic to the complete graph $K_{n'}$. The largest value of n' for which a clique with n' vertices exists is called the ***clique number*** of G and denoted by $\omega(G)$.

An induced subgraph G' on n' vertices of a graph G is a ***coclique*** or ***independent set of vertices*** if G' has no edges. The largest value of n' for which a coclique with n' vertices exists is called the ***vertex independence number*** of G and denoted by $\alpha(G)$.

The ***Shannon capacity*** $\Theta(G)$ of a simple graph G is defined by $\Theta(G) = \sup_\ell \sqrt[\ell]{\alpha(G^\ell)}$

A ***vertex coloring*** of a graph is a partition of the vertex set into cocliques. A coclique in such a partition is called a ***color class***.

The ***chromatic number*** $\chi(G)$ of a graph G is the smallest number of color classes of any vertex coloring of G.

For a simple graph $G = (V, E)$, the ***conductance*** or ***isoperimetric number*** $\Phi(G)$ is defined to be the minimum value of $\delta(V')/|V'|$ over any subset $V' \subset V$ with $|V'| \leq |V|/2$, where $\delta(V')$ equals the number of edges in E with one endpoint in V' and one endpoint outside V'.

An infinite family of graphs with constant degree and isoperimetric number bounded from below is called a family of ***expanders***.

A symmetric real matrix M is said to satisfy the ***Strong Arnold Hypothesis*** whenever there exists no symmetric nonzero matrix X with zero diagonal, such that $MX = \mathbf{0}$, $M \circ X = \mathbf{0}$.

The ***Colin de Verdière parameter*** $\mu(G)$ of a simple graph G is the largest nullity of any generalized Laplacian M of G satisfying the following:

- M has exactly one negative eigenvalue of multiplicity 1.
- The Strong Arnold Hypothesis.

Consider a simple graph G with vertex set $\{v_1, \ldots, v_n\}$. The ***Lovász parameter*** $\vartheta(G)$ is the minimum value of the largest eigenvalue $\lambda_1(M)$ of any real symmetric $n \times n$ matrix M, which satisfies $M[i,j] = 1$ if v_i and v_j are nonadjacent (including the diagonal).

Consider a simple graph G with vertex set $\{v_1, \ldots, v_n\}$. The integer $\eta(G)$ is defined to be the smallest rank of any $n \times n$ matrix M (over any field), which satisfies $M[i,i] \neq 0$ for $i = 1, \ldots, n$ and $M[i,j] = 0$, if v_i and v_j are distinct nonadjacent vertices.

Facts:

1. [2, p.13] A connected graph with r distinct eigenvalues (for the adjacency, the Laplacian or the signless Laplacian matrix) has diameter at most $r-1$.

2. [5, p.90-91], [10, p.83] The chromatic number $\chi(G)$ of a simple graph G with adjacency eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ satisfies: $1 - \lambda_1/\lambda_n \leq \chi(G) \leq 1 + \lambda_1$.

3. [5, p.88] For a simple graph G with n vertices, let m_+ and m_- denote the number of nonnegative and nonpositive adjacency eigenvalues, respectively. Then $\alpha(G) \leq \min\{m_+, m_-\}$.

4. [11, p.204] If G is a k-regular simple graph with adjacency eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ then $\omega(G) \leq \frac{n(\lambda_2+1)}{n-k+\lambda_2}$ and $\alpha(G) \leq \frac{-n\lambda_n}{k-\lambda_n}$.

5. [16] Suppose G is a simple graph with maximum degree Δ and algebraic connectivity μ_2, then the isoperimetric number $\Phi(G)$ satisfies $\mu_2/2 \leq \Phi(G) \leq \sqrt{\mu_2(2\Delta - \mu_2)}$.

6. [14] The Colin de Verdière parameter $\mu(G)$ is minor monotonic, that is, if H is a minor of G, then $\mu(H) \leq \mu(G)$.

7. [14] If G has at least one edge, then $\mu(G) = \max\{\mu(H) \,|\, H$ is a component of $G\}$.

8. [14] The Colin de Verdière parameter $\mu(G)$ satisfies the following:
- $\mu(G) \leq 1$ if and only if G is the disjoint union of paths.
- $\mu(G) \leq 2$ if and only if G is outerplanar.
- $\mu(G) \leq 3$ if and only if G is planar.
- $\mu(G) \leq 4$ if and only if G is linklessly imbeddable.

9. (Sandwich theorems)[15], [12] The parameters $\vartheta(G)$ and $\eta(G)$ satisfy: $\alpha(G) \leq \vartheta(G) \leq \chi(\overline{G})$ and $\alpha(G) \leq \eta(G) \leq \chi(\overline{G})$.

10. [15], [12] The parameters $\vartheta(G)$ and $\eta(G)$ satisfy: $\vartheta(G \cdot H) = \vartheta(G)\vartheta(H)$, and $\eta(G \cdot H) \leq \eta(G)\eta(H)$.

11. [15], [12] The Shannon capacity $\Theta(G)$ of a simple graph G satisfies: $\Theta(G) \leq \vartheta(G)$ and $\Theta(G) \leq \eta(G)$.

12. [15], [12] If G is a k-regular graph with eigenvalues $k = \lambda_1 \geq \ldots \geq \lambda_n$, then $\vartheta(G) \leq -n\lambda_n/(k - \lambda_n)$. Equality holds if G is strongly regular.

13. [15] The Lovász parameter $\vartheta(G)$ can also be defined as the maximum value of $\mathrm{tr}(MJ_n)$, where M is any positive semi-definite $n \times n$ matrix,

satisfying $\mathrm{tr}(M) = 1$ and $M[i,j] = 0$ if v_i and v_j are adjacent vertices in G.

Examples:

1. Suppose G is the Petersen graph. Then $\alpha(G) = 4$, $\vartheta(G) = 4$ (By Fact 12). Thus $\Theta(G) = 4$. Moreover $\chi(G) = 3$, $\chi(\overline{G}) = 5$, $\mu(G) = 5$ (take $M = L_G - 2I$), and $\eta(G) = 4$ (take $M = A_G + I$ over the field with two elements).

2. The isoperimetric number $\Phi(G)$ of the Petersen graph equals 1. Indeed $\Phi(G) \geq 1$, by Fact 5, and any pentagon gives $\Phi(G) \leq 1$.

3. $\mu(K_n) = n - 1$ (take $M = -J$).

4. If G is the empty graph with at least two vertices, then $\mu(G) = 1$ (M must be a diagonal matrix with exactly one negative entry, and the Strong Arnold Hypothesis forbids two or more diagonal entries to be 0).

5. By Fact 12, $\vartheta(C_5) = \sqrt{5}$. If $(v_1, \ldots, v_2)$ are the vertices of C_5, cyclically ordered, then (v_1, v_1), (v_2, v_3), (v_3, v_5), (v_4, v_2), (v_5, v_4) is a coclique of size 5 in $C_5 \cdot C_5$. Therefore $\Theta(C_5) = \sqrt{5}$.

5 Association schemes

Definitions:

A set of graphs $G_0, \ldots, G_d$ on a common vertex set $V = \{v_1, \ldots, v_n\}$ is an ***association scheme*** if the adjacency matrices $A_0, \ldots, A_d$ satisfy:
• $A_0 = I$,
• $\sum_{i=0}^{d} A_i = J$,
• $\mathrm{span}\{A_0, \ldots, A_d\}$ is closed under matrix multiplication.

The numbers $p_{i,j}^k$ defined by $A_i A_j = \sum_{i=0}^{d} p_{i,j}^k A_k$ are called the ***intersection numbers*** of the association scheme.

The algebra spanned by $A_0, \ldots, A_d$ is the ***Bose-Mesner algebra*** of the association scheme.

Consider a connected graph $G_1 = (V, E_1)$ with diameter d. Define $G_i = (V, E_i)$ to be the graph wherein two vertices are adjacent if their distance in G_1 equals i. If $G_0, \ldots, G_d$ is an association scheme, then G_1 is a ***distance-regular graph***.

Let V' be a subset of the vertex set V of an association scheme. The ***inner distribution*** $\mathbf{a} = [a_0, \ldots, a_d]^T$ of V' is defined by $a_i|V'| = \mathbf{c}^T A_i \mathbf{c}$, where $\mathbf{c}$ is the characteristic vector of V'.

Facts:

Facts 1 to 7 below are standard results on association schemes, that can be found in any of the following references: [1], [3], [10].

1. Suppose $G_0, \ldots, G_d$ is an association scheme. For any three integers $i, j, k \in \{0, \ldots, d\}$ and for any two vertices x and y adjacent in G_k, the number of vertices z adjacent to x in G_i and to y in G_j equals the intersection number p_{ij}^k. In particular, G_i is regular of degree $k_i = p_{ii}^0$.

2. The matrices of a Bose-Mesner algebra $\mathcal{A}$ can be diagonalized simultaneously. In other words, there exist a nonsingular matrix S such that SAS^{-1} is a diagonal matrix for every $A \in \mathcal{A}$.

3. A Bose-Mesner algebra has a basis $\{E_0 = \frac{1}{n}J, E_1, \ldots, E_d\}$ of idempotents, that is, $E_iE_j = \delta_{i,j}E_i$ ($\delta_{i,j}$ is the Kronecker symbol).

4. The change-of-coordinates matrix P defined by $A_j = \sum_i P[i,j]E_i$ satisfies:
- $P[i,j]$ is an eigenvalue of A_j with eigenspace $\text{Col}(E_i)$,
- $P[i,0] = 1$, $P[0,i] = k_i$ (the degree of G_i),
- $nk_jP^{-1}[j,i] = m_iP[i,j]$, where $m_i = \text{rank}(E_i)$ (the multiplicity of eigenvalue $P[i,j]$) .

5. (Krein condition) The Bose-Mesner algebra of an association scheme is closed under Hadamard multiplication. The numbers $q_{i,j}^k$, defined by $E_i \circ E_j = \sum_k q_{i,j}^k E_k$ are nonnegative.

6. (Absolute bound) The multiplicities $m_0 = 1, m_1, \ldots, m_d$ of an association scheme satisfy

$$\sum_{k:q_{i,j}^k>0} m_k \leq m_im_j, \quad \text{and} \quad \sum_{k:q_{i,i}^k>0} m_k \leq m_i(m_i+1)/2.$$

7. A connected strongly regular graph is distance-regular with diameter two.

8. [3, p.55] Let V' be a subset of the vertex set V of an association scheme with change-of-coordinates matrix P. The inner distribution $\mathbf{a}$ of V' satisfies $\mathbf{a}^TP^{-1} \geq \mathbf{0}$.

Examples:

1. The change-of-coordinates matrix P of a strongly regular graph with eigenvalues k, ν_2 and ν_3 is equal to

$$\begin{bmatrix} 1 & k & n-k-1 \\ 1 & \nu_2 & -\nu_2-1 \\ 1 & \nu_3 & -\nu_3-1 \end{bmatrix},$$

2. A strongly regular graph with parameters $(28, 9, 0, 4)$ cannot exist, because it violates Fact 5 and 6.

3. The Hamming association scheme $H(d, q)$ has vertex set $V = Q^d$, the set of all vectors with d entries from a finite set Q of size q. Two such vectors are adjacent in G_i if they differ in exactly i coordinate places. The graph G_1 is distance-regular. The matrix P of a Hamming association scheme can be expressed in terms of Kravčuk polynomials, which gives:

$$P[i,j] = \sum_{k=0}^{j} (-1)^k (q-1)^{j-k} \binom{i}{k} \binom{d-i}{j-k}.$$

4. An error correcting code with minimum distance δ is a subset V' of the vertex set V of a Hamming association scheme, such that V' induces a coclique in $G_1, \ldots, G_{\delta-1}$. If $\mathbf{a}$ is the inner distribution of V', then $a_0 = 1$, $a_1 = \ldots = a_{\delta-1} = 0$ and $|V'| = \sum_i a_i$. Therefore the linear programming problem 'Maximize $\sum_{i \geq \delta} a_i$, subject to $\mathbf{a}^T P^{-1} \geq 0$' leads to an upper bound for the size of an error correcting code with given minimum distance. This bound is known as Delsarte's Linear Programming Bound.

5. The Johnson association scheme $J(d, \ell)$ has as vertex set V all subsets of size d of a set of size ℓ ($\ell \geq 2d$). Two vertices are adjacent in G_i if the intersection of the two subsets has size $d - i$. The graph G_1 is distance-regular. The matrix P of a Johnson association scheme can be expressed in terms of Eberlein polynomials, which gives:

$$P[i,j] = \sum_{k=0}^{j} (-1)^k \binom{i}{k} \binom{d-i}{j-k} \binom{\ell-d-i}{j-k}.$$

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