A weakened form of …ctituous play in two-person zero-sum games

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Abstract

Fictitious play can be seen as a numeric iteration procedure for determining the value of a game and corresponding optimal strategies. Although convergence is slow, it needs only a modest computer storage. Therefore it seems to be a good way out for analyzing large games. In this paper we consider a weakened form of …ctitious play, which can be interpreted that players at each stage do not have to make the best choice against the total of past choices of the other player but only an increasingly better one. Theoretical bounds for convergence are derived. Furthermore it is shown that this new form can speed up convergence considerably in practice. The method is related to generalizations in which the game matrix itself becomes better known as the number of stages increases. Finally, the convergence of the strategies themselves is discussed.

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1 Introduction

Consider a two-person zero-sum game in strategic form with pay-off a_{ij} if player I uses strategy i and player II strategy j ($i = 1, \ldots, r$ and $j = 1, \ldots, s$). Let $A = (a_{ij}) \in \mathbb{R}^{r \times s}$ denote the corresponding pay-off matrix.

We are interested in calculation the value $v(A)$ of the game and corresponding optimal strategies of the players for large values of r and s. We have in mind that a_{ij} is a given simple function of (i, j) as is often the case for games arising in practice. For such games the usual LP-techniques are often not feasible. The computer memory needs the size $r \times s$ of all elements of A. Although memory in computers becomes larger and larger, this is often too large. Cases in which A is sparse are rare. Therefore we became interested in an old method refered to as fictitious play. In its basic form only a size $r + s$ of one row and one column of A is needed. The convergence to the game value $v(A)$ is much slower than that with LP. But far large r and s we may say that something is better than nothing. We can get rough estimates in a reasonable time and often this is the first step in the analysis for obtaining easy-to-remember approximations.

The idea of fictitious play goes back to Brown [1949, 1951]. The validity of this method was proved by Robinson [1950]. Shapiro [1958] provided a priori bounds for $v(A)$ of convergence rate $O(n^{-1/(r+s-2)})$. For literature concerning non-zero or more-person games we refer to the recent paper Krisha and Sjöström(1998).

Fictitious play can be seen as an infinite stage learning process where at each stage the players choose a pure strategy which, among the pure strategies, would have been the best against the total of past choices of the other player. In Vrieze and Tijs [1982] the method is extended to the situation in which the game matrix A is not exactly known in advance, but where at each state $n \in \mathbb{N}$ an approximation $A(n)$ is given, where $A(n) \to A$. A priori bounds are not given but its convergence to the game value $v(A)$ is established.

The purpose of this paper is fourfold. Firstly, in section 2 we weaken the idea of playing best against the past of the other player to increasingly better and admit in each stage mixtures of pure strategies as well. In this way convergence can be speed up considerably (this is shown in section 3). We give a priori bounds of the type of Shapiro [1958] leading to sufficient conditions for convergence. The proof is based on the main theorem in section 5 and is worked out in section 6.

Secondly, in section 3 we discuss the speed of convergence of generalized fictitious play. It will appear that in practice convergence takes place far beyond the theoretical Shapiro-bounds. We will illustrate this on random chosenly game matrices and various implementations of increasingly better.

Thirdly, in section 4 we pay attention to the convergence of the corresponding strategies of the players.

Fourthly, in section 7 we will show that the developed framework covers the generalization of Vrieze and Tijs [1982].

The paper concludes in section 8 with some remarks.

2 A generalized form of fictitious play

In this section we define a weaker form of fictitious play and state a priori bounds for the game value.

We write $N = \{1, 2, \ldots\}$ and $N_0 = \{0\} \cup N$. Row-vectors are denoted by upper indices and column vectors by lower indices. So for a given pay-off matrix $A \in R^{r \times s}$ its rows are in R^r and its columns in R_s. The simplex of mixed strategies of player I is $\Delta^r = \{p \in R^r : p_i \geq 0; \Sigma p_i = 1\}$ and that of player II $\Delta_s = \{q \in R_s : q_j \geq 0; \Sigma q_j = 1\}$.

Definition 2.1 For a given pay-off matrix $A \in R^{r \times s}$ we call the sequence $(U(n) \in R_r; V(n) \in R^s)_{n \in N_0}$ a vector system with update sequence $(\lambda(n) \in \Delta^r; \mu(n) \in \Delta_s)_{n \in N}$ if

$$U(n) = U(n-1) + A\mu(n) \tag{2.1}$$

$$V(n) = V(n-1) + \lambda(n)A. \tag{2.2}$$

The corresponding fictitious play sequences $(p(n) \in \Delta^r)_{n \in N}$ of player I and $(q(n) \in \Delta_s)_{n \in N}$ of player II are defined by

$$p(n) = \frac{1}{n} \sum_{k=1} \lambda(k) \tag{2.3}$$

$$q(n) = \frac{1}{n} \sum_{k=1} \mu(k). \tag{2.4}$$

We interprete $U_i(0)$ as player I's initial estimate of what he expects to win should he play his strategy i. Similarly, $V_j(0)$ is the initial estimate of the loss of player II should he play his strategy j.

From (2.1), (2.3) and (2.2), (2.4) it follows

$$U(n) = U(0) + nAq(n) \tag{2.5}$$

$$V(n) = V(0) + np(n)A: \tag{2.6}$$

So, $U_i(n)$ is the estimate of player I's total gain of his strategy i after stage n if player II plays his mixed strategy $q(n)$ build up so far. Similarly for $V_j(n)$ and player II.

There exists a simple relation between the $U(n); V(n)$ and the game values $v(A); v(A')$ where A' denotes the transpose of A. This is contained in the following lemma.

Lemma 2.2 For all $n \in N$:

$$\frac{1}{n}\{\min V(n) - \max V(0)\} \le v(A) \le \frac{1}{n}\{\max U(n) - \min U(0)\} \tag{2.7}$$

$$\frac{1}{n}\{\min U(n) - \max U(0)\} \le v(A') \le \frac{1}{n}\{\max V(n) - \min V(0)\}: \tag{2.8}$$

Proof. We only use (2.5) and (2.6). We have

$$\frac{1}{n}\{\min V(n) - \max V(0)\} \le \frac{1}{n}\min\{V(n) - V(0)\} = \min p(n)A$$
$$\le \min \max_p pA = v(A) \le \max \min_q Aq \le \max Aq(n)$$
$$\le \frac{1}{n}\max\{U(n) - U(0)\} \le \frac{1}{n}\{\max U(n) - \min U(0)\}$$

and this proves (2.7). Similarly, for (2.8):

$$\frac{1}{n}\{\min U(n) - \max U(0)\} \le \frac{1}{n}\min\{U(n) - U(0)\} = \min Aq(n) =$$
$$= \min q'(n)A' \le v(A') \le \max A'p(n) = \max p(n)A$$
$$\le \frac{1}{n}\max\{V(n) - V(0)\} \le \frac{1}{n}\{\max V(n) - \min V(0)\}.$$

∎

Remark. For reasons of symmetry we have included (2.8) in the lemma, although this relation is only used in the proof of lemma 5.3.

Relations (2.7),(2.8) suggest the notations (for $n \in N_0$):

$$\begin{cases} \Delta_{UV}(n) = \max U(n) - \min V(n) \\ \Delta_{VU}(n) = \max V(n) - \min U(n). \end{cases} \tag{2.9}$$

Then (2.7) can be rewritten in two ways

$$-\frac{1}{n}\max V(0) \le v(A) - \frac{1}{n}\min V(n) \le \frac{1}{n}(\Delta_{UV}(n) - \min U(0)) \tag{2.10}$$

$$\frac{1}{n}\min U(0) \le \frac{1}{n}\max U(n) - v(A) \le \frac{1}{n}(\Delta_{UV}(n) + \max V(0)). \tag{2.11}$$

Together with (2.5) and (2.6) it follows that

$$0 \le v(A) - \min p(n)A \le \frac{1}{n}\{\Delta_{UV}(n) + \Delta_{VU}(0)\} \tag{2.12}$$

$$0 \le \max Aq(n) - v(A) \le \frac{1}{n}\{\Delta_{UV}(n) + \Delta_{VU}(0)\}. \tag{2.13}$$

The relations (2.10) - (2.13) make clear that the condition

$$\Delta_{UV}(n)/n \to 0 \tag{2.14}$$

is crucial. Under this condition we see from (2.10) and (2.11) that $\min V(n)/n$ and $\max U(n)/n$ converge to $v(A)$. From (2.12) we get that the sequence $(p(n))_{n\in N}$ of player I is extended maxmin (i.e. for any $\varepsilon > 0$ there exists a N such that $p(n)$ is ε-maxmin for all $n \geq N$). In the same way it follows from (2.13) that $(q(n))_{n\in N}$ of player II is extended minmax. More precise statements follow from (2.10) - (2.13) if we can give bounds for $\Delta_{UV}(n)$.

Without further conditions for the update sequences $\lambda(n); \mu(n)$ we cannot come farther than the following lemma. We introduce the scale factor

$$a = \max_{i;j} a_{ij} - \min_{i;j} a_{ij}. \tag{2.15}$$

We exclude the trivial case of constant pay-offs by assuming throughout this paper that $a > 0$.

Lemma 2.3

$$-\Delta_{VU}(0) \leq \Delta_{UV}(n) \leq \Delta_{UV}(0) + na. \tag{2.16}$$

Proof. The first inequality of (2.16) follows immediately from (2.10). Using (2.5), (2.6) and (2.15) the second inequality follows from

$$\begin{aligned}
\Delta_{UV}(n) &= \max U(n) - \min V(n) \\
&= \max\{U(0) + nAq(n)\} - \min\{V(0) + np(n)A\} \\
&\leq \max U(0) + n \max a_{ij} - \min V(0) - n \min a_{ij} \\
&= \Delta_{UV}(0) + na.
\end{aligned}$$

■

Note that the second inequality in (2.16) is just not sufficient to guarantee that $\Delta_{UV}(n)/n \to 0$. We have to make additional assumptions for the update sequences $\lambda(n)$ and $\mu(n)$.

Basic fictitious play chooses at stage n for player I a pure strategy $i(n)$ for which $U_{i(n)}(n-1) = \max U(n-1)$ and for player II a pure strategy $j(n)$ for which $V_{j(n)}(n-1) = \min V(n-1)$. This corresponds to the unit row vector $\lambda(n) = e^{i(n)} \in \Delta^r$ and the unit column vector $\mu(n) = e_{j(n)} \in \Delta_s$. Then the update $\lambda(n)A$ from $V(n-1)$ to $V(n)$ is row $i(n)$ of A, and the update $A\mu(n)$ from $U(n-1)$ to $U(n)$ is column $j(n)$ of A. In case that these choices are not uniquely determined, mixtures $\lambda(n)$ and $\mu(n)$ of corresponding optimal unit vectors can be considered as well. In the following we introduce an even more weakened form in which strategies close to $\max U(n-1)$ and $\min V(n-1)$ can be included as well. The accuracy is specified by some $\gamma(n) \geq 0$.

Definition 2.4 For given $A \in R^{r\times s}$ we say that the update sequence $(\lambda(n) \in \Delta^r; \mu(n) \in \Delta_s)_{n\in N}$ of the vector system of definition 2.1 has the (non-negative) accuracy sequence $(\gamma(n))_{n\in N}$ if

$$\lambda_i(n) = 0 \text{ if } U_i(n-1) < \max U(n-1) - a\gamma(n) \tag{2.17}$$

$$\mu_j(n) = 0 \text{ if } V_j(n-1) > \min V(n-1) + a\gamma(n); \tag{2.18}$$

where the scale factor a of A is defined in (2.15).

A priori bounds for the game value are given by the following theorem.

Theorem 2.5 Consider the vector system of definition 2.4 with $r + s \ge 3$. If

$$0 \le \gamma(n) \le \delta(n) \text{ with } (\delta(n))_{n \in N} \text{ non-decreasing} \tag{2.19}$$

then

$$\Delta_{UV}(n) \le \Delta_{UV}(0) + a n^{\frac{r+s-3}{r+s-2}} \{c^{r+s} + 2\delta(n)c/(c-1)\}; \tag{2.20}$$

with $c \ge c_0$; where

$$c_0 \approx 1.696 \text{ (the unique real root of } c_0^3 - c_0^2 - 2 = 0\text{)}. \tag{2.21}$$

Proof. The proof is based on the main theorem 5.1 in section 5 and is completed in section 6. ■

Note that (2.20) holds for each $n \in N$. So with (2.10) - (2.13) we have a priori bounds for the differences $v(A) - \min V(n)/n$, $\max U(n)/n - v(A))$ and $v(A) - \min p(n)A$, $\max Aq(n) - v(A)$. The order of convergence to 0 depends on the sum $r + s$ of rows and columns of A and the bound $\delta(n)$ for the accuracy $\gamma(n)$ for the updates. Sufficient for (2.14) is

$$\delta(n) = o(n^{1/(r+s-2)}). \tag{2.22}$$

The special case of theorem 2.5 with $\delta(n) \equiv 0$ is a generalization of Shapiro [1958]: the order of n is the same but his constants are larger: $c = 2$ instead of (2.21) and $a = \max |a_{ij}|$ instead of (2.14). Our proof is based on an extension of the concept of eligibility. In section 6 we will make clear that in this way the order of n cannot be improved. The constant c can be replaced by more accurate expressions. In section 3 we investigate the order of convergence in practice.

3 A numeric evaluation

The weakened form of fictitious play as discussed in section 2 admits all kinds of choices for the initial values $U(0); V(0)$ and the update sequences $\lambda(n); \mu(n)$ dependent on the accuracy sequence $\gamma(n)$. In view of (2.17),(2.18) we write for $n \in N$:

$$I(n) = \{i : U_i(n-1) \ge \max U(n-1) - a\gamma(n)\} \tag{3.1}$$

$$J(n) = \{ j : V_j(n-1) \le \min V(n-1) + a\gamma(n) \}. \tag{3.2}$$

So $\lambda(n)$ is restricted to $I(n)$ and $\mu(n)$ to $J(n)$. We are free in choosing $\lambda(n)$ and $\mu(n)$ on these sets. Convergence to the game value $v(A)$ is proved for $\gamma(n) = \delta(n)$ satisfying (2.22). We will only consider choices independent of the way pure strategies are indexed.

Throughout this section we assume that

$$U(0) = 0, \quad V(0) = 0. \tag{3.3}$$

This seems to be the obvious choice. The bounds (2.10),(2.11) take a simple form.

For $n = 1$, $\lambda(1)$ and $\mu(1)$ are arbitrary since from (3.3) we get $I(1) = \{1, \ldots, r\}$, $J(1) = \{1, \ldots, s\}$.

One simple choice is to take the uniform distribution:

$$\begin{cases} \lambda(1) \text{ uniform on } \{1, \ldots, r\} \\ \mu(1) \text{ uniform on } \{1, \ldots, s\}. \end{cases} \tag{3.4}$$

Another choice is to take $\lambda(1)$ maxmin and $\mu(1)$ minmax under the pure strategies:

$$\begin{cases} \lambda(1) \text{ uniform on } \{ i_0 : \min_j a_{i_0 j} = \max_i \min_j a_{ij} \} \\ \mu(1) \text{ uniform on } \{ j_0 : \max_i a_{i j_0} = \min_j \max_i a_{ij} \}. \end{cases} \tag{3.5}$$

Although (3.5) is somewhat more difficult to start with than (3.4), for games with a saddlepoint convergence can be achieved in one step (take $\gamma(n) = 0$).

For $n \ge 2$ the simpliest choice is

$$\begin{cases} \lambda(n) \text{ uniform on } I(n) \\ \mu(n) \text{ uniform on } J(n). \end{cases} \tag{3.6}$$

The drawback of this choice can be that obtained optimal strategies are disturbed again in the next stage if we take the uniform distribution. Therefore perhaps a better idea is to choose $\lambda(n)$ in such a way that on $I(n)$ we change $p(n-1)$ to $p(n)$ only proportionally; similarly on $J(n)$ for $q(n-1)$ to $q(n)$. More precisely,

$$\begin{cases} \lambda_i(n) = \begin{cases} 1/\#I(n) & \text{if } \sum_{k \in I(n)} p_k(n-1) = 0 \\ p_i(n-1) \Big/ \sum_{k \in I(n)} p_k(n-1) & \text{if } i \in I(n) \end{cases} \\ \mu_j(n) = \begin{cases} 1/\#J(n) & \text{if } \sum_{k \in J(n)} q_k(n-1) = 0 \\ q_j(n-1) \Big/ \sum_{k \in J(n)} q_k(n-1) & \text{if } j \in J(n) \end{cases} \end{cases} \tag{3.7}$$

Indeed, if $\sum_{k \in I(n)} p_k(n-1) > 0$ then $p_i(n)/p_i(n-1) = (n-1)/n + \lambda_i(n)/p_i(n-1)$ does not depend on $i \in I(n)$; similarly for $q_j(n)/q_j(n-1) = (n-1)/n + \mu_j(n)/q_j(n-1)$. To make the arguments more precise, consider the extreme case that $\gamma(n) \equiv 0$ and that at stage n we have that $I(n)$ and $J(n)$ are the sets of the reduced game of A with $p(n-1)$ maxmin and $q(n-1)$ minmax. So $I(n) = I$ and $J(n) = J$ with

$$I = \{i : p_i > 0 \text{ for some } p \in \Delta^{(0)}\} \tag{3.8}$$

$$J = \{j : q_j > 0 \text{ for some } q \in \Delta_{(0)}\}, \tag{3.9}$$

where $\Delta^{(0)}$ denotes the set of optimal maxmin strategies of player I and $\Delta_{(0)}$ the set of minmax strategies of player II (see e.g. Karlin[1959], theorem 3.1.1). Since $q_j(n-1) = 0$ for all $j \notin J = J(n)$ we get from (3.7) that $\mu(n) = q(n-1)$. This implies $q(n) = q(n-1)$ and therefore $U(n)/n = Aq(n) = Aq(n-1) = U(n-1)/(n-1)$. Since $\gamma(n) \equiv 0$ this implies $I(n+1) = I(n)$. In the same way we get $p(n) = p(n-1)$ and $J(n+1) = J(n)$. So (3.7) maintains optimality once it is obtained. More general, we expect that the choice (3.7) does not disturb already good strategies. Of course, the drawback can be that it does not improve bad strategies as well.

The important question is how these choices behave in practice. After some preliminary tests we observed that convergence is much faster than the theoretical bound of section 2 suggests. So we performed a large scale experiment to analyze weakened fictitious play in practice.

For several fixed choices of the dimension (r, s) we generated game matrices A with elements drawn at random. More precisely, we always took $a_{11} = 1$ and $a_{r1} = 0, a_{1s} = 0, a_{rs} = 0$ with probabilities $(r-1)/(rs-1)$, $(s-1)/(rs-1)$, $1-(r+s-2)/(rs-1)$, respectively. The other $rs-2$ elements were drawn from the uniform distribution on $(0, 1)$. So for all such matrices A the scale factor of (2.15) is $a = 1$.

For each matrix A the initial choice was varied between (3.4) and (3.5), the update sequences between (3.6) and (3.7), and the accuracy sequences $\gamma(n)$ according to

$$\gamma(n) = dn^{\delta} \tag{3.10}$$

with values $d = 0, 0.1, 0.2, 0.5, 1$ and $\delta = 0, 0.01, 0.05, 0.1, 0.4$ (in fact we replaced $d = 0$ by $d = 10^{-5}$ for numerical reasons; the value $\delta = 0.5$ was canceled after some trials because too often convergence did not take place). This leads to a design of $2 \times 2 \times 5 \times 5 = 100$ combinations per matrix. For each combination the iteration was continued to the smallest n for which $\Delta_{UV}(n)/n < 10^{-2}$. The ideas behind this rather large value are to save computer time and that in practice more accurate results are not needed at all.

For each combination we registered all kind of data such as the number of iterations n, the computer time, the game value and the LS-estimate of the equation

$$\log \Delta_{UV}(k)/k = r + \rho \log k, \quad k = 1, \ldots, n \tag{3.11}$$

(only if the game had no saddlepoint). According to (2.19),(2.20) and (3.11) the value of ρ should be compared with the theoretical order $(r+s-3)/(r+s-2)-1+\delta = \delta - 1/(r+s-2)$. Graphical inspection of the estimated equation showed a good fit in almost all cases.

To be specific, for $r = s = 5$ our first generated matrix A was

$$A = \begin{bmatrix} 1 & 0.7621 & 0.6154 & 0.4057 & 0 \\ 0.2311 & 0.4565 & 0.7919 & 0.9355 & 0.3529 \\ 0.6068 & 0.0185 & 0.9218 & 0.9169 & 0.8132 \\ 0.4860 & 0.8214 & 0.7382 & 0.4103 & 0.0099 \\ 0.8913 & 0.4447 & 0.1763 & 0.8936 & 0.1389 \end{bmatrix}.$$

This matrix has no saddlepoint. For the combination of basic fictitious play

I: (3.4),(3.6), $d = 0$; $\delta = 0$

we find $n = 3772$ iterations with LS-estimate $\rho = -0.497$. This estimate is much lower than the theoretical bound $\delta - 1/(r+s-2) = -0.125$. For the combination of really weakened fictitious play

II: (3.5),(3.6), $d = 0.5$; $\delta = 0.4$

we need $n = 2847$ iterations giving LS-estimate $\rho = -0.532$. Now the theoretical bound is $\rho = +0.275$, a positive value. So convergence is now not guaranteed by the theory.

Of course we should not draw general conclusions on the base of one 5×5 matrix. We generated 100 game matrices of this size and calculated for all registered values the means per combination (standard deviations were rather large). For the combination I we find a mean of $n = 1927$ iterations and a mean of LS-estimates of $\rho = -0.552$. For II we find similarly means $n = 678.9$ and $\rho = -0.591$. So we see that II performs much better than I. In fact II gives the best means for n and ρ under all 100 combinations. Surprisingly, the worst combination is (3.5),(3.7), $d = 1$; $\delta = 0.4$ with a mean of $n = 9559$ iterations.

We repeated the whole analysis for game matrices of different sizes. Table 3 gives some results for the means of 100 simulated matrices for each size.

TABLE 3

size	n		ρ	
	I	II	I	II
(5,5)	1927	679	-0.552	-0.591
(5,15)	3169	1032	-0.501	-0.535
(10,10)	6017	1770	-0.527	-0.574
(5,100)	2758	956	-0.544	-0.608

In all cases combination II is the best among the 100 combinations (although the effect of replacing (3.5) with (3.4) is small). The size of the matrices has not much influence on the order of convergence. This is quite surprising because the Shapiro-bound depends strongly on this size.

It must be admitted the sizes in table 3 is rather small. Therefore we designed a second experiment with matrices of moderate and large size. On the base of the outcome of the first experiment we varied between (3.4) and (3.5), took (3.6) and $d = 0.5$ fixed and varied $\delta = 0.30;\ 0.35;\ 0.40$. This generates $2 \times 1 \times 1 \times 3 = 6$ combinations. The result for the combinations with the minimal iteration steps is presented in table 3.

TABLE

size	start	δ	n	ρ
(100,100)	(3.4)	0.40	10032	-0.365
(100,500)	(3.4)	0.35	12010	-0.348
(5,1000)	(3.4)	0.35	6071	-0.704
(25,1000)	(3.5)	0.35	5690	-0.426
(100,1000)	(3.5)	0.40	13125	-0.345
(1000,1000)	(3.5)	0.35	3235	-0.458

Again the influence of the choice between (3.4) and (3.5) is small. For moderate sizes not the sum $r + s$ but the difference $|r - s|$ seems to have a reducing effect on the order of convergence.

The generated matrices above are of a rather particular type. They have the property that the optimal strategies are uniquely determined. It is not clear what the effect is of the combinations for games with not uniquely determined optimal strategies. For now on the base of the foregoing analysis it seems save to advise for large matrices weakened fictitious play with (3.4) or (3.5), (3.6), $d = 0.5a;\ \delta = 0.35$. Then the rate of convergence can be expected to lay somewhere between $n^{-1/3}$ and $n^{-1/2}$ independent of the size of the game matrix.

4 Convergence of strategies

In section 2 it has been shown that under the condition (2.14) for A the sequence $(p(n))_{n\in\mathbb{N}}$ of player I is extended maxmin and $(q(n))_{n\in\mathbb{N}}$ of player II is extended minmax.

In general, given extended maxmin $p(n)$ and extended minmax $q(n)$ we have

$$v(A) = \lim_{n\to\infty} \min_q p(n)Aq = \lim_{n\to\infty} p(n)Aq(n) = \lim_{n\to\infty} \max_p pAq(n).$$

So, if $p(n) \to p(0)$ then $p(0)$ is necessarily maxmin and if $q(n) \to q(0)$ then $q(0)$ is minmax.

Since Δ^r is compact, the set of limit points of $(p(n))_{n \in \mathbb{N}}$ is not empty. It follows that any limit point of $(p(n))_{n \in \mathbb{N}}$ is maxmin. So, if A is such that the set of maxmin strategies $\Delta^{(0)}$ contains only one point $p(0)$, then necessarily $p(n) \to p(0)$. However, $\Delta^{(0)}$ can contain more than one point and then $p(n)$ need not to converge. The same remarks apply to $q(n) \in \Delta_s$ and the set $\Delta_{(0)}$ of minmax strategies.

So the following question arises from section 3: under (3.3) and given $\lambda(1); \mu(1)$, can $(\lambda(n); \mu(n))_{n \geq 2}$ be defined in such a way that convergence of $p(n)$ and $q(n)$ takes place?

At first glance the sequences given by (3.7) seem to be a good candidate. This is based on the following implications:

$$\left(\begin{array}{c} \sum_{k \in I(n)} p_k(n-1) > 0 \\ I(n+1) = I(n) \end{array} \right) \Rightarrow \left(\begin{array}{c} \sum_{k \in I(n+1)} p_k(n) > 0 \\ \lambda(n+1) = \lambda(n) \end{array} \right) \tag{4.1}$$

$$\left(\begin{array}{c} \sum_{k \in J(n)} q_k(n-1) > 0 \\ J(n+1) = J(n) \end{array} \right) \Rightarrow \left(\begin{array}{c} \sum_{k \in J(n+1)} q_k(n) > 0 \\ \mu(n+1) = \mu(n): \end{array} \right) \tag{4.2}$$

Clearly, since $np_i(n) = (n-1)p_i(n-1) + \lambda_i(n)$ for all i, we see with (3.7) that for $I(n) = I(n+1)$:

$$\sum_{k \in I(n+1)} np_k(n) = \sum_{k \in I(n)} (n-1)p_k(n-1) + \sum_{k \in I(n)} \lambda_k(n) =$$
$$= (n-1) \sum_{k \in I(n)} p_k(n-1) + 1 > 0$$

and so for $i \in I(n+1) = I(n)$ wih (3.7)

$$\lambda_i(n+1) = \frac{p_i(n)}{\sum_{k \in I(n+1)} p_k(n)} = \frac{(n-1)p_i(n-1) + \lambda_i(n)}{(n-1) \sum_{k \in I(n)} p_k(n-1) + 1} = \lambda_i(n):$$

The proof of (4.2) follows in the same way.

Now suppose that for some $N \geq 2$ we have

$$\sum_{k \in I(N)} p_k(N-1) > 0; \quad I(n) = I(N) \text{ for } n \geq N:$$

Then (4.1) implies

$$p(n) = \frac{N}{n}p(N) + (1 - \frac{N}{n})\lambda(N) \to \lambda(N); \quad n \to \infty:$$

In particular, $\lambda(N)$ is maxmin since it is the limit of extended maxmin $p(n)$. Similar conclusions can be drawn for $q(n)$ using (4.2).

So in this case convergence of $p(n)$ and $q(n)$ have been brought back to the question of convergence of the corresponding sets $I(n)$ and $J(n)$. However, it is still an open question whether this is true or not.

5 The main theorem

In this section we give the main theorem 5.1 on which the bounds of theorem 2.5 are based.

We call the vector system $(U(n); V(n))$ of definition 2.1 initially balanced of

$$\max U(0) = \min V(0) = 0: \tag{5.1}$$

With any $(U(n); V(n))$ there corresponds an initially balanced system $(\bar{U}(n); \bar{V}(n))$ defined by

$$\bar{U}(n) = U(n) - \max U(0).1^r; \ \bar{V}(n) = V(n) - \min V(0).1_s:$$

Since

$$\Delta_{\bar{U}\bar{V}}(n) = \Delta_{UV}(n) - \Delta_{UV}(0);$$

bounds for $\Delta_{\bar{U}\bar{V}}(n)$ can be transformed immediately to bounds for $\Delta_{UV}(n)$.

In the following we only consider initially balanced systems $(U(n); V(n))$. Then (2.20) reads as

$$\Delta_{UV}(n) \le aC_{r;s}(n; \delta(n)); \tag{5.2}$$

where

$$C_{r;s}(n; \delta) = n^{\frac{r+s-3}{r+s-2}}\{c^{r+s} + 2\delta c/(c-1)\}: \tag{5.3}$$

We split up the proof into two parts. The first part is given by theorem 5.1 of this section. Its proof is mainly game-theoretic. This main theorem only specifies sufficient conditions for the function $C_{r;s}(n; \delta)$. The second part is the proof of theorem 2.5 and this is purely analytical. In section 6 we show that $C_{r;s}(n; \delta)$ given by (5.3) satisfies the conditions of theorem 5.1.

Theorem 5.1 Consider the vector system $(U(n); V(n))$ of definition 2.4 with $\delta(n)$ satisfying (2.19) and suppose that it is initially balanced in the sense of (5.1). For given integers r_0, s_0 with $1 \le r_0 \le r$, $1 \le s_0 \le s$ let $C_{g,h}(n;\delta)$ be defined for (integer) g, h with $r_0 \le g \le r$, $s_0 \le h \le s$, for $n \in \mathbb{N}$ and for $\delta > 0$ such that

$$\left\{ \begin{array}{l} C_{g,h}(n;\delta) \text{ is non-decreasing in } \delta \text{ if } g + h < r + s \\ C_{g,h}(n;\delta(n)) \ge 1 \text{ if } g + h > r_0 + s_0. \end{array} \right. \tag{5.4}$$

If for any member $(\hat{U}(n), \hat{V}(n))$ of the set of all initially balanced vector systems with respect to all submatrices $\in \mathbb{R}^{r_0 \times s_0}$ of A (with the fixed a and $\gamma(n)$ of definition 2.4) we have:

$$\Delta_{\hat{U}\hat{V}}(n) \le aC_{r_0,s_0}(n;\delta(n)); \tag{5.5}$$

then (5.2) is fulfilled as well, provided that for each fixed g, h the following condition is fulfilled: for each $n > C_{g,h}(n;\delta(n))$ we can find an integer T with $0 < T < n$ such that

$$\left\{ \begin{array}{l} \text{if } g > 1, h > 1: \\ \quad 2\left\lceil \frac{n-1}{2T} \right\rceil \max\{C_{g-1,h}(T;\delta(n)), C_{g,h-1}(T;\delta(n))\} + \\ \qquad\qquad + 2T + 2\delta(n) \le C_{g,h}(n;\delta(n)) \\ \text{if } g = 1, h > 1: \\ \quad \left\lceil \frac{n-1}{T} \right\rceil C_{1,h-1}(T;\delta(n)) + T + \delta(n) \le C_{1,h}(n;\delta(n)) \\ \text{if } g > 1, h = 1: \\ \quad \left\lceil \frac{n-1}{T} \right\rceil C_{g-1,1}(T;\delta(n)) + T + \delta(n) \le C_{g,1}(n;\delta(n)). \end{array} \right. \tag{5.6}$$

Proof. The lemma 5.4 below and its remark thereafter is just the case that $r_0 + s_0 = r + s - 1$. The general case follows by induction, starting with (r_0, s_0) going through pairs (g, h) with $r_0 + s_0 + 1$ and so on up to the pair (r, s). ■

Remark 1. In (5.5) for the trivial case $r_0 = s_0 = 1$ we can take $C_{1,1}(n;\delta) \equiv 0$. (Note that this does not contradict (5.4)).

Remark 2. Explicit bounds can be derived as well for $(r_0, 1)$ and $(1, s_0)$ with arbitrary r_0 and s_0. We omit the details. So in general this theorem can be used in several ways.

Before we can state lemma 5.4 we need two preparetory lemma's.

Lemma 5.2 Consider the vector system $(U(n), V(n))$ of definition 2.4. Take fixed integers T, n with $0 < T < n$.

a) Let $r \ge 2$. Suppose strategie i of player I is not eligible in the interval $[n - T, n]$, i.e.

$$U_i(t-1) < \max U(t-1) - a\gamma(t); \quad n - T \le t \le n. \tag{5.7}$$

For $\tau = 0, \ldots, T$ let

$$\left\{ \begin{array}{l} \hat{V}(\tau) = V(n - T + \tau) - \min V(n - T).1^s \\ \hat{U}(\tau) = U^{(i)}(n - T + \tau) - \max U(n - T).1_{r-1}. \end{array} \right. \tag{5.8}$$

Then $(\hat{U}(\tau) \in \mathbb{R}_{r-1}, \hat{V}(\tau) \in \mathbb{R}^s)_{0 \le \tau \le T}$ is the start of an initially balanced vector system with accuracy coefficients $(\gamma(n - T + \tau))_{1 \le \tau \le T}$ with respect to $A^{(i)} \in \mathbb{R}^{(r-1)\times s}$. (Here the upper index (i) means dropping the i^{th} row of A.)

b) Let $s \geq 2$. Suppose strategy j of player II is not eligible in the interval $[n - T; n]$, i.e.

$$V_j(t - 1) > \min V(t - 1) + a\gamma(t); \quad n - T \leq t \leq n: \tag{5.9}$$

For $\tau = 0; \dots; T$. Let

$$\left\{ \begin{array}{l} \hat U(\tau) = U(n - T + \tau) - \max U(n - T) \cdot 1_r \\ \hat V(\tau) = V_{(j)}(n - T + \tau) - \min V(n - T) \cdot 1^{s-1}: \end{array} \right. \tag{5.10}$$

Then $(\hat U(\tau) \in R_r, \hat V(\tau) \in R^{s-1})_{0 \leq \tau \leq T}$ is the start of an initially balanced vector system with accuracy coefficients $(\gamma(n - T + \tau))_{1 \leq \tau \leq T}$ with respect to $A_{(j)} \in R^{r \times (s-1)}$. (Here the lower index (j) means dropping the j^{th} column of A:)

Proof.

a) Since i is not eligible it follows from (5.7) that for all $0 \leq \tau \leq T$:

$$U_i(n - T + \tau - 1) < \max U(n - T + \tau - 1) - a\gamma(n - T + \tau);$$

implying $\lambda_i(n - T + \tau) = 0$. Define $\hat\gamma(\tau) = \gamma(n - T + \tau)$ and

$$\hat\lambda_k(\tau) = \lambda_k(n - T + \tau); \quad k \in \{1; \dots; i - 1; i + 1; \dots; r\}$$

$$\hat\mu_k(\tau) = \mu_k(n - T + \tau); \quad k \in \{1; \dots; s\}:$$

Then we have

$$\begin{aligned} V(n - T + \tau) &= V(n - T + \tau - 1) + \lambda(n - T + \tau)A \\ &= V(n - T + \tau - 1) + \hat\lambda(\tau)A^{(i)} \end{aligned}$$

with $\hat\lambda_k(\tau) = 0$ if

$$U_k(n - T + \tau - 1) < \max U(n - T + \tau - 1) - a\gamma(n - T + \tau);$$

and

$$\begin{aligned} U(n - T + \tau) &= U(n - T + \tau - 1) + A^{1}(n - T + \tau) \\ &= U(n - T + \tau - 1) + A\hat\mu(\tau) \end{aligned}$$

with $\hat\mu_k(\tau) = 0$ if

$$V_k(n - T + \tau - 1) > \min V(n - T + \tau - 1) + a\gamma(n - T + \tau):$$

With (5.8) this implies

$$\hat V(\tau) = \hat V(\tau - 1) + \hat\lambda(\tau)A^{(i)}$$

with $\hat\lambda_k(\tau) = 0$ if $\hat U_k(\tau - 1) < \max \hat U(\tau - 1) - a\hat\gamma(\tau)$ and

$$\hat U(\tau) = \hat U(\tau - 1) + A^{(i)}\hat\mu(\tau)$$

with $\hat\mu_k(\tau) = 0$ if $\hat V_k(\tau - 1) > \min \hat V(-1) + a\hat\gamma(\tau)$. Clearly, $\max \hat U(0) = 0$ and $\min \hat V(0) = 0$ and so $(\hat U(n); \hat V(n))$ is initially balanced.

b) Since j is not eligible it follows from (5.9) that for all $0 \le \tau \le T$

$$V_j(n - T + \tau - 1) > \min V(n - T + \tau - 1) + a\gamma(n - T + \tau);$$

implying $\mu_j(n - T + \tau) = 0$. Define $\hat{\gamma}(\tau) = \gamma(n - T + \tau)$ and

$$\hat{\lambda}_k(\tau) = \lambda_k(n - T + \tau) \quad ; \quad k \in \{1, \ldots, r\}$$

$$\hat{\mu}_k(\tau) = \mu_k(n - T + \tau) \quad ; \quad k \in \{1, \ldots, j-1, j+1, \ldots, s\}.$$

Then we have

$$\begin{aligned} U(n - T + \tau) &= U(n - T + \tau - 1) + A\mu(n - T + \tau) \\ &= U(n - T + \tau - 1) + A_{(j)}\hat{\mu}(\tau) \end{aligned}$$

with $\hat{\mu}_k(\tau) = 0$ if

$$V_k(n - T + \tau - 1) > \min V(n - T + \tau - 1) + a\gamma(n - T + \tau);$$

and

$$\begin{aligned} V(n - T + \tau) &= V(n - T + \tau - 1) + \lambda(n - T + \tau)A \\ &= V(n - T + \tau - 1) + \hat{\lambda}(\tau)A \end{aligned}$$

with $\hat{\lambda}_k(\tau) = 0$ if

$$U_k(n - T + \tau - 1) < \max U_k(n - T + \tau - 1) - a\gamma(n - T + \tau).$$

With (5.10) this implies

$$U(\tau) = \hat{U}(\tau - 1) + A_{(j)}\hat{\mu}(\tau)$$

with $\hat{\mu}_k(\tau) = 0$ if $\hat{V}_k(\tau - 1) > \min \hat{V}(\tau - 1) + a\hat{\gamma}(\tau)$ and

$$\hat{V}(\tau) = \hat{V}(\tau - 1) + \hat{\lambda}(\tau)A_{(j)}$$

with $\hat{\lambda}_k(\tau) = 0$ if $\hat{V}_k(\tau - 1) < \max \hat{U}(\tau - 1) - a\hat{\gamma}(\tau)$. Clearly, $\max \hat{U}(0) = 0$ and $\min \hat{V}(0) = 0$.

■

Lemma 5.3 Consider the initially balanced vector system $(U(n), V(n))$ of definition 2.4 satisfying (2.19) and (5.1) and let $r \ge 2$; $s \ge 2$. Suppose for $(g, h) = (r, s - 1)$ and $(r - 1, s)$ that

$$C_{g,h}(n; \delta) \text{ is non-decreasing in } \delta \qquad (5.11)$$

and for any initially balanced vector system $(\hat{U}(n); \hat{V}(n))$ of submatrices $\in R^{(r-1)\times s}$ or $R^{r\times(s-1)}$ of Δ we have

$$\Delta_{\hat{U}\hat{V}}(n) \leq aC_{g;h}(n; \delta(n)). \tag{5.12}$$

Take integer T, n with $0 < T < n$. Then, either

$$\Delta_{UV}(n) \leq 2a(T + \gamma(n)) \tag{5.13}$$

or

$$\Delta_{UV}(n) - \Delta_{UV}(n - T) \leq a \max\{C_{r-1;s}(T; \delta(n)), C_{r;s-1}(T; \delta(n))\}. \tag{5.14}$$

Proof. We introduce (compare the notation (2.9))

$$\left\{ \begin{array}{l} \Delta_{UU}(n) = \max U(n) - \min U(n) \\ \Delta_{VV}(n) = \max V(n) - \min V(n). \end{array} \right. \tag{5.15}$$

It follows that

$$\Delta_{UV}(n) + \Delta_{VU}(n) \leq \Delta_{UU}(n) + \Delta_{VV}(n).$$

From (2.8) in lemma 2.2 we get $\Delta_{VU}(n) + \Delta_{UV}(0) \geq 0$. So with (5.2) we see that $\Delta_{VU}(n) \geq 0$. Hence,

$$\Delta_{UV}(n) \leq \Delta_{UU}(n) + \Delta_{VV}(n).$$

Suppose (5.13) is false. Then either $\Delta_{UU}(n) > a(T + \gamma(n))$ or $\Delta_{VV}(n) > a(T + \gamma(n))$ or both. Then it remains to prove that (5.14) holds.

a) Suppose $\Delta_{UU}(n) > a(T + \gamma(n))$. Let i, j be such that $U_j(n) = \max U(n)$ and $U_i(n) = \min U(n)$. Then we get with (2.4) and (2.5) that for $0 \leq w \leq T$:

$$\begin{aligned} U_i(n-w) - U_j(n-w) &= U_i(n) - U_j(n) - \sum_{k=n-w+1} \{(A^1(k))_j - (A^1(k))_i\} \\ &\geq \max U(n) - \min U(n) - wa = \Delta_{UU}(n) - wa \\ &> a(T-w) + a\gamma(n) \geq a\gamma(n). \end{aligned}$$

Hence, i not eligible on $[n - T, n]$. Define the deleted system $(\hat{U}(n); \hat{V}(n))$ with respect to $A^{(i)}$ as in lemma 5.2,a. Then with (5.12) we get

$$\Delta_{\hat{U}\hat{V}}(T) \leq C_{r-1;s}(T; \delta(n-T)) \leq C_{r-1;s}(T; \delta(n)).$$

Since

$$\max \hat{U}(T) = \max U^{(i)}(n) - \max U^{(i)}(n-T) = \max U(n) - \max U(n-T)$$

$$\min \hat{V}(T) = \min V(n) - \min V(n-T)$$

this gives

$$\Delta_{UV}(n) - \Delta_{UV}(n-T) \leq aC_{r-1,s}(T, \delta(n));$$

proving one part of (5.14).

b) Suppose $\Delta_{VV}(n) > a(T + \gamma(n))$. Let i, j be such that $V_j(n) = \max V(n)$ and $V_i(n) = \min V(n)$. Then we get with (2.3) and (2.6) that for $0 \leq w \leq T$:

$$\begin{aligned} V_j(n-w) - V_i(n-w) &= V_j(n) - V_i(n) - \sum_{k=n-w+1} \{(\lambda(k)A))_j - (\lambda(k)A))_i\} \\ &\geq \max V(n) - \min V(n) - wa = \Delta_{VV}(n) - wa \\ &> a(T-w) + a\gamma(n) \geq a\gamma(n): \end{aligned}$$

Hence, j not eligible on $[n-T, n]$. Define $(\hat{U}(n), \hat{V}(n))$ with respect to $A_{(j)}$ as in lemma 5.2,b. Then with (5.12) we get

$$\Delta_{\hat{U}\hat{V}}(T) \leq C_{r-1,s}(T, \delta(n-T)) \leq C_{r-1,s}(T, \delta(n)):$$

Since

$$\max \hat{U}(T) = \max U(n) - \max U(n-T)$$

$$\min \hat{V}(T) = \min V_{(j)}(n) - \min V^{(i)}(n-T) = \min V(n) - \min V(n-T)$$

this gives

$$\Delta_{UV}(n) - \Delta_{UV}(n-T) \leq aC_{r,s-1}(T, \delta(n));$$

proving the other part of (5.14).

■

Remark. In the same way it follows:

for $r = 1, s > 1$: either

$$\Delta_{UV}(n) \leq a(T + \gamma(n))$$

or

$$\Delta_{UV}(n) - \Delta_{UV}(n-T) \leq aC_{r,s-1}(T, \delta(n));$$

for $r > 1, s = 1$: either

$$\Delta_{UV}(n) \leq a(T + \gamma(n))$$

or

$$\Delta_{UV}(n) - \Delta_{UV}(n-T) \leq aC_{r-1,s}(T, \delta(n)):$$

Lemma 5.4 Let $r \geq 2, s \geq 2$. Suppose for $(g,h) = (r,s-1)$ and $(r-1,s)$ that (5.11) and (5.12) are fulfilled and that

$$C_{g,h}(n,\delta(n)) \geq 1. \tag{5.16}$$

Then (5.2) holds under the following condition: for each $n > C_{r,s}(n,\delta(n))$ we can find an integer T with $0 < T < n$ such that

$$2\left\lfloor\frac{n-1}{2T}\right\rfloor \max\{C_{r-1,s}(T,\delta(n)), C_{r,s-1}(T,\delta(n))\} + 2T + 2\delta(n) \leq C_{r,s}(n,\delta(n)). \tag{5.17}$$

Proof. If $n \leq C_{r,s}(n,\delta(n))$ then with (2.16) and (5.1):

$$\Delta_{UV}(n) \leq an \leq aC_{r,s}(n,\delta(n))$$

and this proves (5.2). So we need only consider the case that $n > C_{r,s}(n,\delta(n))$. Then from (5.16) it follows that $n \geq 2$. Choose T with $0 < T < n$ according to (5.17).

We follow the terminology of Shapiro[1958]. In view of lemma 5.3 we call n an integer of the first kind if $\Delta_{UV}(n) \leq 2a(T + \gamma(n))$. Otherwise we call n an integer of the second kind. Since $\Delta_{UV}(n) \leq na$ we see that n is of the first kind for all $n \leq 2T$.

Take $q = \left\lfloor\frac{n-1}{2T}\right\rfloor \geq 0$. Then $n - 2qT \leq 2T$, so $n - 2qT$ is of the first kind. Then among $n-2qT, n-(2q-1)T, \ldots, n-T, n$ there is a largest integer $n - \tau T$ of the first kind. Write

$$\Delta_{UV}(n) = \sum_{i=1}^{\tau} \{\Delta_{UV}(n-(i-1)T) - \Delta_{UV}(n-iT)\} + \Delta_{UV}(n-\tau T).$$

Since $n - iT + T > n - \tau T$ for all $i = 1, \ldots, \tau$ we have that $n - iT + T$ is of the second kind. Hence, since $0 \leq \tau \leq 2q \leq 2[(n-1)/(2T)]$, we get with (2.20) and lemma 5.3:

$$\begin{aligned}
\Delta_{UV}(n) &\leq a\tau \cdot \max\{C_{r-1,s}(T,\delta(n)), C_{r,s-1}(T,\delta(n))\} + 2a(T+\gamma(n)) \\
&\leq 2a[(n-1)/(2T)] \cdot \max\{C_{r-1,s}(T,\delta(n)), C_{r,s-1}(T,\delta(n))\} + 2a(T+\delta(n)) \\
&\leq aC_{r,s}(n,\delta(n))
\end{aligned}$$

and this proves (5.2). ■

Remark. If $r = 1, s > 1$ or $r > 1, s = 1$ then we can proceed in a similar way. We have to modify the definition of integers of the first kind by using the inequality $\Delta_{UV}(n) \leq a(T + \gamma(n))$. Then $n - qT \leq T$ and so $n - qT$ is of the first kind. This leads to:

$r = 1, s > 1$:

$$\left\lfloor\frac{n-1}{T}\right\rfloor C_{1,s-1}(T,\delta(n)) + T + \delta(n) \leq C_{1,s}(n,\delta(n))$$

$r > 1, s = 1$:

$$\left\lfloor\frac{n-1}{T}\right\rfloor C_{r-1,1}(T,\delta(n)) + T + \delta(n) \leq C_{r,1}(n,\delta(n)).$$

6 The proof of theorem 2.5

In this section we will show how theorem 2.5 follows from the main theorem 5.1. In particular, we have to prove that $C_{r,s}(n;\delta)$ given by (5.3) satisfies the conditions of theorem 5.1. We will not simply verify this by substitution but we describe the way the expressions are derived.

We apply theorem 5.1 for $r_0 = s_0 = 1$. Then the condition (5.5) is fulfilled in a trivial way (see the first remark of theorem 5.1). We restrict ourselves to look at functions $C_{g,h}$ only depending on the sum $g + h$. So we write

$$C_{g,h}(n;\delta) = C_{g+h}(n;\delta). \tag{6.1}$$

Then (5.4) can be replaced by the slightly stronger condition

$$\left\{\begin{array}{l} C_k(n;\delta) \text{ is non-decreasing in } \delta \text{ for } k < r + s \\ C_k(0) \geq 1 \text{ for } k > 2. \end{array}\right. \tag{6.2}$$

In view of (6.2) the condition containing (5.6) can be strengthened to:

for each $n > C_k(n;0)$ we can find an integer $0 < T < n$ such that for all $k = 4,\ldots,r+s$:

$$\frac{n}{T}C_{k-1}(T;\delta(n)) + 2T + 2\delta(n) \leq C_k(n;\delta(n)) \tag{6.3}$$

where $C_2(n;\delta) \equiv 0$.

With $k \geq 3$ we try the form

$$C_k(n;\delta) = (1 + \gamma_k\delta)C_k(n) \tag{6.4}$$

for suitable γ_k and with

$$1 \leq C_k(1) \leq C_k(n). \tag{6.5}$$

Clearly, (6.2) is satisfied.

The next step in the construction is to try T such that for $n > C_k(n)$ we have

$$\frac{n}{T}C_{k-1}(T) \leq \alpha_k C_k(n) \tag{6.6}$$

$$T \leq \beta_k C_k(n) \tag{6.7}$$

for suitable α_k and β_k.

Lemma 6.1 Under the assumptions (6.6),(6.7) the condition (6.3) is fulfilled provided that

$$\alpha_k + 2\beta_k \leq 1,\ k \geq 4 \text{ and } \alpha_3 = 0, \beta_3 \leq 1 \tag{6.8}$$

$$\alpha_k\gamma_{k-1} + 2/C_k(1) \leq \gamma_k, \tag{6.9}$$

where $\gamma_2 = 0$.

Proof. Using (6.4) we see that (6.3) is fulfilled if

$$\frac{n}{T}C_{k-1}(T)(1+\gamma_{k-1}\delta(n)) + 2T + 2\delta(n) \le C_k(n)(1+\gamma_k\delta(n)).$$

With (6.6), (6.7) this is implied by

$$\alpha_k C_k(n)(1+\gamma_{k-1}\delta(n)) + 2\beta_k C_k(n) + 2\delta(n) \le C_k(n)(1+\gamma_k\delta(n)).$$

This in turn is implied by

$$(\alpha_k + 2\beta_k)C_k(n) \le C_k(n)$$

$$(\alpha_k\gamma_{k-1}C_k(n) + 2)\delta(n) \le \gamma_k C_k(n)\delta(n).$$

Using (6.5) we see that (6.8) and (6.9) are even stronger. ■

For a moment we look only at (6.6),(6.7) for $n \to \infty$, trying the form

$$C_k(n) \sim n^{\phi_k}; \quad T \sim n^{\psi_k}.$$

Then

$$\frac{n}{T}C_{k-1}(T) \sim \frac{n}{T}T^{\phi_{k-1}} \sim n^{\psi_k(\phi_{k-1}-1)+1}.$$

This leads to

$$\psi_k(\phi_{k-1} - 1) \le \phi_k; \quad \psi_k \le \phi_k$$

or (for $\phi_{k-1} < 1$):

$$\frac{1-\phi_k}{1-\phi_{k-1}} \le \psi_k \le \phi_k.$$

The most balanced choice for ψ_k is forcing the equalities, leading to $\psi_k = \phi_k$. With the minimal choice $\phi_3 = 0$ this leads to

$$\phi_k = \frac{k-3}{k-2}.$$

Therefore we continue to look for T satisfying (6.6),(6.7) by trying

$$C_k(n) = c^k n^{\frac{k-3}{k-2}} \tag{6.10}$$

with $c \ge 1$ still to be determined. Note that

$$n > C_k(n) \iff n > c^{k(k-2)}.$$

For $k = 3$ (6.6) is trivial and (6.7) leads to $T \le \beta_3 c^3$. The conditions (6.8),(6.9) hold for $\alpha_3 = 0$ and $\beta_3 \le 1$. So (6.3) is fulfilled for $T = 1$ since $c \ge 1$. So the following lemma looks for α_k, β_k not depending on $k \ge 4$.

Lemma 6.2 ($k \ge 4$). For $n > C_k(n)$ there exists an integer T with $0 < T < n$ such that

$$t_k n^{\frac{k-3}{k-2}} \le T \le T_k n^{\frac{k-3}{k-2}} \tag{6.11}$$

with

$$t_k = c^{(k-\varphi)\frac{k-3}{k-2}}; \quad T_k = c^{k-\psi} \tag{6.12}$$

where

$$\psi > 0;\ 0 < \varphi \le 4;\ \psi \le \varphi + 1;\ \psi \le \frac{1}{2}\varphi + 2;\ c^{8-\psi} - c^{6-\frac{1}{2}\varphi} \ge 1. \tag{6.13}$$

For such T the conditions (6.6) and (6.7) are satisfied for

$$\alpha_k = c^{\varphi-1}\beta_k; \quad \beta_k = c^{-\psi}. \tag{6.14}$$

Proof. With (6.10)-(6.13) we see

$$T \le T_k n^{\frac{k-3}{k-2}} = c^{-k} C_k(n) \le C_k(n) \le n.$$

Furthermore, $n > c^{k(k-2)}$ implies

$$(T_k - t_k) n^{\frac{k-3}{k-2}} = (T_k - t_k) c^{k(k-3)} \ge 1$$

since with (6.13) it follows for all $k \ge 4$ that

$$\begin{aligned} &(k-\psi)(k-2) - (k-\varphi)(k-3) \\ &= (\varphi - \psi + 1)k + 2\psi - 3\varphi \ge 4(\varphi - \psi + 1) + 2\psi - 3\varphi = \varphi - 2\psi + 4 \ge 0. \end{aligned}$$

Therefore T in (6.11) can be chosen to be an integer.

Now condition (6.7) holds since with (6.10) - (6.12) and (6.14):

$$T \le c^{k-\psi} n^{\frac{k-3}{k-2}} = c^{-k} C_k(n) = \beta^k C_k(n).$$

Finally, from (6.11)

$$n \le t_k^{-\frac{k-2}{k-3}} T^{\frac{k-2}{k-3}}$$

and so condition (6.6) is satisfied since with (6.14) and (6.7):

$$\begin{aligned} \frac{n}{T} C_{k-1}(T) &\le t_k^{-\frac{k-2}{k-3}} T^{-\frac{1}{k-3}} . c^{k-1} T^{\frac{k-4}{k-3}} = c^{\varphi-1} T \\ &\le c^{\varphi-1}\beta_k C_k(n) = \alpha_k C_k(n). \end{aligned}$$

■

For α_k, β_k given by (6.14) we try to satisfy (6.8) for a suitable choice of $c \geq 1$ and $(\phi; \psi)$ in the range specified by (6.13).

From (6.14) it follows that (6.8) is satisfied provided that

$$(c^{\phi-1} + 2)c^{-\psi} \leq 1$$

or, equivalently,

$$c^{\psi} - c^{\phi-1} - 2 \geq 0.$$

A suitable choice of $(\phi; \psi)$ in the allowed range (6.13) for which c can be chosen almost as small as possible is

$$\phi = 3; \quad \psi = 3. \tag{6.15}$$

This has been verified numerically. So for c_0 we can choose the unique real root of

$$c_0^3 - c_0^2 - 2 = 0 \tag{6.16}$$

and this is the choice (2.21). Finally, we have to verify (6.9) with $C_k(1) = c^k$ and $\alpha_k = c^{\phi-1}\beta_k = c^{\phi-1-\psi} = c^{-2}$ for a suitable choice of the $\nu_k; k \geq 4$ and ν_3. This leads to

$$c^{-2}\nu_{k-1} + 2c^{-k} \leq \nu_k; \quad k \geq 3.$$

Clearly, this is satisfied for

$$\nu_k = \left(\frac{2c}{c-1}\right) c^{-k}. \tag{6.17}$$

So with (6.4),(6.10) and (6.17) we see that

$$C_k(n; \delta) = n^{\frac{k-3}{k-2}}(c^k + 2\delta c/(c-1)).$$

In view of (6.1) and (5.3) this completes the proof of theorem 2.5.

7 Approximated pay-offs

The definition of vector systems can be extended straightforward to a sequence $(A(n) \in \mathbb{R}^{r \times s})_{n \in \mathbb{N}}$ of pay-off matrices instead of a fixed matrix $A \in \mathbb{R}^{r \times s}$. So (2.1), (2.2) generalize to

$$U(n) = U(n-1) + A(n)\mu(n) \tag{7.1}$$

$$V(n) = V(n-1) + \lambda(n)A(n). \tag{7.2}$$

The definition of $p(n); q(n)$ remain (2.3),(2.4). Bounds for the game value $v(A)$ of A can be derived for such extended systems, considering $A(n)$ as an approximation for A. Introduce (elementswise)

$$\alpha(n) = \max \left| \sum_{k=1}^{n} \{A(k) - A\} \right| . \tag{7.3}$$

Then it is easily seen that the relations (2.10)-(2.13) can be generalized to

$$-\frac{1}{n}\{\max V(0) + \alpha(n)\} \le v(A) - \frac{1}{n}\min V(n) \le \frac{1}{n}\{\Delta_{UV}(n) - \min U(0) + \alpha(n)\} \tag{7.4}$$

$$\frac{1}{n}\{\min U(0) - \alpha(n)\} \le \frac{1}{n}\max U(n) - v(A) \le \frac{1}{n}\{\Delta_{UV}(n) + \max V(0) + \alpha(n)\} \tag{7.5}$$

$$0 \le v(A) - \min p(n)A \le \frac{1}{n}\{\Delta_{UV}(n) + \Delta_{VU}(0) + 2\alpha(n)\} \tag{7.6}$$

$$0 \le \max Aq(n) - v(A) \le \frac{1}{n}\{\Delta_{UV}(n) + \Delta_{VU}(0) + 2\alpha(n)\}. \tag{7.7}$$

Clearly, for convergence to $v(A)$ we need again (2.14), but now also

$$\alpha(n)/n \to 0 \tag{7.8}$$

i.e. Caesaro-convergence of $A(n)$ to A.

Choose $\lambda(n); \mu(n)$ as in definition 2.4. Then theorem 2.5 maintains to hold provided that we take $c > c_0$ and that we impose some rather complicated extra order condition (dependent on $c - c_0$) that the difference $A(n) - A$ does not tend to ∞ too fast. Sufficient for this is simply the condition

$$A(n) \text{ bounded in } n. \tag{7.9}$$

So for the special case that $\gamma(n) = 0$ our conditions (7.8),(7.9) are weaker than the condition $A(n) \to A$ of Vrieze and Tijs [1982].

We only sketch the proof by indicating the necessary adaptions in the proofs of sections 5,6. In lemma 5.2 we have only to replace $A^{(i)}$ by $A^{(i)}(n - T + \tau)$ and $A_{(j)}$ by $A_{(j)}(n - T + \tau)$. In lemma 5.3 this has the effect that (5.13) should be changed to

$$\Delta_{UV}(n) \le 2a(T + \gamma(n) + 2\beta(n; n - T)) \tag{7.10}$$

where (componentswise)

$$\beta(n; m) = \frac{1}{a} \max_{m<k\le n} |A(k) - A|. \tag{7.11}$$

With this we should consider in part a) of the proof the case that $\Delta_{UU}(n) > a(T + \gamma(n) + 2\beta(n, n-T))$.

The effect in lemma 5.4 is that in the left hand side of 5.16 we should add the term $4\beta(n, n-T)$ (in the remark of this lemma only $2\beta(n, n-T)$).

Now consider the final choice in section 6. The left hand side of (6.3) contains the extra term $4\beta(n, n-T)$ (for $k = 3$ only $2\beta(n, n-T)$). So, for $k \geq 4$; we have to choose C_k simply in such a way that the difference of the right hand side and the left hand side in the original inequality (6.3) does not exceed $4\beta(n, n-T)$. By inspecting the final choice of T and C_k for general $c \geq c_0$ it is easily seen that under the condition (7.9) this can be achieved for all n and $k \geq 4$ by taking c sufficiently large. (The case $k = 3$ should be checked separately: $\beta(n, n-1) \leq (1-\beta_3)c^3$ can certainly be satisfied for c sufficiently large.)

The method above makes clear how precise bounds in the style of (2.20) should be derived. Expressions become rather complicated and therefore we have restricted ourselves to the convergence of the game value.

8 Conclusions and remarks

In this paper we could give thorough arguments for introducing weakened fictitious play. A priori bounds leading to sufficient conditions for convergence could be derived. A numerical analysis showed that weakening the basic form of fictitious play can speed up convergence. An open question is under which conditions the strategies themselves converge. This will be a point for further research.

A related question is wether fictitious play can be applied directly to games in extensive form as well, without a transformation to the corresponding strategic form. For games arising in practice the extensive form is primarily given and transformation often leads to very large game matrices.

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