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By A.J.J. Talman, Z. Yang

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Dolf Talman² and Zaifu Yang³

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²A.J.J. Talman, Department of Econometrics & Operations Research and CentER, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands. E-mail: talman@uvt.nl

³Z. Yang, Faculty of Business Administration, Yokohama National University, Yokohama 240-8501, Japan. E-mail: zyang@business.ynu.ac.jp

Abstract

In this paper, a simplicial algorithm is introduced to compute a coincidence of two mappings as stated in Ky Fan's theorem. Fan's theorem says: Let X be a non-empty compact and convex set in the n -dimensional Euclidean space \mathbb{R}^n and let ϕ and ψ be two upper semi-continuous mappings from X to the collection of non-empty compact, convex subsets of \mathbb{R}^n . Suppose that for every $x \in X$ and every $d \in \mathbb{R}^n$ satisfying $d^\top x = \max\{d^\top y \mid y \in X\}$, there exist $v \in \phi(x)$ and $w \in \psi(x)$ such that $d^\top v \geq d^\top w$. Then there exists an x^* in X satisfying $\phi(x^*) \cap \psi(x^*) \neq \emptyset$. Such a point x^* is called a coincidence. As a result, our algorithm leads to a constructive proof of this well-known and powerful existence theorem.

Keywords: coincidence, fixed point, zero point, simplicial algorithm, upper semi-continuity

AMS subject classifications: Primary, 54H25, 65K10; Secondary, 49J53, 68W25

JEL classifications: C62, C63

1 Introduction

Brouwer's and Kakutani's fixed point theorems and their variants are powerful tools to show the existence of a solution to various problems in economics, game theory and engineering; see for example Debreu (1959) and Arrow and Hahn (1971). Yet, for a number of (new) economic and game-theoretic problems these tools appeared insufficient and Ky Fan's coincidence theorem (1972), a more powerful theorem, has to be invoked; see for example Ichiishi (1983), Vohra (1991), Yang (2001), and Florenzano (2003).

For many problems, one may not be contented just to know the existence of a solution, but one would like to pinpoint where a solution is located in order to evaluate and analyse the performance of the underlying model, or to appraise the effects of policy, or technical parameter changes in the model. However, in reality, for many problems it is impossible to find a closed form analytical solution. It is therefore essential to develop numerical computational methods to approximate solutions to the underlying model. Much of the recent literature on the computation of fixed points or economic equilibria has its root in the pioneering work of Scarf (1967, 1973), which introduced the first algorithm to compute a fixed point, that was guaranteed to converge. Subsequent algorithms have been developed by Eaves (1972), Merrill (1972), van der Laan and Talman (1979), and Wright (1981) among others, which have substantially refined Scarf's original algorithm, to accelerate its speed and to extend its applicability. As a consequence, Brouwer's and Kakutani's theorems and many of their variants can be proved in constructive ways; see e.g., Todd (1976) and van der Laan (1981). More recently, algorithms have been proposed to compute robust or stable fixed points, or continua of fixed points; see Herings, Talman and Yang (2001), van der Laan, Talman and Yang (1998), and Yang (1996). Allgower and Georg (1990), Doup (1988), Todd (1976), and Yang (1999) provide comprehensive treatments on simplicial algorithms at various stages.

In this paper we propose a simplicial algorithm to compute a coincidence of two mappings as stated in Fan's theorem (1972). Fan's coincidence theorem gives a sufficient condition under which two mappings ϕ and ψ defined on the same convex and compact set X in \mathbb{R}^n have a point at which the two images of this point under ϕ and ψ have a non-empty intersection. The condition says that for every x in X it has to hold that for every element d of the normal cone of X at x there exist an element v in the image $\phi(x)$ and an element w in the image $\psi(x)$ satisfying that $d^\top v \geq d^\top w$. The problem of computing a coincidence is considerably more delicate and difficult than, for example, the computation of a fixed point of a mapping from X to X . First of all, many extensions of Brouwer's theorem such as Kakutani's theorem are in essence equivalent variants of Brouwer's theorem in the sense that they are or can be directly derived from Brouwer's theorem, whereas for Fan's coincidence theorem this does not seem to be the case. In fact, the latter theorem is

proved in a totally different way. Second, the boundary condition stated in Fan's theorem differs considerably from those in Brouwer's theorem and many of its extensions. Fan's theorem requires a weak separation condition at each element of the normal cone at any point of X . For different elements of the normal cone the separation condition may hold for different elements of the images. The question is then how to deal with this complication in an algorithm. Precisely because of this, the usual simplicial approximation and other techniques do not apply here.

To circumvent the computational impasse, we embed the set X into an elaborately-designed full-dimensional compact and convex set Q containing X in its interior and satisfying that at each point in its boundary the normal cone is just a half-line. We also extend the mappings to Q in a proper way. Then we propose a simplicial algorithm to operate on a cube P containing the set Q in its interior.

This paper is organized as follows. In Section 2 we present Fan's coincidence theorem and derive several closely related existence results. In Section 3 we propose the simplicial algorithm which will be used to approximate a coincidence of two mappings and we prove its convergence. In Section 4 we give a constructive proof of Fan's theorem.

2 Fan's Coincidence Theorem

Let Y be an arbitrary non-empty set in the n -dimensional Euclidean space \mathbb{R}^n . For $x \in Y$, the set

$$N(Y, x) = \{y \in \mathbb{R}^n \mid (x - x')^\top y \geq 0 \text{ for all } x' \in Y\}$$

denotes the normal cone of Y at x . Its polar cone

$$T(Y, x) = \{z \in \mathbb{R}^n \mid z^\top y \leq 0 \text{ for all } y \in N(Y, x)\}$$

denotes the tangent cone of Y at x . If Y is compact and convex, $N(Y, \cdot)$ is an upper semi-continuous, convex-valued and closed-valued mapping on Y and $T(Y, \cdot)$ is a convex-valued and closed-valued mapping on Y and, for every $y \in Y$, both $N(Y, y)$ and $T(Y, y)$ are non-empty.

The notion \mathbf{N} denotes the set of all positive integers and I_k denotes the set of the first k positive integers. The notions 0^n , 1^n and $E(n)$ stand for the vector of zeros and ones of dimension n and the $n \times n$ identity matrix, respectively. Given a subset D of \mathbb{R}^n , $\text{bd}(D)$ and $\text{int}(D)$ represent the sets of (relative) boundary and interior points of D , respectively, and $\text{co}(D)$ represents the convex hull of D .

Let X be an arbitrary non-empty set in \mathbb{R}^n and let ϕ be a point-to-set mapping from X to the collection of non-empty subsets of \mathbb{R}^n . A point $x^* \in X$ is called a *zero point* of ϕ if

$0^n \in \phi(x^*)$, a *fixed point* of ϕ if $x^* \in \phi(x^*)$, a *coincidence* of ϕ and some other mapping ψ on X if $\phi(x^*) \cap \psi(x^*) \neq \emptyset$. Well known conditions for the existence of a fixed or zero point were given by Kakutani (1942) and for the existence of a coincidence of two mappings by Fan (1972). The following theorem gives a sufficient condition for the existence of a zero point and is an equivalent form of Fan's coincidence theorem. The objective of this paper is to give a constructive proof for this theorem.

Theorem 2.1 *Let ϕ be an upper-semicontinuous point-to-set mapping from the non-empty convex and compact set X in \mathbb{R}^n to the collection of non-empty compact and convex subsets of \mathbb{R}^n . Suppose that for every $x \in X$ and every $v \in N(X, x)$, there is a $y \in \phi(x)$ satisfying $v^\top y \leq 0$. Then there exists a zero point of ϕ in X .*

From Theorem 2.1 we immediately obtain the following results. The first one is the coincidence existence theorem of Fan (1972) and is equivalent to it.

Theorem 2.2 *Let ϕ and ψ be two upper semi-continuous mappings from the non-empty convex and compact set X in \mathbb{R}^n to the collection of non-empty compact and convex subsets of \mathbb{R}^n . Suppose that for every $x \in X$ and every $d \in \mathbb{R}^n$ satisfying $d^\top x = \max\{d^\top y \mid y \in X\}$, there exist $u \in \phi(x)$ and $w \in \psi(x)$ such that $d^\top u \geq d^\top w$. Then there exists a coincidence of ϕ and ψ in X .*

Proof: Define the mapping γ on X by $\gamma(x) = \psi(x) - \phi(x)$ for all $x \in X$. Clearly, being the difference of two such mappings, γ is an upper semi-continuous mapping from X to the collection of non-empty compact and convex subsets of \mathbb{R}^n . Since $d^\top x = \max\{d^\top y \mid y \in X\}$ implies $d \in N(X, x)$, γ satisfies the conditions of Theorem 2.1. Hence, there exists a zero point of γ in X . By construction, every zero point of γ is a coincidence of the mappings ϕ and ψ . \square

Notice that Theorem 2.1 also immediately follows from Theorem 2.2 if we take one of the two mappings to be the mapping that assigns the origin to every $x \in X$. The next existence theorem can be seen as a direct generalization of Kakutani's fixed point theorem.

Theorem 2.3 *Let ϕ be an upper-semicontinuous point-to-set mapping from the non-empty convex and compact set X in \mathbb{R}^n to the collection of non-empty compact and convex subsets of \mathbb{R}^n . Suppose that for every $x \in X$ it holds that $\phi(x) \cap X \neq \emptyset$. Then there exists a fixed point of ϕ in X .*

Proof: For given $x \in X$, take some $y \in \phi(x) \cap X$. Since $y \in X$ we have that $v^\top y \leq v^\top x$ for all $v \in N(X, x)$. Hence, the mappings ϕ and ψ on X , where ψ is defined by $\psi(x) = \{x\}$ for all $x \in X$, satisfies the conditions of Theorem 2.2. Therefore, there exists a coincidence of ϕ and ψ in X . Clearly, any coincidence of ϕ and ψ is a fixed point of ϕ in X . \square

Notice that in this theorem we only require that $\phi(x) \cap X \neq \emptyset$ for every $x \in X$. The image $\phi(x)$ may contain elements outside the set X . Clearly, when ϕ is a fixed point mapping in the sense that for every $x \in X$ it holds that $\phi(x) \subseteq X$, we obtain Kakutani's fixed point theorem.

Corollary 2.4 *Let ϕ be an upper-semicontinuous point-to-set mapping from the non-empty convex and compact set X in \mathbb{R}^n to the collection of non-empty compact and convex subsets of X . Then there exists a fixed point of ϕ in X .*

The next result says that if for every x in X the image $\phi(x)$ has a non-empty intersection with the tangent cone $T(X, x)$ of X at x , a zero point of ϕ must exist.

Corollary 2.5 *Let ϕ be an upper-semicontinuous point-to-set mapping from the non-empty convex and compact set X in \mathbb{R}^n to the collection of non-empty compact and convex subsets of \mathbb{R}^n . Suppose that for every $x \in X$ it holds that $\phi(x) \cap T(X, x) \neq \emptyset$. Then there exists a zero point of ϕ in X .*

Proof: For any $x \in X$ it holds that $T(X, x) \subseteq \{y \in \mathbb{R}^n \mid v^\top y \leq 0\}$ for every $v \in N(X, x)$. Hence, ϕ satisfies the conditions of Theorem 2.1. \square

3 A Simplicial Algorithm

In this section we propose a simplicial algorithm which will lead to a constructive proof of Theorem 2.1. For $x \in \mathbb{R}^n$, let $p(x)$ be the orthogonal projection of x on X , i.e.,

$$\|x - p(x)\|_2 \leq \|x - y\|_2 \text{ for all } y \in X.$$

Since X is a closed and convex set, $p(\cdot)$ is a continuous function on \mathbb{R}^n . The next lemma shows that the vector $x - p(x)$ is an element of the normal cone of X at $p(x)$.

Lemma 3.1 *For every $x \in \mathbb{R}^n$ it holds that $x - p(x) \in N(X, p(x))$.*

Proof: For $x \in X$ it holds that $p(x) = x$. Since $0^n \in N(X, x)$, we immediately obtain $x - p(x) \in N(X, x)$. Suppose $x \notin X$. Take any $y \in X$ and $\lambda \in (0, 1]$. Since X is convex, $\lambda y + (1 - \lambda)p(x) \in X$. By definition of $p(x)$ it holds that

$$\|x - p(x)\|_2 \leq \|x - (\lambda y + (1 - \lambda)p(x))\|_2.$$

Hence,

$$0 \leq -2\lambda(x - p(x))^\top (y - p(x)) + \lambda^2 \|y - p(x)\|_2^2.$$

Dividing by λ yields

$$2(x - p(x))^\top (y - p(x)) \leq \lambda \|y - p(x)\|_2.$$

Taking the limit for λ going to zero, we obtain

$$(y - p(x))^\top (x - p(x)) \leq 0.$$

Since $y \in X$ is arbitrary, this implies that $x - p(x) \in N(X, p(x))$. □

Let the set Q be defined by

$$Q = \{q \in \mathbb{R}^n \mid \|q - p(q)\|_2 \leq 1\}.$$

Lemma 3.2 *The set Q is a full-dimensional, compact and convex subset of \mathbb{R}^n , containing X in its interior.*

Proof: Clearly, Q is a full-dimensional set in \mathbb{R}^n containing X in its interior. Since X is compact, Q is also compact. To prove convexity of Q , take any $q^1, q^2 \in Q$ and $0 \leq \lambda \leq 1$ and let

$$q(\lambda) = \lambda q^1 + (1 - \lambda)q^2$$

and

$$p(\lambda) = \lambda p(q^1) + (1 - \lambda)p(q^2).$$

Since X is convex, we have that $p(\lambda) \in X$. Moreover,

$$\|q(\lambda) - p(\lambda)\|_2 \leq \lambda \|q^1 - p(q^1)\|_2 + (1 - \lambda) \|q^2 - p(q^2)\|_2 \leq 1.$$

Therefore, $q(\lambda) \in Q$, i.e., Q is a convex set. □

For $q \in Q$, let $v(q) = q - p(q)$, and let B^n be the unit ball in \mathbb{R}^n . By construction, $v(q) \in B^n$ for every $q \in Q$, $\|v(q)\|_2 = 1$ if and only if $q \in \text{bd}(Q)$, and $v(q) = 0^n$ if and only if $q \in X$. For $v \in \mathbb{R}^n$, define the set $C(v)$ by $C(v) = \{y \in \mathbb{R}^n \mid y = \alpha v, \alpha \geq 0\}$. When $v \neq 0^n$, $C(v)$ is a half-line.

Lemma 3.3 *For every $q \in \text{bd}(Q)$, it holds that $N(Q, q) = C(v(q))$ and $N(Q, q) \subseteq N(X, p(q))$.*

Proof: Take any point $q \in \text{bd}(Q)$. Let $B(p(q)) = \{x \in \mathbb{R}^n \mid \|x - p(q)\|_2 \leq 1\}$. By definition of Q , $B(p(q)) \subseteq Q$ and $q \in \text{bd}(B(p(q)))$, and therefore $N(Q, q) \subseteq N(B(p(q)), q)$. However, $q \in \text{bd}(B(p(q)))$ implies $N(B(p(q)), q) = C(q - p(q)) = C(v(q))$. Hence, $N(Q, q) \subseteq C(v(q))$. On the other hand, since Q is convex and $q \in \text{bd}(Q)$, the normal cone $N(Q, q)$ of Q at q contains at least one half-line, which must be then $C(v(q))$. Finally, since according to Lemma 3.2 it holds that $v(q) \in N(X, p(q))$, it follows that $C(v(q)) \subseteq N(X, p(q))$. \square

From the lemma it follows that for every point on the boundary of Q the normal cone of Q at that point is a half-line. Notice that since Q is full-dimensional, the normal cone of Q at an interior point of Q is just the origin. In this way a point $q \in Q$ represents a unique point $x = p(q)$ in X and a unique element $v = q - p(q)$ in $N(X, x)$. Also, every combination of a point x in X and an element v in $N(X, x)$ (with length at most equal to 1) is represented by a unique point $q = x + v$ in Q . Hence, the set Q is an full-dimensional expansion of X with smooth boundary.

Let $P = \{x \in \mathbb{R}^n \mid l \leq x \leq u\}$ be an n -dimensional cube in \mathbb{R}^n for some given vectors l and u in \mathbb{R}^n with $u_i > l_i$, for all $i \in I_n$, containing Q in its interior. Let $e(i)$, $i \in I_n$, denote the i th unit vector in \mathbb{R}^n , and let $I = \{-n, \dots, -1, 1, \dots, n\}$. For $i \in I_n$, define $a^i = e(i)$ and $b_i = u_i$, and $a^{-i} = -e(i)$ and $b_{-i} = -l_i$. Then P can be reformulated as

$$P = \{x \in \mathbb{R}^n \mid a^{i^\top} x \leq b_i \text{ for all } i \in I\}.$$

Clearly, P is a simple full-dimensional polytope, and none of the constraints is redundant. Let \mathcal{I} be the collection of non-empty subsets J of I such that $|J| \leq n$ and $-j \notin J$ whenever $j \in J$. For each $J \in \mathcal{I}$, define

$$F(J) = \{x \in P \mid a^{i^\top} x = b_i, \text{ for all } i \in J\}.$$

Clearly, for every $J \in \mathcal{I}$, the set $F(J)$ is an $(n - |J|)$ -dimensional face of P , where $|J|$ denotes the number of elements in J .

Let q^0 be an arbitrary point in the interior of P . The point q^0 will be the starting point of the algorithm to be described below. Notice that we allow q^0 to lie outside X or even outside Q . For any $J \in \mathcal{I}$, let $cF(J)$ be the convex hull of the point q^0 and $F(J)$. Since q^0 lies in the interior of P , the dimension of $cF(J)$, $J \in \mathcal{I}$, is equal to $n - |J| + 1$. Now we first describe a simplicial subdivision or triangulation of the polytope P which underlies the algorithm.

For a nonnegative integer t , a t -dimensional simplex or t -simplex, denoted by σ , on \mathbb{R}^n is defined by the convex hull of $t + 1$ affinely independent points x^1, \dots, x^{t+1} in \mathbb{R}^n . We often write $\sigma = \sigma(x^1, \dots, x^{t+1})$ and call x^1, \dots, x^{t+1} the vertices of σ . A $(t - 1)$ -simplex being the convex hull of t vertices of a t -simplex σ is called a facet of σ . The facet

$\tau(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^{t+1})$ is called the facet of $\sigma(x^1, \dots, x^{t+1})$ opposite to the vertex x^i of σ . For k , $0 \leq k \leq t$, a k -simplex being the convex hull of $k + 1$ vertices of σ is said to be a k -dimensional face or k -face of σ . A finite collection \mathcal{T} of n -simplices in \mathbb{R}^n is called a triangulation of the set P if

- (i) P is the union of all simplices in \mathcal{T} ;
- (ii) The intersection of any two simplices of \mathcal{T} is either the empty set or a common face of both;

The diameter of a simplex $\sigma(x^1, \dots, x^{n+1})$ is the maximum Euclidean distance between any two points in σ and is denoted by $diam(\sigma)$. The mesh size of a triangulation \mathcal{T} is defined as

$$mesh(\mathcal{T}) = \max_{\sigma \in \mathcal{T}} \{diam(\sigma)\}.$$

We are interested in triangulations of P with arbitrary (small) mesh size such that every subset $cF(J)$, $J \in \mathcal{I}$, is subdivided into t -simplices, with $t = n - |J| + 1$. For example, we may take the V -triangulation with arbitrary mesh size introduced by Doup and Talman (1987) for triangulating a simplotope. The set P , being the Cartesian product of n intervals, is a special case of a simplotope.

Let τ be a facet of a t -simplex σ on $cF(J)$, where $t = n - |J| + 1$, $J \in \mathcal{I}$. Then either τ lies on the boundary of $cF(J)$ and is only a facet of σ or τ does not lie on the boundary of $cF(J)$ and is a facet of exactly one other t -simplex on $cF(J)$. If τ lies on the boundary of $cF(J)$, then either τ lies on the face $F(J)$ of P or τ is equal to the 0-dimensional simplex $\{q^0\}$ or τ is a $(t - 1)$ -dimensional simplex on $cF(J')$ for some unique $J' \in \mathcal{I}$ satisfying $J \subset J'$ and $|J'| = |J| + 1$.

Let ϕ be a mapping on X satisfying the conditions of Theorem 2.1, let Q and P be as constructed above, and let \mathcal{T} be a simplicial subdivision of the set P with arbitrary mesh size as described above. For $v \in B^n$, let $\pi(v)$ be defined by $\pi(v) = \mathbb{R}^n$ if $v = 0^n$ and $\pi(v) = \{y \in \mathbb{R}^n \mid y^\top v \leq 0\}$ otherwise. Notice that π is an upper semi-continuous mapping from B^n to the collection of nonempty closed and convex subsets of \mathbb{R}^n . Now we consider the point-to-set mapping $\bar{\phi}$ from P to \mathbb{R}^n defined by

$$\bar{\phi}(x) = \begin{cases} \{p(x) - x\}, & \text{if } x \in P \setminus Q, \\ \text{co}(\{p(x) - x\} \cup [\phi(p(x)) \cap \pi(x - p(x))]), & \text{if } x \in \text{bd}(Q), \\ \phi(p(x)) \cap \pi(x - p(x)), & \text{if } x \in \text{int}(Q). \end{cases}$$

One can easily verify that since X and ϕ satisfy the conditions of Theorem 2.1, the mapping $\bar{\phi}$ is an upper semi-continuous mapping on P with non-empty convex and compact images in \mathbb{R}^n . To any vertex x of a simplex σ of \mathcal{T} we assign the vector label $f(x)$, where $f(x)$ is

an arbitrarily chosen element in $\bar{\phi}(x)$. Now we extend f piecewise linearly on each simplex of \mathcal{T} , i.e., if $x = \sum_{i=1}^{n+1} \lambda_i x^i$ in some simplex $\sigma(x^1, \dots, x^{n+1})$ of \mathcal{T} for some $\lambda_i \geq 0$, $i \in I_{n+1}$, with $\sum_{i=1}^{n+1} \lambda_i = 1$, then $f(x) = \sum_{i=1}^{n+1} \lambda_i f(x^i)$. Clearly, $f(\cdot)$ is affine on each simplex of \mathcal{T} and is a continuous function from P to \mathbb{R}^n . We call f the *piecewise linear approximation* of $\bar{\phi}$ with respect to \mathcal{T} .

A row vector is said to be *lexicopositive* if it is a non-zero vector and its first non-zero entry is positive. A matrix is said to be *lexicopositive* if all its rows are lexicopositive.

Definition 3.4 Let $\tau(x^1, \dots, x^t)$ be a facet of a t -simplex on $cF(J)$, where $J \in \mathcal{I}$ with $J = \{j_{t+1}, \dots, j_{n+1}\}$, $t = n - |J| + 1$. The $(n+1) \times (n+1)$ matrix

$$A_{\tau, J} = \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ -f(x^1) & \cdots & -f(x^t) & a^{j_{t+1}} & \cdots & a^{j_{n+1}} \end{bmatrix}$$

is the label matrix of τ with respect to J . The simplex τ is J -complete if $A_{\tau, J}^{-1}$ exists and is lexicopositive.

Definition 3.5 Let $\sigma(x^1, \dots, x^{n+1})$ be an n -simplex on P . The $(n+1) \times (n+1)$ matrix

$$A_{\sigma} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ -f(x^1) & -f(x^2) & \cdots & -f(x^{n+1}) \end{bmatrix}$$

is the label matrix of σ . The simplex σ is complete if A_{σ}^{-1} exists and is lexicopositive.

Notice that if for a J -complete simplex τ we change the ordering of the columns of the matrix $A_{\tau, J}$, the inverse of the resulting matrix still exists and is lexicopositive. Clearly, if, for some $J \in \mathcal{I}$, a $(t-1)$ -simplex $\tau(x^1, \dots, x^t)$ is a J -complete facet of a simplex $\sigma(x^1, \dots, x^{t+1})$ on $cF(J)$, then the system of $n+1$ linear equations with $n+2$ variables

$$\sum_{i=1}^{t+1} \lambda_i \begin{pmatrix} 1 \\ -f(x^i) \end{pmatrix} + \sum_{j \in J} \mu_j \begin{pmatrix} 0 \\ a^j \end{pmatrix} = \begin{pmatrix} 1 \\ 0^n \end{pmatrix} \quad (*)$$

has a solution $(\lambda, \mu) = (\lambda_1, \dots, \lambda_{t+1}, (\mu_j)_{j \in J})$ satisfying $\lambda_i \geq 0$ for all $i \in I_{t+1}$, $\sum_{i=1}^{t+1} \lambda_i = 1$, and $\mu_j \geq 0$ for all $j \in J$. Let x be defined by $x = \sum_{i=1}^{t+1} \lambda_i x^i$ at a nonnegative solution (λ, μ) of $(*)$. Then x lies in σ and $f(x) = \sum_{j \in J} \mu_j a^j$. Similarly, if a simplex $\sigma(x^1, \dots, x^{n+1})$ is a complete n -simplex on P , then the system of $n+1$ linear equations with $n+1$ variables

$$\sum_{i=1}^{n+1} \lambda_i \begin{pmatrix} 1 \\ -f(x^i) \end{pmatrix} = \begin{pmatrix} 1 \\ 0^n \end{pmatrix} \quad (**)$$

has a unique solution $\lambda^* = (\lambda_1^*, \dots, \lambda_{n+1}^*)$ satisfying $\lambda_i^* \geq 0$ for all $i \in I_{n+1}$ and $\sum_{i=1}^{n+1} \lambda_i^* = 1$. Let x^* be defined by $x^* = \sum_{i=1}^{n+1} \lambda_i^* x^i$. Then x^* lies in σ and $f(x^*) = 0^n$, i.e., x^* is a zero point of f in P .

We now show that $\{q^0\}$ is a J -complete 0-simplex for a unique index set $J \in \mathcal{I}$ containing n indices.

Lemma 3.6 *There exists a unique element J^0 of \mathcal{I} with $|J^0| = n$ such that $\{q^0\}$ is a J^0 -complete 0-simplex.*

Proof: Let $c = -f(q^0)$, $\tau^0 = \{q^0\}$ and

$$J^0 = \{i \mid c_i \leq 0\} \cup \{-i \mid c_i > 0\}.$$

Let $s_i = 1$ if $c_i \leq 0$ and $s_i = -1$ if $c_i > 0$. Then we have

$$A_{\tau^0, J^0} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ c & s_1 e(1) & s_2 e(2) & \cdots & s_n e(n) \end{bmatrix}.$$

Its inverse can be explicitly given as below.

$$A_{\tau^0, J^0}^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \bar{c} & s_1 e(1) & s_2 e(2) & \cdots & s_n e(n) \end{bmatrix},$$

where $\bar{c} = (\bar{c}_1, \dots, \bar{c}_n)^\top$ with $\bar{c}_i = -s_i c_i$ for every $i \in I_n$. Clearly, the matrix A_{τ^0, J^0}^{-1} is lexicopositive, since $-s_i c_i \geq 0$ for every $i \in I_n$ and $s_i = 1$ if $s_i c_i = 0$. Hence, τ^0 is J^0 -complete. By definition, τ^0 is a 0-dimensional simplex and lies on $cF(J^0)$. Moreover, there does not exist any other $J \in \mathcal{I}$ with $|J| = n$ such that the inverse of $A_{\tau^0, J}$ exists and is lexicopositive. \square

The following lemma is well-known in linear programming theory and can easily be proved. Let R be a matrix. We denote its i th row by R_i and its j th column by R_j .

Lemma 3.7 *Let $R = (R_{\cdot 1}, \dots, R_{\cdot n+1})$ be any non-singular $(n+1) \times (n+1)$ matrix and let x be any vector in \mathbb{R}^{n+1} . Let $k \in I_{n+1}$ and $\bar{R} = (R_{\cdot 1}, \dots, R_{\cdot k-1}, x, R_{\cdot k+1}, \dots, R_{\cdot n+1})$. Then either $(R^{-1}x)_k = 0$ and \bar{R} is singular, or $(R^{-1}x)_k \neq 0$, \bar{R} is non-singular and \bar{R}^{-1} is given by*

$$\bar{R}^{-1} = \begin{bmatrix} (R^{-1})_{\cdot 1} - \frac{(R^{-1}x)_1}{(R^{-1}x)_k} (R^{-1})_k \\ \vdots \\ (R^{-1})_{\cdot k-1} - \frac{(R^{-1}x)_{k-1}}{(R^{-1}x)_k} (R^{-1})_k \\ \frac{1}{(R^{-1}x)_k} (R^{-1})_k \\ (R^{-1})_{\cdot k+1} - \frac{(R^{-1}x)_{k+1}}{(R^{-1}x)_k} (R^{-1})_k \\ \vdots \\ (R^{-1})_{\cdot n+1} - \frac{(R^{-1}x)_{n+1}}{(R^{-1}x)_k} (R^{-1})_k \end{bmatrix}.$$

Using this lemma, the following lemmas can be proved.

Lemma 3.8 *Let σ be a t -simplex on $cF(J)$ for some $J \in \mathcal{I}$, where $t = n - |J| + 1$. If σ has a J -complete facet τ , then exactly one of the following three cases occurs:*

- (1) The simplex σ is a complete n -simplex on P ;
- (2) There exists a unique $\bar{J} \in \mathcal{I}$ so that σ is a \bar{J} -complete simplex on $cF(\bar{J})$, where $\bar{J} = J \setminus \{j\}$ for some $j \in J$;
- (3) The simplex σ has exactly one other J -complete facet $\bar{\tau}$.

Proof: Without loss of generality, let x^{t+1} be the vertex of σ opposite to τ , let $J = \{j_{t+1}, \dots, j_{n+1}\}$, and let $y = A_{\tau, J}^{-1}(1, -f(x^{t+1})^\top)^\top$. Notice that $y \neq 0^{n+1}$. Let $K = \{i \in I_{n+1} \mid y_i > 0\}$. We first prove $|K| > 0$. Since $A_{\tau, J}y = (1, -f(x^{t+1})^\top)^\top$, we have $\sum_{i=1}^t y_i = 1$. This implies that there exists at least one index $i \in I_t$ such that $y_i > 0$. Hence K is non-empty.

Consider the ratio vectors $(1/y_j)(A_{\tau, J}^{-1})_j$ for all $j \in K$. Choose $k \in K$ such that the k th ratio vector is the minimum in the lexicographic order over all such ratio vectors. Since $A_{\tau, J}^{-1}$ is regular, k is uniquely determined. Now we consider the following two cases (i) and (ii).

(i) If $k \in I_{n+1} \setminus I_t$, then let $l = j_k$ and $\bar{J} = J \setminus \{l\}$. If $\bar{J} = \emptyset$, then σ is a complete n -simplex on P . Otherwise, $\bar{J} \in \mathcal{I}$ and σ is on $cF(\bar{J})$. Let R be the matrix obtained from $A_{\tau, J}$ by replacing its k th column by $(1, -f(x^{t+1})^\top)^\top$. It follows from Lemma 3.7 that R^{-1} exists and is lexicopositive. By reordering the columns of R we get $A_{\sigma, \bar{J}}$ whose inverse exists and is lexicopositive. So, σ is \bar{J} -complete.

(ii) If $k \in I_t$, then let $\bar{\tau}$ be the facet of σ opposite to the vertex x^k . Using Lemma 3.7, it follows from the choice of k that $A_{\bar{\tau}, J}^{-1}$ exists and is lexicopositive. Hence $\bar{\tau}$ is a J -complete $(t-1)$ -simplex on $cF(J)$.

It follows immediately from Lemma 3.7 that if any column other than the k th column is replaced, then the inverse of the resulting matrix is not lexicopositive. \square

Lemma 3.9 *Let τ be a J -complete $(t-1)$ -simplex on $cF(\bar{J})$ for some $J \in \mathcal{I}$, where $t = n - |J| + 1$, and $\bar{J} = J \cup \{l\} \in \mathcal{I}$ for some $l \in I \setminus J$, then exactly one of the following two cases occurs:*

- (1) There exists a unique set $J' \in \mathcal{I}$ with $J' \neq J$ so that τ is a J' -complete $(t-1)$ -simplex on $cF(J')$, where $|J'| = |J|$ and $J' \subset \bar{J}$.
- (2) The simplex τ has exactly one \bar{J} -complete facet τ' .

Proof: Let $J = \{j_{t+1}, \dots, j_n\}$, $x = (0, a^{l^\top})^\top$ and $y = A_{\tau, J}^{-1}x$. Note that $y \neq 0^{n+1}$. Let $K = \{i \in I_{n+1} \mid y_i > 0\}$. Note that $A_{\tau, J}y = (0, a^{l^\top})^\top$. We first show that K is non-empty. Suppose that $y_i = 0$ for all $i \in I_t$. Then there must exist some parameters y_i for $i = t+1, t+2, \dots, n+1$, such that $a^l = \sum_{i=t+1}^{n+1} y_i a^{j_i}$, and y_i must be non-zero for some i . This

implies that the vectors a^l, a^j for all $j \in J$ are linearly dependent. This contradicts that $J \cup \{l\} \in \mathcal{I}$. Hence there exists at least one index $i \in I_t$ such that $y_i \neq 0$. If there exists an index $j \in I_t$ such that $y_j < 0$, then there must exist an index $i \in I_t$ such that $y_i > 0$ since $\sum_{k=1}^t y_i = 0$. Hence K is non-empty.

Consider the ratio vectors $(1/y_j)(A_{\tau,J}^{-1})_j$ for all $j \in K$. Choose $k \in K$ such that the k th ratio vector is the minimum in the lexicographic order over all such ratio vectors. Since $A_{\tau,J}^{-1}$ is regular, k is uniquely determined. Now we consider the following two cases (i) and (ii).

(i) If $k \in I_{n+1} \setminus I_t$, then let $p = j_k$ and $J' = J \cup \{l\} \setminus \{p\}$. Clearly, $J' \in \mathcal{I}$, $J' \neq J$, $|J'| = |J|$, $J' \subset \bar{J}$, and τ lies on $cF(J')$. Let R be the matrix obtained from $A_{\tau,J}$ by replacing its k th column by x . It follows from Lemma 3.7 that R^{-1} exists and is lexicopositive. Clearly, $A_{\tau,J'} = R$. So, τ is a J' -complete $(t-1)$ -simplex on $cF(J')$.

(ii) If $k \in I_t$, then let τ' be the facet of σ opposite to the vertex x^k . Clearly, τ' is a $(t-2)$ -simplex on $cF(\bar{J})$. Let R be the matrix obtained from $A_{\tau,J}$ by replacing its k th column by x . It follows from Lemma 3.7 that R^{-1} exists and is lexicopositive. By reordering the columns of R we get $A_{\tau',\bar{J}}$, whose inverse also exists and is lexicopositive. So, τ' is \bar{J} -complete.

Again it follows from Lemma 3.7 that if any other column is replaced, then the new matrix is no longer lexicopositive. \square

The next lemma says that any complete simplex on P lies on $cF(\{h\})$ for some unique $h \in I$ and also has a $\{h\}$ -complete facet.

Lemma 3.10 *Let $\sigma(x^1, \dots, x^{n+1})$ be a complete n -simplex on P . Then there exists a unique $h \in I$ such that σ lies on $cF(\{h\})$. Furthermore, σ has precisely one facet τ which is $\{h\}$ -complete.*

Proof: Since the union of the sets $cF(\{j\})$, $j \in I$, is P and the intersection of their interiors is empty, there exists a unique $h \in I$ such that σ lies on $cF(\{h\})$. Let $x = (0, a^{h^\top})^\top$ and $y = A_\sigma^{-1}x$. Let $K = \{i \in I_{n+1} \mid y_i > 0\}$. We will show that K is non-empty. Note that $y \neq 0^{n+1}$ since $x \neq 0^n$ and A_σ is non-singular. Hence, there exists at least one index $i \in I_{n+1}$ such that $y_i \neq 0$. If there exists an index $j \in I_{n+1}$ such that $y_j < 0$, then there must exist an index $i \in I_{n+1}$ such that $y_i > 0$ since $\sum_{i=1}^n y_i = 0$. Hence K is non-empty.

Consider the ratio vectors $(1/y_j)(A_\sigma^{-1})_j$ for all $j \in K$. Choose $k \in K$ such that the k th ratio vector is the minimum in the lexicographic order over all such ratio vectors. Since A_σ^{-1} is regular, k is uniquely determined. Now let τ be the facet of σ opposite to the vertex x^k . Let R be the matrix obtained from A_σ by replacing its k th column by x . It follows from Lemma 3.7 that R^{-1} exists and is lexicopositive. By reordering the columns

of R we get $A_{\tau, \{h\}}$ whose inverse also exists and is lexicopositive. So, τ is an $\{h\}$ -complete $(n - 1)$ -simplex on $cF(\{h\})$.

Again it follows from Lemma 3.7 that if any other column is replaced, then the new matrix is no longer lexicopositive. \square

In the following we will show that starting at q^0 there exists a finite sequence of adjacent J -complete simplices on $cF(J)$ for varying J , $J \in \mathcal{I}$, which leads to a complete n -simplex σ on P . First we show that a J -complete simplex on $cF(J)$ can not lie on the boundary of P .

Lemma 3.11 *If $\tau(x^1, \dots, x^t)$ is a J -complete $(t - 1)$ -simplex on $cF(J)$ for some $J \in \mathcal{I}$ where $t = n - |J| + 1$, then τ does not lie on the boundary of P .*

Proof: Suppose to the contrary that τ is a J -complete simplex on the boundary of P . Since τ is on $cF(J)$, τ must be a subset of $F(J)$. Hence, all vertices of τ are outside Q . So, $f(x^i) = p(x^i) - x^i$ for all $i = 1, \dots, t$, and $a^{j\top} x^i = b_j$ for all $j \in J$ and $i = 1, \dots, t$. Since $a^{j\top} p(x^i) < b_j$ we obtain that $a^{j\top} f(x^i) < 0$ for all $j \in J$ and all $i = 1, \dots, t$. Because τ is J -complete, according to $(*)$ we have

$$\sum_{i=1}^t \lambda_i f(x^i) = \sum_{j \in J} \mu_j a^j \tag{3.1}$$

for some $\lambda_i \geq 0$, $i = 1, \dots, t$, $\mu_j \geq 0$ for all $j \in J$, with $\sum_{i=1}^t \lambda_i = 1$. By premultiplying equation (3.1) with any vector a^i , $i \in J$, we obtain

$$\begin{aligned} 0 &> \sum_{h=1}^t \lambda_h a^{i\top} f(x^h) \\ &= \sum_{j \in J} \mu_j a^{i\top} a^j \\ &= \mu_i \\ &\geq 0, \end{aligned}$$

yielding a contradiction. The properties of the vectors a^i , $i \in J$, imply the above equalities. \square

We construct now a graph $G = (N, E)$ where N denotes a set of nodes and E denotes a set of edges. A simplex σ is called a *node* if it is either a J -complete $(n - |J|)$ -simplex on $cF(J)$ for some $J \in \mathcal{I}$ or a complete n -simplex. Two nodes are said to be *adjacent* if both are J -complete facets of the same $(n - |J| + 1)$ -simplex on $cF(J)$, or if one is a J -complete facet of the other and the other is a J' -complete $(n - |J| + 1)$ -simplex on $cF(J)$, or if one is a $\{j\}$ -complete facet of the other and the other is a complete n -simplex on $cF(\{j\})$. The notion $e = \{\sigma_1, \sigma_2\}$ is called an *edge* if the two nodes σ_1 and σ_2 are adjacent. The degree of a node σ in G is defined to be the number of nodes being adjacent to σ , denoted by

$deg(\sigma)$. A finite sequence of adjacent simplices in G from σ_0 to σ_l is defined as the tuple $(\sigma_0, \sigma_1, \dots, \sigma_l)$, where $\sigma_0, \sigma_1, \dots, \sigma_l$ are nodes in G and $e_i = \{\sigma_{i-1}, \sigma_i\}$ are edges in G for all $i \in I_l$.

Theorem 3.12 *Let \mathcal{T} be a triangulation of P . Then there exists a finite sequence of adjacent simplices in G from $\{q^0\}$ to a complete n -simplex.*

Proof: From Lemma 3.6 it follows that $\{q^0\}$ is a J^0 -complete 0-simplex for some unique set $J^0 \in \mathcal{I}$ with $|J^0| = n$. Since $\{q^0\}$ lies on the boundary of $cF(J^0)$, there exists a unique 1-simplex σ on $cF(J^0)$ having $\{q^0\}$ as its facet. By Lemma 3.8, either σ is a complete 1-simplex or σ is a $J^0 \setminus \{j\}$ -complete 1-simplex for some unique $j \in J^0$, or σ has exactly one other J^0 -complete facet τ . Hence, there exists a unique node being adjacent to $\{q^0\}$. That is, $deg(\{q^0\}) = 1$.

Let σ be a complete n -simplex on P . According to Lemma 3.10 σ lies in $cF(\{k\})$ for some unique $k \in I$ and has a unique $\{k\}$ -complete facet. That is, $deg(\sigma) = 1$.

In all other cases, we will show that $deg(\tau) = 2$ if τ is a node. Let τ be a J -complete $(n - |J|)$ -simplex on $cF(J)$ for some $J \in \mathcal{I}$. Then, either τ does not lie on the boundary of $cF(J)$ or τ lies on the boundary of $cF(J)$. If τ does not lie on the boundary of $cF(J)$, then τ is a facet of precisely two $(n - |J| + 1)$ -simplices on $cF(J)$. It follows from Lemma 3.8 that τ is adjacent to exactly two nodes. If τ lies on the boundary of $cF(J)$, there exists exactly one $(n - |J| + 1)$ -simplex σ on $cF(J)$ having τ as its facet. By Lemma 3.8 either σ is a complete n -simplex or a \bar{J} -complete $(n - |\bar{J}|)$ -simplex on $F(\bar{J})$ for some unique $\bar{J} \in \mathcal{I}$ with $|\bar{J}| = |J| - 1$ and has no other J -complete facets, or σ has exactly one other J -complete facet. This yields one adjacent node to τ . On the other hand, since τ lies on the boundary of $cF(J)$, it follows from Lemma 3.11 that τ does not lie on the boundary of P . Hence, since $\tau \neq \{q^0\}$, τ lies on $cF(\tilde{J})$ for some unique set $\tilde{J} \in \mathcal{I}$ with $|\tilde{J}| = |J| + 1$. By Lemma 3.9 either τ is J' -complete for some unique set $J' \in \mathcal{I}$ with $|J'| = |J|$ and $J' \neq J$, or τ has exactly one \tilde{J} -complete facet. It follows again that in both these cases there exists exactly one node adjacent to τ . This concludes that τ has exactly two adjacent nodes. That is, we have $deg(\tau) = 2$.

As shown above, the degree of each node in the graph $G = (N, E)$ is at most two. Since the number of simplices on P is finite, the number of nodes in G is finite. Since $deg(\{q^0\}) = 1$, there exists a finite sequence of adjacent nodes starting from $\{q^0\}$. The end node of this sequence must be a node of degree 1 and different from $\{q^0\}$. The only possibility is that this node is a complete n -simplex. \square

The algorithm is such that it generates the sequence of adjacent simplices described in the theorem. From the theorem it follows that starting at the point q^0 , the algorithm generates a finite sequence of adjacent J -complete $(n - |J|)$ -simplices on $cF(J)$ for varying

$J \in \mathcal{I}$, leading to a complete n -simplex σ^* on P . In the next section we show that when the mesh size of the triangulation of P goes to zero the sequence of complete simplices generated by the algorithm contains a convergent subsequence of points in P , of which the projections on the set X converge to a zero point of the mapping ϕ .

4 A Constructive Existence Proof

By making use of the results obtained in Section 3 we will give a constructive proof for Theorem 2.1. To achieve this, a sequence of triangulations \mathcal{T}^r , $r \in \mathbb{N}$, with mesh size converging to zero is taken. Applying the algorithm described in the previous section, for every $r \in \mathbb{N}$, a complete n -simplex σ_r^* on P is obtained. According to (**), σ_r^* contains a zero point q^r of the piecewise linear approximation f^r of $\bar{\phi}$ with respect to \mathcal{T}^r . In the next theorem we show that the sequence $(p(q^r))_{r \in \mathbb{N}}$ has a subsequence converging to a zero point of ϕ . Recall that $p(\cdot)$ is the orthogonal projection on X and is a continuous function.

Theorem 4.1 *Let $\phi : X \mapsto \mathbb{R}^n$ be a point-to-set mapping satisfying the conditions in Theorem 2.1. For $r \in \mathbb{N}$, let \mathcal{T}^r be a triangulation of P with mesh size smaller than $\frac{1}{r}$ and let q^r be the zero point in P of f^r found by the algorithm. Then there exists a convergent subsequence of $(q^r)_{r \in \mathbb{N}}$ and any convergent subsequence converges to a point whose projection on X is a zero point of ϕ in X .*

Proof: Since P is a compact set and $(q^r)_{r \in \mathbb{N}}$ is a sequence in P , this sequence has a convergent subsequence converging to some q^* in P . We will show that $p(q^*)$ is a zero point of ϕ . Without loss of generality we assume that $(q^r)_{r \in \mathbb{N}}$ converges to $q^* \in P$. Since $p(\cdot)$ is a continuous function, the sequence $(p(q^r))_{r \in \mathbb{N}}$ converges to $p(q^*)$. Since the mesh size of the sequence of triangulations $(\mathcal{T}^r)_{r \in \mathbb{N}}$ of P converges to zero when r goes to infinity and since $\bar{\phi}$ is upper semi-continuous, compact-valued and convex-valued, the system of equations (**) at q^r will reduce in the limit for r going to infinity, after taking subsequences if necessary, to $f^* = 0^n$ with $f^* \in \bar{\phi}(q^*)$. Hence, q^* is a zero point of $\bar{\phi}$. Let $v^* = q^* - p(q^*)$. From Lemma 3.3 it follows that $v^* \in N(X, p(q^*))$. We consider the following cases.

1) In case $q^* \in P \setminus Q$, $f^* = 0^n$ implies $p(q^*) = q^*$. Since $p(q^*)$ is in X and q^* is not in X , we obtain a contradiction.

2) In case $q^* \in \text{bd}(Q)$, we have $f^* = \mu^*(p(q^*) - q^*) + (1 - \mu^*)f = 0^n$ for some μ^* , $0 \leq \mu^* \leq 1$, and some $f \in \phi(p(q^*)) \cap \pi(v^*)$. For $\mu^* = 1$ this case reduces to case 1). For $0 \leq \mu^* < 1$, we obtain $f = \lambda^*v^*$ with $\lambda^* = \mu^*/(1 - \mu^*) \geq 0$. Hence, $f \in \pi(v^*) \cap C(v^*)$. The latter intersection is equal to $\{0^n\}$ and therefore $f = 0^n$. Consequently, $x^* = p(q^*)$ is a zero point of ϕ .

3) In case $q^* \in \text{int}(Q) \setminus X$, we have $0^n = f^* \in \phi(p(q^*))$. Hence, $x^* = p(q^*)$ is a zero point of ϕ .

4) In case $q^* \in X$, we have $q^* = p(q^*)$. This implies that $0^n = f^* \in \phi(q^*)$. Hence, $x^* = q^*$ is a zero point of ϕ . \square

This result shows that any convergent subsequence of points generated by the algorithm for a sequence of triangulations of P with mesh size going to zero, converges to a point whose projection on X is a zero point of ϕ . When for a given mesh size the accuracy of approximation is not satisfactory the algorithm can be restarted for a triangulation of P with a smaller mesh size with new starting point for example equal to the previously found approximation. In this way a priori given level of accuracy can be reached within a finite number of restarts.

In case a coincidence of two mappings ϕ and ψ on X is being computed, a vertex x of a simplex of the underlying triangulation \mathcal{T} of P is assigned the vector label $w - u$ for some $w \in \psi(p(x))$ and $u \in \phi(p(x))$ satisfying $v(q)^\top u \geq v(q)^\top w$ if $q \in Q$, and the vector label $p(x) - x$ if x is not in Q . Since $x - p(x)$ is an element of $N(X, p(x))$ such a u and w always exist when x lies in Q . Notice that if X is lower-dimensional, typically no vertex of any simplex in \mathcal{T} lies in X .

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