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SIMPLE COMBINATORIAL OPTIMISATION COST GAMES

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Abstract

In this paper we introduce the class of simple combinatorial optimisation cost games, which are games associated to $\{0,1\}$-matrices. A coalitional value of a combinatorial optimisation game is determined by solving an integer program associated with this matrix and the characteristic vector of the coalition. For this class of games, we will characterise core stability and totally balancedness. We continue by characterising exactness and largeness. Finally, we conclude the paper by applying our main results to minimum colouring games and minimum vertex cover games.

Keywords: Combinatorial optimisation game, core stability, totally balancedness, largeness, exactness.

JEL Classification Number: C71.

1 Introduction

A combinatorial optimisation game is a cooperative game associated to a $\{0,1\}$-matrix and a cost vector. A coalitional value of a combinatorial optimisation game is determined by solving an integer program associated with the matrix, the cost vector, and the characteristic vector of the coalition. The class of combinatorial optimisation games, first introduced in [3], includes, for example, maximum flow games, maximum matching games, minimum vertex cover games and minimum colouring games. In [3] it is shown that the core of a combinatorial optimisation game is non-empty if and only if the outcome of the integer program associated with the grand coalition is equal to the outcome of its relaxation. Using this result, [3] studies algorithmic aspects of the core of several subclasses of combinatorial optimisation games. In the follow-up paper [4], totally balancedness of combinatorial optimisation games is considered.

The class of minimum colouring games, a subclass of combinatorial optimisation games, is studied in [1] and [8]. The latter paper shows that the core of a minimum colouring game coincides with the convex hull of the characteristic vectors of maximum cliques, in case the underlying graph is perfect. Besides that, that paper studies well-known one-point solution concepts as the nucleolus, the τ-value and the Shapley value.

A characterisation of core stability of minimum colouring games on the class of perfect graphs is one of the main results of [1]. In cooperative game theory, characterising core stability is a notoriously difficult problem. Only a few classes of games with stable cores are known. These classes include convex games ([9]), chain-component additive games ([11]) and assignment games ([10]). Besides the characterisation of core stability, [1] also considers related concepts as exactness, extendibility and largeness. These concepts are shown to be equivalent on the class of perfect graphs.

In this paper we introduce the class of simple combinatorial optimisation cost games. The word simple here refers to the fact that we only consider situations with the cost vector consisting of ones only. For this class of games, we will characterise core stability in terms of the core. Subsequently, we characterise totally balancedness in terms of the underlying matrix. In particular we show that a simple combinatorial optimisation cost game is totally balanced if and only if the transpose of its underlying matrix is perfect. We continue by characterising exactness and largeness in terms of properties of the underlying matrix. We conclude the paper by applying our main results to minimum colouring games and minimum vertex cover games. For the class of minimum colouring games we are able to prove the main results of [1] in an alternative short way. Finally, we characterise core stability and totally balancedness of minimum vertex cover games, and we show that largeness, extendibility and exactness are equivalent for these games.

The remainder of this paper is organised as follows. In Section 2 we recall some elementary concepts from cooperative game theory. In Section 3 we introduce simple combinatorial optimisation cost games, and we present our main results. In Sections 4 and 5 we apply our main results to the classes of minimum colouring games and minimum vertex games, respectively.

2 Preliminaries

A *transferable utility cost game*, or game for short, (N, c) consists of a finite player set N and a map $c : 2^N \to \mathbb{R}$ assigning to each coalition $S \subseteq N$ a cost $c(S)$. By assumption, $c(\emptyset) = 0$. For each $S \subseteq N$, the *subgame* (S, c_S) is the game with player set S and $c_S(T) = c(T)$ for each $T \subseteq S$. The *core* of a game (N, c), denoted by $C(c)$, consists of the allocations of $c(N)$ such that no coalition has an incentive to split off from the grand coalition. Formally, $C(c) = \{x \in \mathbb{R}^N : \sum_{i\in N} x_i = c(N),\ \sum_{i\in S} x_i \leq c(S)$ for each $S \subseteq N\}$. The core of a game can be empty. A game with a non-empty core is called *balanced*. A balanced game is *totally balanced* if all its subgames are balanced as well. The *imputation set* is the set $I(c) = \{x \in \mathbb{R}^N : \sum_{i\in N} x_i = c(N),\ x_i \leq c(\{i\})$ for each $i \in N\}$, and the *upper core* is given by $U(c) = \{x \in \mathbb{R}^N : \sum_{i\in S} x_i \leq c(S)$ for each $S \subseteq N\}$. Obviously, $C(c) \subseteq I(c)$ and $C(c) \subseteq U(c)$.

Now let (N, c) be a game and let $x, y \in I(c)$. Then x is said to *dominate* y via $S \subseteq N$ if $x_i < y_i$ for each $i \in S$ and $\sum_{i\in S} x_i \geq c(S)$. Intuitively, S will not agree to allocation y since the reachable allocation x is strictly better for each member of S. The core of (N, c) is called *stable* if each imputation outside the core is dominated by a core element via some coalition. A game (N, c) is said to be *exact* if for each $S \subseteq N$ there is an $x \in C(c)$ with $\sum_{i\in S} x_i = c(S)$. The core of (N, c) is *large* if for each $y \in U(c)$ there is an $x \in C(c)$ with $y \leq x$. The game (N, c) is called *extendible* if each core element of each subgame can be extended to a core element, i.e. if for each $S \subseteq N$ and $y \in C(c_S)$ there is an $x \in C(c)$ with $x_i = y_i$ for each $i \in S$.

In [6] it is shown that largeness of the core is sufficient for extendibility and that extendibility on its turn is sufficient for core stability. Also observe that totally balanced extendible games are exact.

3 Simple combinatorial optimisation cost games

In this section we present our main results. First we introduce simple combinatorial optimisation cost games, and then we continue with characterising core stability in terms of a core property. Subsequently, we characterise totally balancedness in terms of the underlying matrix and use this result to characterise exactness. The final result of this section is a characterisation of largeness.

Let A be a $\{0, 1\}$-matrix, with its row set indexed by N, its column set indexed by M, and each row and each column containing at least one non-zero entry. Let $w : M \to \mathbb{R}$ be a weight vector on the columns of A. The *combinatorial optimisation cost game*, as introduced in [3], associated with

A and w is defined by

$$c(S) = \min\{wx : Ax \geq 1_S, x \in \{0,1\}^M\}$$

for each $S \subseteq N$. In this paper we will only consider situations with $w = 1_M$. The associated games are referred to as *simple combinatorial optimisation cost games*. The following result is due to [3].

Theorem 3.1 ([3]) Let A be a $\{0,1\}$-matrix with its row set indexed by N, its column set indexed by M, and each row and each column containing at least one non-zero entry. Let (N, c) be its associated simple combinatorial optimisation cost game. Then $C(c) \neq \emptyset$ if and only if $\min\{\sum_{i \in M} x_i : Ax \geq 1_N, x \in \{0,1\}^M\} = \min\{\sum_{i \in M} x_i : Ax \geq 1_N, x \geq 0\}$. In case the core is non-empty, $z \in C(c)$ if and only if z is an optimal solution of the dual of $\min\{\sum_{i \in M} x_i : Ax \geq 1_N, x \geq 0\}$.

The previous theorem included a description of the core of simple combinatorial optimisation cost games. The next lemma gives a description of the nonnegative part of the upper core.

Lemma 3.1 Let A be a $\{0,1\}$-matrix with its row set indexed by N, its column set indexed by M, and each row and each column containing at least one non-zero entry. Let (N, c) be its associated simple combinatorial optimisation cost game. Then $U(c) \cap \mathbb{R}^N_+ = \{x \geq 0 : xA \leq 1_M\}$.

Proof: Obviously $U(c) \cap \mathbb{R}^N_+ \subseteq \{x \geq 0 : xA \leq 1_M\}$ since the columns of A can be seen as characteristic vectors of coalitions with cost equal to 1. So it remains to show $\{x \geq 0 : xA \leq 1_M\} \subseteq U(c) \cap \mathbb{R}^N_+$. Let $x \geq 0$ be such that $xA \leq 1_M$. Let $S \subseteq N$. Let $M' \subseteq M$ be an index set of columns minimising the cost of S, i.e. $c(S) = |M'|$ and $A 1_{M'} \geq 1_S$. With abuse of notation, let A_j be the coalition whose characteristic vector is the j-th column of A, for each $j \in M'$. Then,

$$\sum_{i \in S} x_i \leq \sum_{j \in M'} \sum_{i \in A_j} x_i \leq \sum_{j \in M'} 1 = |M'| = c(S).$$

The first inequality is satisfied because $x \geq 0$, and the second because $xA \leq 1_M$. We conclude that $\sum_{i \in S} x_i \leq c(S)$ for each $S \subseteq N$, and therefore $x \in U(c)$. □

In the upcoming part of this section we will characterise core stability of simple combinatorial optimisation cost games. Before we state and prove our characterisation, it is convenient to prove two lemmas. The first lemma uses a concept called essential extendibility and shows that this concept yields a sufficient condition for core stability of any cost game. The second lemma provides a necessary condition for core stability. We first introduce essential extendibility.

Let (N, c) be a cost game. A coalition $S \subseteq N$, $S \neq \emptyset$, is called *essential* for (N, c) if for each proper partition P of S it is satisfied that $\sum_{W \in P} c(W) > c(S)$. The set of essential coalitions of (N, c) is denoted by $\mathcal{E}$. A game (N, c) is said to be *essential extendible* if for each $S \in \mathcal{E}$ and each $y \in C(c_S)$ there is an $x \in C(c)$ with $y_i = x_i$ for each $i \in S$. So if (N, c) is extendible, then it is essential extendible as well. The following lemma states that essential extendibility is sufficient for core stability.

Lemma 3.2 ([11]) Let (N, c) be essential extendible. Then $C(c)$ is stable.

The next lemma is helpful in order to characterise core stability as well.

Lemma 3.3 Let (N, c) be a game. If $C(c)$ is stable, then for each $i \in N$ there is a $y \in C(c)$ with $y_i = c(\{i\})$.

Proof: We prove the lemma by contradiction. Assume that $C(c)$ is stable, but that $i \in N$ is such that $y_i < c(\{i\})$ for each $y \in C(c)$. Let $y \in \text{argmax}\{x_i : x \in C(c)\}$. Let $j \in N \backslash \{i\}$, $\varepsilon = c(\{i\}) - y_i > 0$ and define $z \in \mathbb{R}^N$ by

$$z_l = \begin{cases} y_l, & \text{if } l \in N \backslash \{i, j\}; \\ y_i + \varepsilon, & \text{if } l = i; \\ y_j - \varepsilon, & \text{if } l = j. \end{cases}$$

Since $z_i = y_i + \varepsilon > y_i$ and because y_i is the maximum that is allocated to i in any core allocation, $z \notin C(c)$. Observe that $z \in I(c)$ since $z_k \leq 1$ for each $k \in N$ and $\sum_{k \in N} z_k = c(N)$. Since $C(c)$ is stable, there is an $S \subseteq N$ and a $w \in C(c)$ with $\sum_{l \in S} w_l = c(S)$ and $w_l < z_l$ for each $l \in S$. Obviously, $i \in S$, $j \notin S$ and $S \backslash \{i\} \neq \emptyset$. Indeed, coalitions not containing i are satisfied with z, since these coalitions are already satisfied with y. This implies that $i \in S$. Similarly, coalitions containing j are satisfied with z since these are satisfied with y. So, $j \notin S$. Finally, because $z_i = c(\{i\})$, it follows that $S \neq \{i\}$. Now note that

$$c(S) = \sum_{k \in S} w_k = \sum_{k \in S \backslash \{i\}} w_k + w_i \leq \sum_{k \in S \backslash \{i\}} w_k + y_i < \sum_{k \in S \backslash \{i\}} z_k + y_i = \sum_{k \in S \backslash \{i\}} y_k + y_i \leq c(S).$$

The first equality holds because $\sum_{k \in S} w_k = c(S)$. The first inequality is satisfied because $x_i \leq y_i$ for each $x \in C(c)$. The strict inequality is because w dominates z via S, which implies $w_k < z_k$ for each $k \in S$. The third equality is because $z_k = y_k$ for each $k \in S \backslash \{i\}$. The last inequality is due to $y \in C(c)$. □

Now we are ready to state and prove our characterisation of core stability of simple combinatorial optimisation cost games.

Theorem 3.2 Let A be a $\{0,1\}$-matrix with its row set indexed by N, its column set indexed by M, and each row and each column containing at least one non-zero entry. Let (N, c) be its associated simple combinatorial optimisation cost game. The following assertions are equivalent:

1. For each $i \in N$ there is a $y \in C(c)$ with $y_i = 1$;
2. (N, c) is essential extendible;
3. $C(c)$ is stable.

Proof: We only show $1 \Rightarrow 2$. The implication $2 \Rightarrow 3$ is proved in Lemma 3.2 and $3 \Rightarrow 1$ follows from Lemma 3.3 by using that $c(\{i\}) = 1$ for each $i \in N$.

Assume that 1 is satisfied. Let $S \in \mathcal{E}$. Then $c(S) = 1$. By assumption, for each $i \in S$ there is a $y^i \in C(c)$ with $y^i_i = 1$. Since $c(S) = 1$, and $y \geq 0$ for each $y \in C(c)$ it follows that $y^i_j = 0$ for each $j \in S \backslash \{i\}$. Now let $x \in C(c_S)$ and construct $z = \sum_{i \in S} x_i y^i$. Since $x_i \geq 0$ and $y^i \in C(c)$ for each $i \in S$, it follows that $z \in C(c)$. Now note that z extends x. □

From Theorem 3.2 we can conclude that exactness is a sufficient condition for core stability of combinatorial optimisation cost games.

Corollary 3.1 Let A be a $\{0,1\}$-matrix with its row set indexed by N, its column set indexed by M, and in each row and each column at least one non-zero entry. Let (N, c) be its associated simple combinatorial optimisation cost game. If (N, c) is exact, then its core is stable.

Proof: If (N, c) is exact, then for each $i \in N$ there is an $x \in C(c)$ with $x_i = c(\{i\}) = 1$. Hence, (N, c) satisfies the first assertion of Theorem 3.2 and we conclude that $C(c)$ is stable. □

In the upcoming part of this section we will characterise exactness of simple combinatorial optimisation cost games. It is convenient to first study totally balancedness, and to introduce the concepts of totally dual integrality and perfect matrices.

Let B be a matrix whose row set is indexed by M and whose column set is indexed by N, and let $b \in \mathbb{R}^M_+$. The linear system $Bx \leq b$, $x \geq 0$ is *totally dual integral* if the linear program $\max\{cx : Bx \leq b, x \geq 0\}$ has an integral dual solution for all $c \in \mathbb{Z}^N$ for which it has an optimal solution.

Let A be a $\{0,1\}$-matrix with its row set indexed by M, its column set indexed by N, in each row and each column at least one non-zero entry. Then A is called *perfect* if the polytope $\{x \geq 0 : Ax \leq 1_M\}$ has only integral extreme points. The following theorem is a well-known characterisation of perfect matrices.

Theorem 3.3 ([7]) Let A be a $\{0,1\}$-matrix with its row set indexed by M, its column set indexed by N, and in each row and each column at least one non-zero entry. The following assertions are equivalent:

1. A is perfect;

2. the linear system $Ax \leq 1_M$, $x \geq 0$ is totally dual integral;

3. $\max\{cx : Ax \leq 1_M, x \geq 0\}$ is an integer for each $c \in \{0,1\}^N$.

The following theorem easily follows from Theorem 3.3.

Theorem 3.4 Let A be a $\{0,1\}$-matrix with its row set indexed by N, its column set indexed by M, and in each row and each column at least one non-zero entry. Let (N, c) be its associated simple combinatorial optimisation cost game. Then (N, c) is totally balanced if and only if A^T is perfect.

Proof: First we show the "if"-part. Assume that A^T is perfect. According to Theorem 3.3, the linear system $xA \leq 1_M$, $x \geq 0$ is totally dual integral. So $\max\{cx : xA \leq 1_M, x \geq 0\}$ has an integral optimal dual solution for each $c \in \mathbb{Z}^N$. In particular, $\max\{cx : xA \leq 1_M, x \geq 0\}$ has an integral optimal dual solution for each $c \in \{0,1\}^N$. This implies that $\min\{\sum_{i \in M} x_i : Ax \geq c, x \geq 0\} = \min\{\sum_{i \in M} x_i : Ax \geq c, x \in \{0,1\}^M\}$ for each $c \in \{0,1\}^N$. Or equivalently, $\min\{\sum_{i \in M} x_i : Ax \geq 1_S, x \geq 0\} = \min\{\sum_{i \in M} x_i : Ax \geq 1_S, x \in \{0,1\}^M\}$ for each $S \subseteq N$. Since each subgame of a simple combinatorial optimisation cost game is again a simple combinatorial optimisation cost game, it follows according to Theorem 3.1 that (S, c_S) is balanced for each $S \subseteq N$. So (N, c) is totally balanced.

It remains to show the "only if"-part. Assume that (N, c) is totally balanced. Then Theorem 3.1 implies that $\min\{\sum_{i \in M} x_i : Ax \geq 1_S, x \geq 0\} = \min\{\sum_{i \in M} x_i : Ax \geq 1_S, x \in \{0,1\}^M\}$ for each $S \subseteq N$. So $\max\{\sum_{i \in S} x_i : xA \leq 1_M, x \geq 0\} = \min\{\sum_{i \in M} x_i : Ax \geq 1_S, x \geq 0\}$ is an integer for each $S \subseteq N$. This implies that $\max\{cx : xA \leq 1_M, x \geq 0\}$ is an integer for each $c \in \{0,1\}^N$. According to Theorem 3.3, A^T is perfect. □

Using Lemma 3.1 and Theorem 3.4 it is straightforward to show that totally balanced simple combinatorial optimisation cost games have integral cores. Indeed, if (N, c) is totally balanced, then A^T is perfect. This means that each extreme point of $\{x \geq 0 : xA \leq 1_M\}$ is integral. Because $\{x \geq 0 : xA \leq 1_M\} = U(c) \cap \mathbb{R}^N_+$ and because $C(c)$ is a facet of $U(c) \cap \mathbb{R}^N_+$, each extreme point of $C(c)$ is integral as well.

We proceed this section with a characterisation of exactness, but first we need one more lemma.

Lemma 3.4 Let A be a $\{0,1\}$-matrix with its row set indexed by N, its column set indexed by M, and in each row and each column at least one non-zero entry. Let (N, c) be its associated simple combinatorial optimisation cost game. If (N, c) is exact, then for all $y \in \{0,1\}^N$ with $yA \leq 1_M$ there is a $z \in C(c) \cap \{0,1\}^N$ with $y \leq z$.

Proof: Let (N, c) be exact. Let $y \in \{0,1\}^N$ be such that $yA \leq 1_M$. Since (N, c) is a simple combinatorial optimisation cost game, it follows that $\sum_{i \in N} y_i \leq c(N)$. If $\sum_{i \in N} y_i = c(N)$, then, according to Lemma 3.1, $y \in C(c)$ and we are done. So assume that $\sum_{i \in N} y_i < c(N)$. We will show that there is a $w \in \{0,1\}^N$ with $wA \leq 1_M$, $\sum_{i \in N} w_i = \sum_{i \in N} y_i + 1$ and $y \leq w$. A recursive argument then provides the $z \in C(c) \cap \{0,1\}^N$ with $y \leq z$.

First define $S = \{i \in N : y_i = 1\}$. Because $yA \leq 1_M$ and $y \geq 0$, we conclude from Lemma 3.1 that $y \in U(c)$. Hence, $|S| = \sum_{i \in S} y_i \leq c(S)$. By definition of (N, c), $c(S) \leq |S|$. So, $c(S) = |S|$. Because (N, c) is exact, there is an $x \in C(c)$ with $\sum_{i \in S} x_i = c(S)$. So $x_i = 1$ for each $i \in S$. If x is integral, then $x \in C(c) \cap \{0,1\}^N$, $y \leq x$ and we are done. So assume that x is not integral. Let $i \in N$ be such that $0 < x_i < 1$. Clearly, $i \notin S$. Now define $\bar{x} \in \mathbb{R}^N$ by $\bar{x}_j = x_j$ if $j \in S \cup \{i\}$ and $\bar{x}_j = 0$ otherwise. Furthermore define $w \in \{0,1\}^N$ by $w_j = 1$ if $j \in S \cup \{i\}$ and $w_j = 0$ otherwise.

It follows from $x \in C(c)$ that $xA \leq 1_M$. Hence, $\bar{x}A \leq 1_M$ as well. This implies that each column of A has at most one non-zero entry in the rows associated with $S \cup \{j\}$. Therefore, $wA \leq 1_M$. Now note that $w \in \{0,1\}^N$ is such that $wA \leq 1_M$, $\sum_{i \in N} w_i = \sum_{i \in N} y_i + 1$ and $y \leq w$. □

We are now ready to prove our characterisation of exactness.

Theorem 3.5 Let A be a $\{0,1\}$-matrix with its row set indexed by N, its column set indexed by M, and in each row and each column at least one non-zero entry. Let (N, c) be its associated simple combinatorial optimisation cost game. The following assertions are equivalent:

1. (N, c) is exact;
2. A^T is perfect and for each $y \in \{0,1\}^N$ with $yA \leq 1_M$ there is a $z \in C(c) \cap \{0,1\}^N$ with $y \leq z$.

Proof: We only prove $2 \Rightarrow 1$, since $1 \Rightarrow 2$ follows from the combination of Theorem 3.4 (since exact games are totally balanced) and Lemma 3.4.

Let $S \subseteq N$. Since A^T is perfect, it follows from Theorem 3.4 that (N, c) is totally balanced. Therefore, $C(c_S) \neq \emptyset$. Let x^S be an extreme point of $C(c_S)$ and extend x^S to the vector x in $\mathbb{R}^N$ by assigning zeros to the players in $N \backslash S$. Note that $x \geq 0$, $xA \leq 1_M$, $\sum_{i \in S} x_i = c(S)$ and $x_i = 0$ for each $i \in N \backslash S$. Using Lemma 3.1, x is an extreme point of $\{y \geq 0 : yA \leq 1_M\}$. Since A^T is perfect, we conclude that x is integral. According to our assumption there is a $z \in C(c) \cap \{0,1\}^N$ with $x \leq z$. Note that $\sum_{i \in S} z_i = c(S)$. So, (N, c) is exact. □

We conclude this section with a characterisation of largeness of the core.

Proposition 3.1 Let A be a $\{0,1\}$-matrix with its row set indexed by N, its column set indexed by M, and in each row and each column at least one non-zero entry. Let (N, c) be its associated simple combinatorial optimisation cost game. Then $C(c)$ is large if and only if for all $y \geq 0$ with $yA \leq 1_M$ there is a $z \in C(c)$ with $y \leq z$.

Proof: First we show the "only if"-part. Assume that $C(c)$ is large. Let $y \geq 0$ be such that $yA \leq 1_M$. According to Lemma 3.1, $y \in U(c)$. Hence, there is a $z \in C(c)$ with $y \leq z$.

It remains to show the opposite direction. Assume that for all $y \geq 0$ with $yA \leq 1_M$ there is a $z \in C(c)$ with $y \leq z$. Let $y \in U(c)$. Since columns of A are characteristic vectors of coalitions with cost equal to 1, we conclude that $yA \leq 1_M$. Now define $\bar{y}$ by $\bar{y}_i = \max\{0, y_i\}$ for each $i \in N$. We will first show that $\bar{y}A \leq 1_M$ by contradiction. Suppose that $i \in M$ is such that $(\bar{y}A)_i > 1$. Define

$T = \{j \in N : A_{ji} = 1\} \cap \{j \in N : y_j > 0\}$. Then $\sum_{j \in T} y_j = \sum_{j \in T} \bar{y}_j > 1 = c(T)$. But this implies that $y \notin U(c)$, which is clearly a contradiction. Therefore $\bar{y}A \leq 1_M$. Also note that $\bar{y} \geq 0$. So by assumption there is a $z \in C(c)$ with $\bar{y} \leq z$. Note that $y \leq z$ as well. □

We conclude this section with several examples. The first example shows that largeness need not be sufficient for exactness for the class of simple combinatorial optimisation cost games.

Example 3.1 Consider the matrix A and its associated simple combinatorial optimisation cost game (N, c).

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is straightforward to check that the subgame $(\{2,4,5\}, c_{\{2,4,5\}})$ is not balanced, since $c(\{2,4\}) = c(\{2,5\}) = c(\{4,5\}) = 1$ and $c(\{2,4,5\}) = 2$. So (N, c) is not exact as well. However, $C(c)$ is large. Indeed, let $y \geq 0$ be such that $yA \leq 1_M$. Define $z = (y_1, y_2, 1 - y_1 - y_2, y_4, y_5, 1 - y_4 - y_5)$. Note that $y \leq z$ and that $z \in C(c)$. According to Proposition 3.1, $C(c)$ is large. ◇

Example 3.1 in extremis shows that extendibility does not imply exactness (since largeness is sufficient for extendibility). The next example shows that largeness and extendibility are not equivalent.

Example 3.2 Consider the matrix A and the associated simple combinatorial optimisation cost game (N, c).

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

The core of (N, c) is the convex hull of $(1,0,0,0,0,1)$, $(0,1,0,0,1,0)$ and $(0,0,1,1,0,0)$. It is clear that the core is not large, since $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0) \in U(c)$ is not exceeded by a core element. However, this game is extendible. Indeed, let $S \subseteq N$ be such that $C(c_S) \neq \emptyset$, and let $x \in C(c_S)$. First suppose that $c(S) = 1$. Note that for all $i \in S$ there is a $z^i \in C(c)$ with $z^i_i = 1$ and $z^i_j = 0$ for each $j \in S \backslash \{i\}$. Now note that $z = \sum_{i \in S} x_i z^i$ is a core element of (N, c) that extends x.

Now suppose that $c(S) = 2$. Then x can be extended to an element of $C(c)$ by assigning zero to the players outside S. ◇

4 Minimum colouring games

In this section we consider minimum colouring games. In [1] core stability is characterised for these games on the class of perfect graphs. Moreover, it is shown that largeness, extendibility and exactness are equivalent on the class of perfect graphs and these concepts are characterised. In this section we reprove their results using the results obtained in Section 3. First we recall some elementary graph theory.

Let $G=(V,E)$ be a graph. A *stable set* is a subset of vertices inducing a graph with no edge. A *clique* is a vertex set inducing a complete graph. A clique is a maximum clique if the graph does not contain a clique of larger cardinality. The size of a maximum clique is denoted by $\omega(G)$. A *colouring* of G is a map $c: V \to \mathbb{N}$ such that $c(u) \neq c(v)$ for each $\{u,v\} \in E$. A minimum colouring is a colouring with minimum possible $|c(V)|$. The *chromatic number* $\chi(G)$ is the cardinality $|c(V)|$ of a minimum colouring $c: V \to \mathbb{N}$. For each $S \subseteq V$, the subgraph induced by S is the graph $G[S]=(S,E_S)$, where $E_S=\{\{i,j\} \in E : i,j \in S\}$. A graph is called *perfect* if $\omega(G[S])=\chi(G[S])$ for each $S \subseteq V$.

Let $G=(V,E)$ be a graph. The *minimum colouring game* as introduced in [3] is defined by $N=V$ and $c(S)=\chi(G[S])$ for each $S \subseteq N$. Given a graph G, let M be the set of stable sets of G. Furthermore, let A be the vertex-stable set incidence matrix of G. That is, $A_{ij}=1$ if vertex i is in stable set j and $A_{ij}=0$ otherwise. Observe that $c(S)=\min\{\sum_{i\in M} x_i : Ax \geq 1_S, x \in \{0,1\}^M\}$ for each $S \subseteq N$. So minimum colouring games are simple combinatorial optimisation cost games. The following result is due to [8].

Theorem 4.1 ([8]) Let $G=(V,E)$ be a perfect graph. The core of a minimum colouring game is the convex hull of the characteristic vectors of maximum cliques of G.

So minimum colouring games have non-empty cores in case the underlying graph is perfect. In fact, since induced subgraphs of perfect graphs are again perfect, minimum colouring games are totally balanced in case the underlying graph is perfect. In the remainder of this section we reprove several results of [1], using the theory developed in Section 3. The first result is a characterisation of core stability.

Theorem 4.2 ([1]) Let $G=(V,E)$ be a perfect graph. The associated minimum colouring game has a stable core if and only if each vertex belongs to a maximum clique of G.

The proof of this theorem is omitted since it is a straightforward combination of Theorems 3.2 and 4.1.

In the remainder of this section we reprove the characterisation of largeness, extendibility and exactness obtained by [1]. It is convenient to use the following theorem of [2] and [5] dealing with clique polytopes of perfect graphs. Here the clique polytope of a graph is the convex hull of characteristic vectors of cliques.

Theorem 4.3 ([2],[5]) Let $G=(V,E)$ be a graph and let A be its vertex-stable set incidence matrix. Then G is perfect if and only if the clique polytope of G is determined by $\{y \geq 0 : yA \leq 1_M\}$.

Theorem 4.4 ([1]) Let $G=(V,E)$ be a perfect graph. Let (N,c) be the associated minimum colouring game. The following assertions are equivalent:

1. $C(c)$ is large;
2. (N,c) is extendible;
3. (N,c) is exact;
4. Every clique of G is contained in a maximum clique.

Proof: The implication $1 \Rightarrow 2$ is again due to [6], and $2 \Rightarrow 3$ holds for all totally balanced games. So we first prove $3 \Rightarrow 4$. Assume that (N,c) is exact. Let $V' \subseteq V$ be a clique of G and let $y \in \{0,1\}^N$ be the characteristic vector of this clique. Note that $yA \leq 1_M$. From Lemma 3.4

it follows that there is a $z \in C(c) \cap \{0,1\}^N$ with $y \leq z$. According to Theorem 4.1, z is the characteristic vector of a maximum clique of G, say V''. Observe that $V' \subseteq V''$.

It remains to show $4 \Rightarrow 1$. Assume that every clique of G is contained in a maximum clique. Let A be the vertex-stable set incidence matrix of G and define $P = \{y \geq 0 : yA \leq 1_M\}$. According to Proposition 3.1 we need to show for each $y \in P$ that there is a $z \in C(c)$ with $y \leq z$. Clearly, it is sufficient to show for each extreme point y of P that there is a $z \in C(c)$ with $y \leq z$. So let $y \in P$ be an extreme point of P. According to Theorem 4.3, using the perfection of G, it follows that y is an extreme point of the clique polytope of G. That is, y is the characteristic vector of a clique of G. Let $V' \subseteq V$ be this clique. By assumption, there is a maximum clique $V'' \subseteq V$ containing V'. Let z be the characteristic vector of V''. Observe that $y \leq z$. Theorem 4.1 implies that $z \in C(c)$. So $C(c)$ is large. □

5 Minimum vertex cover games

In this section we study minimum vertex cover games. Since minimum vertex cover games are simple combinatorial optimisation cost games, we can apply the results obtained in Section 3 to these games. In particular, we will characterise core stability and totally balancedness of minimum vertex cover games in terms of the underlying graph. Furthermore we show that exactness, extendibility and largeness are equivalent, and we characterise these concepts.

First we introduce minimum vertex cover games, as defined in [3]. Let $G = (V, E)$ be a graph. In the associated *minimum vertex cover game* (N, c), each edge represents a player, and the cost of a coalition $S \subseteq E$ is the cardinality of a minimum vertex cover of $G_S = (V, S)$. (A vertex cover is a subset of the vertices such that each edge is adjacent to at least one vertex of this subset.) The size of a minimum vertex cover is denoted by $\tau(G)$. A matching is a set of pairwise non-adjacent edges. The size of a maximum matching is denoted by $\nu(G)$. Let A be the edge-vertex incidence matrix of G. So $A_{ij} = 1$ if edge i is incident to vertex j, and $A_{ij} = 0$ otherwise. Formally, the minimum vertex cover game (N, c) is given by $N = E$ and

$$c(S) = \min\{\sum_{i \in V} x_i : Ax \geq 1_S, x \in \{0,1\}^V\}$$

for each $S \subseteq N$. The following result is due to [3].

Theorem 5.1 ([3]) Let $G = (V, E)$ be a graph, and let (N, c) be its associated minimum vertex cover game. Then $C(c) \neq \emptyset$ if and only if $\tau(G) = \nu(G)$. If $C(c) \neq \emptyset$, then an imputation is in the core if and only if it is a convex combination of characteristic vectors of maximum matchings in G. (So there is a one-to-one correspondence between maximum matchings and extreme points of $C(c)$.)

Using the results from the previous section it is straightforward to characterise core stability of minimum vertex cover games.

Theorem 5.2 Let $G = (V, E)$ be a graph, and let (N, c) be its associated minimum vertex cover game. Then the following statements are equivalent:

1. G is bipartite and each edge is member of a maximum matching;
2. $\nu(G) = \tau(G)$ and each edge is member of a maximum matching;
3. $C(c)$ is stable.

Proof: Implication 1 $\Rightarrow$ 2 is immediate, since König's matching theorem states that for each bipartite graph G it holds that $\nu(G) = \tau(G)$. The equivalence between statements 2 and 3 follows from Theorems 3.2 and 5.1. It remains to show 2 $\Rightarrow$ 1.

Assume that $G = (V, E)$ is such that $\tau(G) = \nu(G)$, and that each edge is member of a maximum matching. We need to show that G is bipartite. To this end, let $V' \subseteq V$ be a minimum vertex cover, and define $W = V \backslash V'$. We show that the V' and W are independent sets, i.e. sets of vertices for which the induced subgraph does not contain any edges, which proves that G is bipartite.

Let $i, j \in W$ with $i \neq j$. If $\{i, j\} \in E$, then V' is not a vertex cover. Hence, $\{i, j\} \notin E$. We conclude that W forms an independent set.

Let $i, j \in V'$ with $i \neq j$. Suppose that $e = \{i, j\} \in E$. According to our assumption, there is a maximum matching $E' \subseteq E$ containing e. So $|V'| = \tau(G) = \nu(G) = |E'|$. Since $|V'| = |E'|$, $e = \{i, j\} \in E'$ and $i, j \in V'$, it follows there is a $\{k, l\} \in E'$ with $k, l \notin V'$. This implies that $\{k, l\} \in E$ for $k, l \in W$, contradicting that W is an independent set. So we conclude that $\{i, j\} \notin E$ for each $i, j \in V'$, $i \neq j$. So V' is an independent set as well. □

Totally balancedness of minimum vertex cover games is characterised in [4]. We include this characterisation since it assists the proof of Theorem 5.4.

Theorem 5.3 ([4]) Let $G = (V, E)$ be a graph, and let (N, c) be its associated minimum vertex cover game. Then (N, c) is totally balanced if and only if G is bipartite.

As a final result we show that largeness, extendibility and exactness are equivalent on the class of minimum vertex cover games. Moreover, we provide a characterisation of these concepts in terms of the underlying graph.

Theorem 5.4 Let $G = (V, E)$ be a graph, and let (N, c) be its associated minimum vertex cover game. The following statements are equivalent:

1. $C(c)$ is large;
2. (N, c) is extendible;
3. (N, c) is exact;
4. G is bipartite and each matching is contained in a maximum matching.

Proof: The implication 1 $\Rightarrow$ 2 is proved in [6] for all games. So we first show 2 $\Rightarrow$ 3. Assume that (N, c) is extendible. In order to show that (N, c) is exact, it is sufficient to show that (N, c) is totally balanced. First observe that extendibility of (N, c) is sufficient for core stability. This implies according to Theorem 5.2 that G is bipartite. According to Theorem 5.3, (N, c) is totally balanced.

We now prove the implication 3 $\Rightarrow$ 4. Assume that (N, c) is exact. Then (N, c) is totally balanced. According to Theorem 5.3, G is bipartite. It remains to show that each matching is contained in a maximum matching. Let $E' \subseteq E$ be a matching, and let $y \in \{0, 1\}^E$ be its characteristic vector. Let A be the edge-vertex incidence matrix of G. Then $yA \leq 1_M$. According to Lemma 3.4, there is a $z \in C(c) \cap \{0, 1\}^N$ with $y \leq z$. Let $E'' \subseteq E$ be the matching whose characteristic vector is z. According to Theorem 5.1 E'' is a maximum matching containing E'.

It remains to show 4 $\Rightarrow$ 1. Assume that G is bipartite and that each matching is contained in a maximum matching. Define $P = \{y \geq 0 : yA \leq 1_M\}$. According to Proposition 3.1 we need to show for each $y \in P$ that there is a $z \in C(c)$ with $y \leq z$. Clearly, it is sufficient to show for each extreme point y of P that there is a $z \in C(c)$ with $y \leq z$. So let $y \in P$ be an extreme point of P. Since G is bipartite, it follows that y is integral. (Because G is bipartite if and only if its edge-vertex incidence matrix is totally unimodular.) That means that y is the characteristic vector of a matching in G. Let $E' \subseteq E$ be this matching. By assumption, there is a maximum matching $E'' \subseteq E$ with containing E'. Let z be the characteristic vector of E''. Clearly, $y \leq z$. From Theorem 5.1 it follows that $z \in C(c)$. □

We conclude this section with an example that shows that for minimum vertex cover games core stability is not equivalent to largeness, extendibility and exactness.

Example 5.1 Let $G = (V, E)$ be given by $V = \{a, b, c, d, e, f\}$ and $E = \{\{a, d\}, \{a, f\}, \{b, d\}, \{b, e\}, \{c, e\}, \{c, f\}\}$. Let (N, c) be its associated minimum vertex cover game. Note that $C(c)$ is stable since the graph is bipartite, and because each edge is part of a maximum matching. However, since the matching $\{\{a, d\}, \{c, e\}\}$ is not part of a maximum matching, it follows that $C(c)$ is not large.

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