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Abstract

This paper introduces a new class of transferable-utility games, called multi-issue allocation games. These games arise from various allocation situations and are based on the concepts underlying the bankruptcy model, as introduced by O'Neill (1982). In this model, a perfectly divisible good (estate) has to be divided amongst a given set of agents, each of whom has some claim on the estate. Contrary to the standard bankruptcy model, the current model deals with situations in which the agents' claims are multi-dimensional, where the dimensions correspond to various issues.

It is shown that the class of multi-issue allocation games coincides with the class of (nonnegative) exact games. The run-to-the-bank rule is introduced as a solution for multi-issue allocation situations and turns out to be Shapley value of the corresponding game. Finally, this run-to-the-bank rule is characterised by means of a consistency property.

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1 Introduction

Bankruptcy problems were first introduced by O'Neill (1982) and have been subsequently analysed in a variety of contexts. In a bankruptcy situation, one has to divide a given amount of money (estate) amongst a set of agents, each of whom has a claim on the estate. The total amount claimed typically exceeds the estate available, so not all the claims of the agents can be fully satisfied.

The example originally given by O'Neill (and which is inspired by some passages in the Talmud) is that of a bequest: a man dies, leaving behind an estate which is not sufficiently large to satisfy all promises made to his heirs in his will. Another example is that of a firm going bankrupt, whose assets are insufficient to satisfy all creditors' outstanding claims.

O'Neill proposes a particular solution to this problem, which he calls the method of recursive completion. This solution turns out to be the Shapley value of a corresponding bankruptcy game, which is a transferable-utility game where the value of each coalition is the amount of money that is left of the estate after all the claims of the agents outside that coalition are satisfied. Aumann and Maschler (1985) and Curiel et al. (1987) proposed and characterised two further solutions that coincide with the nucleolus and τ-value of the corresponding bankruptcy game, respectively.

O'Neill's bankruptcy model has been applied to a wide array of economic problems, e.g., taxation problems (Young (1988)), surplus-sharing problems (Moulin (1987)), cost-sharing problems (Moulin (1988)), apportionment of indivisible good(s) problems (Young (1994)) and priority problems (Moulin (2000) and Young (1994)).

The bankruptcy model relates to a particular kind of allocation problem. An allocation problem arises whenever a bundle of goods (resources, rights, costs, burdens) is held in common by a group of individuals and must be allotted to them individually. An allocation situation has two ingredients: the goods to be distributed and the claimants amongst whom they are to be allotted. Young (1994) introduced a general framework with the central concept of a "type" of a claimant: "The type of a claimant is a complete description of the claimant for purposes of the allocation, and determines the extent of a claimant's entitlement to the good". A type of a claimant therefore involves a complete description of the claimant in several dimensions or attributes. These attributes are accepted as the benchmark against which allocations are to be judged and can take on many forms, depending on the particular allocation situation at hand. E.g., the allocation of public housing typically depends on such attributes as financial need, family size and time spent on a waiting list. Looking from this general point of view, one can say that the bankruptcy model deals with all allocation problems in which there is one perfectly divisible good (money) to be allocated and the type of each agent can be characterised by a single (monetary) claim on that good.

In a general rationing framework, Kaminski (2000) also considers bankruptcy situations in which the type of each claimant is not one-dimensional, as is the case in O'Neill, but multi-dimensional. In the environment he presents, a type is a vector of claims, the components of which have di®erent legal statuses. As a result, di®erent priorities are assigned to the various components of an agent's claim vector.

The model we present in this paper also characterises the types of the claimants in a multi-dimensional way by means of a vector of claims. Contrary to Kaminski however, the multidimensionality of claims is not the necessary consequence of some exogenously given priorities. In our model, we regard each claim component as originating from a particular issue. An issue, which in the terminology used above takes on the role of attribute, constitutes a reason on the basis of which the estate is to be divided. Crucially, such a reason should be well founded and be accepted as such by all parties involved. A particular way one can interpret an issue is in terms of a will. In fact, O'Neill (1982) hints at this point of view in the bequest example, where the deceased leaves behind not just a single will, but a number of (contradictory) wills, each of which contains promises to one or more of his heirs.

To illustrate the terminology of our new model, consider the following example. The central government has to decide how to allocate the taxpayers' money to various public services. The system of government is such that it doesn't allocate this money directly to these services, but indirectly through various government departments. Each department (agent/player) has a number of claims on the amount of money available (estate), arising from those public services (issues) for which it has responsibility. Some of these services are provided by just a single department (e.g., tax collection viz. the Department of Finance), while more departments may be responsible for other services (e.g., foreign trade viz. the Departments of Economic A®airs, Foreign A®airs and Defence).

Another multi-dimensional extension of the bankruptcy model is provided by Lerner (1998). In that paper, a pie is allocated amongst groups, not necessarily disjoint, rather than users.

The outline of this paper is as follows. In Section 2, we introduce multi-issue allocation situations and de¯ne two corresponding cooperative games. These games are constructed from a pessimistic point of view, as are standard bankruptcy games. In order to determine the value of a particular coalition, we let the players outside that coalition decide in which order the issues are to be addressed. One important assumption in our framework is that once we start paying out money according to one particular issue, this issue must ¯rst be fully dealt with before we move on to the next. As illustrated by our public service example, handing out the estate takes place primarily on the basis of issues, whereas the individual claims of the players are of secondary concern. Because of the importance of issues, it seems natural to assume that they are dealt with consecutively. But that still leaves some freedom within each issue: in our ¯rst game (called Proportional game), we distribute the money within each issue proportional to the claims in that issue, while in the second game (called Queue game), we take an even more pessimistic view by allowing the players outside the coalition to choose also the order in which the claims within each issue are satis¯ed.

The computation of the second of our multi-issue allocation games turns out to be a less than straightforward combinatorial optimisation problem. In the Appendix, we provide algorithms to determine the worth of coalitions in both approaches.

Properties of multi-issue allocation games are presented in Section 3. The main result is that the class of multi-issue allocation games coincides with the class of non-negative exact games.

In Section 4, we analyse run-to-the-bank rules as solutions for multi-issue allocation situations. These rules are based on the interpretation behind the method of recursive completion for bankruptcy situations (cf. O'Neill (1982)). As the name suggests, the players hold a race to the person or institution administering the estate. Upon arrival, each player can choose an order on the issues that is most favourable to him. By averaging over all possible orders of arrival, we obtain a run-to-the-bank rule. One new aspect of this kind of rule, which is not present in the method of recursive completion, is that new players arriving do not only take into account their own payo®s, but also have to make some compensation payments.

The two run-to-the-bank rules we introduce in this fashion di®er in the way they treat claims within each issue. The ¯rst one (the P-rule) divides the money assigned to a particular issue proportionally, while the second one (the Q-rule) chooses an \optimistic" order on the players. The two run-to-the-bank rules turn out to be the Shapley value of the corresponding P-game and Q-game, respectively.

Finally, in Section 5, we characterise both run-to-the-bank rules by means of (P- and Q-)consistency. In the context of bankruptcy games, the term consistency has been used for a number of di®erent properties. Our de¯nition of consistency is similar to the one used by O'Neill (1982). It is based on the idea that applying a solution concept to a particular problem and applying the same solution concept to some speci¯c subproblems and aggregating the solutions of these subproblems should yield the same outcome. In order to properly de¯ne such a consistency property, we extend the domain of a solution concept to a wider class of problems, i.e., the class of multi-issue allocation situations with awards.

2 The Model

A cooperative game with transferable utility, or TU-game, is described by a pair (N, v), where $N = \{1, \ldots, n\}$ denotes the set of players and $v : 2^N \to \mathbb{R}$ is the characteristic function, assigning to every coalition $S \subset N$ of players a value $v(S)$, representing the total monetary payoff the members of this group can obtain when they cooperate. By convention, $v(\emptyset) = 0$.

A bankruptcy situation (cf. O'Neill (1982)) is a triple (N, E, c), where $N = \{1, \ldots, n\}$ is the set of players, $E > 0$ is the estate to be divided and $c \in \mathbb{R}^N_{++}$ is the vector of claims such that $\sum_{i \in N} c_i > E$. Every bankruptcy situation (N, E, c) gives rise to a bankruptcy game (N, v), where the value of a coalition $S \subset N$ is given by

$$v(S) = \max\{E - \sum_{i \in N \setminus S} c_i, 0\}.$$

So $v(S)$ is that part of the estate that is left for the players in S after the claims of all other players have been satisfied.

A multi-issue allocation situation is a triple (N, E, C), where $N = \{1, \ldots, n\}$ is the set of players, $E > 0$ is the estate under contest and $C \in \mathbb{R}^{r \times n}_{+}$ is the matrix of claims. Every row in C represents an issue and the set of issues is denoted by $R = \{1, \ldots, r\}$. An element $c_{ki} \geq 0$ represents the amount that player $i \in N$ claims according to issue $k \in R$. If a player is not involved in a particular issue, his claim corresponding to that issue equals zero.

Every bankruptcy situation (N, E, c) gives rise to a multi-issue allocation situation with $C \in \mathbb{R}^{n \times n}$ the diagonal matrix with the claims c_i on the diagonal. [1]

With respect to the matrix of claims C, we assume the following:

- Every issue gives rise to a claim: $\sum_{i \in N} c_{ki} > 0$ for all $k \in R$.
- Every player is involved in at least one issue: $\sum_{k \in R} c_{ki} > 0$ for all $i \in N$.
- The allocation problem is nontrivial: $\sum_{k \in R} \sum_{i \in N} c_{ki} > E$.

For ease of notation, we define $c_k = \sum_{i \in N} c_{ki}$ to be the total of claims according to issue $k \in R$. Similarly, we define $c_{kS} = \sum_{i \in S} c_{ki}$ for all coalitions $S \subset N$ and $c_{Ki} = \sum_{k \in K} c_{ki}$ for all sets of issues $K \subset R$. An ordering of the players in N is a bijection $\sigma : \{1, \ldots, n\} \to N$, where $\sigma(i)$ denotes which player in N is at position i. The set of all $n!$ permutations of N is denoted by $\Pi(N)$. Similarly, the set of permutations of the set of issues R is denoted by $\Pi(R)$.

[1] In fact, one can generalise a bankruptcy situation in more than one way. The method described here is one that results in the same game for both approaches we follow in this paper.

As stated in the introduction, we make the basic assumption that once we are paying out money according to one particular issue, this issue must first be fully dealt with before we move on to the next. In addition, we consider two approaches on how to handle the claims within each issue. As a result, we define two multi-issue allocation games, a proportional game v^P based on Assumption 2.1 and a queue game v^Q based on Assumption 2.2.

Assumption 2.1 If some money is allocated to the players on the basis of a particular issue, the amount of money each of the players gets is proportional to his claim according to that issue.

In order to define the proportional game v^P, we first compute the maximum amount the players in a coalition $S \subset N$ can get when the issues are dealt with according to Assumption 2.1. We do this by considering all orders on the issues, so let $\tau \in \Pi(R)$. Now the players in S first address the first t issues completely, where $t = \max\{t' \mid \sum_{s=1}^{t'} c_{\tau(s)} \leq E\}$. The part of the estate that is left, $E' = E - \sum_{s=1}^{t} c_{\tau(s)}$, is divided proportional to the claims according to issue $\tau(t+1)$. So in total, the players in S receive

$$f_S^P(\tau) = \sum_{s=1}^{t} c_{\tau(s),S} + \frac{c_{\tau(t+1),S}}{c_{\tau(t+1)}} E'. \tag{2.1}$$

The value of coalition $S \subset N$ is the amount of money they get when the players in $N \backslash S$ choose that order on the issues that gives them the highest payoff:

$$v^P(S) = E - \max_{\tau \in \Pi(R)} f_{N \backslash S}^P(\tau). \tag{2.2}$$

Using the identity $f_S^P(\tau) + f_{N \backslash S}^P(\tau) = E$, (2.2) can be rewritten as

$$v^P(S) = \min_{\tau \in \Pi(R)} f_S^P(\tau).$$

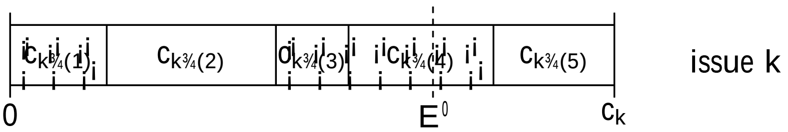
The queue game v^Q is based on Assumption 2.2.

Assumption 2.2 If a particular coalition allocates some money to the players on the basis of a particular issue, this coalition can also decide in which order the claims corresponding to that issue are satisfied.

To define the queue game, we first define an auxiliary function $g(S, k, \sigma, E')$, which describes how much money the players in $S \subset N$ get according to issue $k \in R$ if the order on the players is $\sigma \in \Pi(N)$ and the estate is E' with $E' < c_k$. The first q players get their entire claim, where $q = \max\{q' \mid \sum_{p=1}^{q'} c_{k\sigma(p)} \leq E'\}$. The function g is then defined by

$$g(S;k;\sigma;E') = \begin{cases} E' - \sum_{\substack{p=1 \\ \sigma(p) \in N\setminus S}}^{q} c_{k\sigma(p)} & \text{if } \sigma(q+1) \in S \\ \sum_{\substack{p=1 \\ \sigma(p) \in S}}^{q} c_{k\sigma(p)} & \text{if } \sigma(q+1) \notin S \end{cases} \tag{2.3}$$

The computation of $g(S;k;\sigma;E')$ is illustrated in the following example with five players.



Coalition S consists of players $\sigma(1)$, $\sigma(3)$ and $\sigma(4)$ and corresponds to the shaded area. The estate E' is such that only the claims of first three players can be fully satisfied ($q = 3$). Furthermore, $\sigma(q+1) \in S$, so (2.3) yields $g(S;k;\sigma;E') = E' - c_{k\sigma(2)}$, the area to the left of E' that is not claimed by $N\setminus S$.

Next, we compute the maximum amount the players in a coalition $S \subset N$ can get if the order on the issues is $\tau \in \Pi(R)$. As in the proportional case, the first t issues are fully dealt with and the remainder E' is distributed according to some order $\sigma \in \Pi(N)$ on the players, using Assumption 2.2.

$$f_S^Q(\sigma;\tau) = \sum_{s=1}^{t} c_{\tau(s);S} + g(S;\tau(t+1);\sigma;E'):$$

The value of coalition S is then given by

$$v^Q(S) = E - \max_{\tau \in \Pi(R)} \max_{\sigma \in \Pi(N)} f_{N\setminus S}^Q(\sigma;\tau): \tag{2.4}$$

Again, using the identity $f_S^Q(\sigma;\tau) + f_{N\setminus S}^Q(\sigma;\tau) = E$, (2.4) can be restated in terms of a minimum:

$$v^Q(S) = \min_{\tau \in \Pi(R)} \min_{\sigma \in \Pi(N)} f_S^Q(\sigma;\tau):$$

It is immediately clear that the optimal order on the players that coalition $N\setminus S$ will choose puts the players of $N\setminus S$ in front. So, (2.4) reduces to

$$v^Q(S) = E - \max_{\tau \in \Pi(R)} f_{N\setminus S}^Q(\tau);$$

where

$$f_{N\setminus S}^Q(\tau) = f_{N\setminus S}^Q(\bar\sigma;\tau) = \sum_{s=1}^{t} c_{\tau(s);N\setminus S} + \min\{c_{\tau(t+1);N\setminus S};E'\} \tag{2.5}$$

with $\sigma \in \Pi(N)$ such that $\sigma^{-1}(N\backslash S) = \{1,\ldots,|N\backslash S|\}$.

In the appendix, we present two algorithms to compute v^P and v^Q given any multi-issue allocation situation (N,E,C).

3 Properties of Multi-issue Allocation Games

In this section we look at some of the properties that multi-issue allocation games of both types possess. First, we prove that the worth of a coalition in the queue game is smaller than the worth of that coalition in the corresponding proportional game. This means that the queue approach is more pessimistic than the proportional approach.

Proposition 3.1 Let (N,E,C) be a multi-issue allocation situation with corresponding games (N,v^P) and (N,v^Q). Then $v^Q(S) \leq v^P(S)$ for all $S \subset N$.

Proof: Let $S \subset N$ and let $\tau^* \in \Pi(R)$ be the order on the issues where the maximum in (2.2) is obtained. For any order $\tau \in \Pi(R)$, $\min\{c_{\tau(t+1),N\backslash S}, E'\}$ in (2.5) exceeds $\frac{c_{\tau(t+1),N\backslash S}}{c_{\tau(t+1)}} E'$ in (2.1). So, in particular, this is the case for τ^*. But then certainly, $\max_{\tau\in\Pi(R)} f^Q_{N\backslash S}(\tau) \geq f^P_{N\backslash S}(\tau^*)$ and hence, $v^Q(S) \leq v^P(S)$. □

The core of a TU-game (N,v) is defined as

$$C(v) = \{x \in \mathbb{R}^N \mid \sum_{i\in N} x_i = v(N), \forall_{S\subset N}: \sum_{i\in S} x_i \geq v(S)\}$$

and (N,v) is called exact (cf. Driessen and Tijs (1985)) if for all $S \subset N$ there exists an $x \in C(v)$ such that $\sum_{i\in S} x_i = v(S)$.

Theorem 3.2 Let (N,E,C) be a multi-issue allocation situation. Then both corresponding games (N,v^P) and (N,v^Q) are exact.

Proof: Let $S \subset N$ and let $\tau^* \in \Pi(R)$ be such that $f^P_S(\tau^*)$ is minimal. Define $x = (f^P_i(\tau^*))_{i\in N}$. Then $\sum_{i\in N} x_i = E = v^P(N)$ and $\sum_{i\in T} x_i = f^P_T(\tau^*) \geq \min_{\tau\in\Pi(R)} f^P_T(\tau) = v^P(T)$ for all coalitions $T \subset N$. So, $x \in C(v^P)$. Furthermore, $\sum_{i\in S} x_i = f^P_S(\tau^*) = v^P(S)$. Hence, (N,v^P) is exact. The proof for (N,v^Q) is similar. □

In the proof of Theorem 3.2 we showed that $(f^P_i(\tau^*))_{i\in N}$ is a core element of the proportional game v^P for certain $\tau^* \in \Pi(R)$. This property can be extended to all orders on the issues, so for all $\tau \in \Pi(R)$ we have

$$(f_i^P(\tau))_{i\in N} \in C(v^P)$$

and similarly for the queue game, for all $\tau \in \Pi(R)$ and $\sigma \in \Pi(N)$,

$$(f_i^Q(\sigma;\tau))_{i\in N} \in C(v^Q).$$

Theorem 3.3 Let $(N;v)$ be a nonnegative exact game. Then there exists a multi-issue allocation situation $(N;E;C)$ such that both corresponding games v^P and v^Q equal v.

Proof: If $|N| = 1$, the result is obvious. Otherwise, define $E = v(N)$ and take for all $S \subsetneq N, S \neq \emptyset$ an $x^S \in C(v)$ such that $\sum_{i\in S} = v(S)$. Interpret these core elements as issues and gather them (as rows) in the $(2n-2)\times n$ claim matrix C. Because $\sum_{i\in N} c_{ki} = E$ for all $k \in R$, no issue is addressed partially and v^P and v^Q coincide.
Now, let $S \subset N$. By construction, there is a row $k' \in R$ such that $c_{k'S} = v(S)$ and because all issues are core elements of v, $c_{kS} \geq v(S)$ for all $k \in R$. Hence, $v^P(S) = \min_{\tau\in\Pi(R)} f_S^P(\tau) = \min_{k\in R} c_{kS} = v(S)$. Therefore, v, v^P and v^Q coincide. □

From Theorems 3.2 and 3.3 we conclude that the class of multi-issue allocation games coincides with the class of nonnegative exact games.

A TU-game $(N;v)$ is called balanced if it has a nonempty core and totally balanced if the core of every subgame is nonempty, where the subgame corresponding to some coalition $T \subset N, T \neq \emptyset$ is the game $(T;V^T)$ with $V^T(S) = V(S)$ for all $S \subset T$.

Proposition 3.4 Let $(N;v)$ be an exact TU-game. Then $(N;v)$ is totally balanced.

Proof: Let $T \subset N, T \neq \emptyset$. By exactness, there exists an $x \in C(v)$ such that $\sum_{i\in T} x_i = v(T)$. But then $x_T \in \mathbb{R}^T$ is an efficient allocation of $v^T(T)$ such that $x_S \geq v^T(S)$ for all $S \subset T$. Hence, $x_T \in C(v^T)$ and $(N;v)$ is totally balanced. □

As a corollary, both multi-issue allocation games are totally balanced.

A TU-game $(N;v)$ is called superadditive if for all coalitions $S, T \subset N$ we have

$$v(S) + v(T) \leq v(S \cup T)$$

and convex if for all coalitions $U \subset N$ and all $S \subset T \subset N\backslash U$ we have

$$v(S \cup U) - v(S) \leq v(T \cup U) - v(T).$$

For a TU-game $(N;v)$, the utopia vector $M(v)$ is defined by

$$M_i(v) = v(N) - v(N \setminus \{i\})$$

for all $i \in N$, and the minimal right vector $m(v)$ by

$$m_i(v) = \max_{S: i \in S} \left[v(S) - \sum_{j \in S; j \neq i} M_j(v) \right]$$

for all $i \in N$. $(N; v)$ is called quasi-balanced if $m(v) \leq M(v)$ and $\sum_{i \in N} m_i(v) \leq v(N) \leq \sum_{i \in N} M_i(v)$. The following proposition comes from Driessen and Tijs (1985).

Proposition 3.5 Let $(N; v)$ be a TU-game with $x \in C(v)$. Then $(N; v)$ is quasi-balanced and $m(v) \leq x \leq M(v)$.

For an arbitrary game $(N; v)$, we define the core cover by

$$CC(v) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N); m(v) \leq x \leq M(v)\}:$$

It follows from Proposition 3.5 that $CC(v) \supseteq C(v)$. A TU-game $(N; v)$ is called semi-convex if it is superadditive and

$$m_i(v) = v(\{i\})$$

for all $i \in N$.

For every order $\sigma \in \Pi(N)$, we define the marginal vector $m^\sigma(v)$ recursively by

$$m^\sigma_{\sigma(k)}(v) = v(\{\sigma(1), \ldots, \sigma(k)\}) - v(\{\sigma(1), \ldots, \sigma(k-1)\})$$

for all $k = 1, \ldots, n$. The Shapley value of $(N; v)$ is defined as (cf. Shapley (1953))

$$\Phi(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi N} m^\sigma(v):$$

The following results come from Driessen and Tijs (1985).

Proposition 3.6 Let $(N; v)$ be a TU-game. Then:

- If $(N; v)$ is convex, then it is exact.
- If $(N; v)$ is exact, then it is semi-convex.
- If $(N; v)$ is semi-convex and $|N| \leq 3$, then it is convex.

As a corollary, both multi-issue allocation games v^P and v^Q are semi-convex (and hence, superadditive) and multi-issue allocation games with 3 players or less are convex. Example 3.7 shows that 4-player multi-issue allocation games need not be convex.

Example 3.7 Consider the multi-issue allocation situation with player set $N = \{1, \ldots, 4\}$, estate $E = 12$ and claim matrix

$$C = \begin{bmatrix} 4 & 0 & 0 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix}.$$

The corresponding proportional and queue games are as follows:

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	N
$v^P(S)$	3	1	1	3	6	6	6	2	6	6	7	9	9	7	12
$v^Q(S)$	0	0	0	0	4	4	4	0	4	4	4	8	8	4	12

Now take $U = \{3\}$, $S = \{1\}$ and $T = \{1, 2\}$. Then

$$v^P(S \cup U) - v^P(S) = 6 - 3 = 3 > 1 = 7 - 6 = v^P(T \cup U) - v^P(T)$$

and

$$v^Q(S \cup U) - v^Q(S) = 4 - 0 = 4 > 0 = 4 - 4 = v^Q(T \cup U) - v^Q(T).$$

Hence, both v^P and v^Q do not satisfy convexity. ◁

A well known property of a convex game is that its Shapley value belongs to the core. Rabie (1981) shows that this does not hold in general for exact games. However, Theorem 3.8 shows that the Shapley value of a nonnegative exact game belongs to the core cover.

Theorem 3.8 Let (N, v) be a nonnegative exact game. Then $\Phi(v) \in CC(v)$.

Proof: First, use Theorem 3.3 to construct a multi-issue allocation situation (N, E, C) such that $v^P = v$. Next, let $i \in N$. Then supperadditivity implies $v^P(S) - v^P(S \backslash \{i\}) \geq v^P(\{i\}) = m_i(v^P)$ for all $S \subset N$ such that $i \in S$. Furthermore,

$$\begin{aligned} v^P(S) - v^P(S \backslash \{i\}) &= \left[E - \max_{\tau \in \Pi(R)} f^P_{N \backslash S}(\tau) \right] - \left[E - \max_{\tau \in \Pi(R)} f^P_{N \backslash S \cup \{i\}}(\tau) \right] \\ &= \max_{\tau \in \Pi(R)} f^P_{N \backslash S \cup \{i\}}(\tau) - \max_{\tau \in \Pi(R)} f^P_{N \backslash S}(\tau) \\ &\leq \max_{\tau \in \Pi(R)} f^P_{N \backslash S}(\tau) + \max_{\tau \in \Pi(R)} f^P_{\{i\}}(\tau) - \max_{\tau \in \Pi(R)} f^P_{N \backslash S}(\tau) \end{aligned}$$

$$\begin{aligned} &= E - \left[E - \max_{\tau \in \Pi(R)} f^P_{\{i\}}(\tau) \right] \\ &= v^P(N) - v^P(N \setminus \{i\}) \\ &= M_i(v^P). \end{aligned}$$

Hence, the marginal contribution of i to every coalition $S : i \in S$ is bounded by $m_i(v^P)$ and $M_i(v^P)$. Because the Shapley value is the average of these marginal contributions, $\Phi(v^P) \in CC(v^P)$ and hence, $\Phi(v) \in CC(v)$. □

A population monotonic allocation scheme (cf. Sprumont (1990)), or pmas, is a set of vectors $x^S \in \mathbb{R}^S$ for all $S \subset N, S \neq \emptyset$ such that

$$\sum_{i \in S} x_i^S = v(S) \text{ for all } S \subset N, S \neq \emptyset \tag{3.1}$$

and

$$x_i^S \leqslant x_i^T \text{ for all } S \subset T \subset N, i \in S. \tag{3.2}$$

Sprumont shows that every convex game has a pmas. This does not hold for exact games, as is shown by the following example.

Example 3.9 Consider the multi-issue allocation situation with player set $N = \{1, \ldots, 4\}$, estate $E = 22$ and claim matrix

$$C = \begin{bmatrix} 6 & 6 & 5 & 3 \\ 12 & 0 & 2 & 6 \end{bmatrix}.$$

The corresponding queue game is as follows:

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	N
$v^Q(S)$	6	0	2	3	12	11	9	2	6	8	14	15	16	8	22

To show that v^Q has no pmas, suppose $(x_i^S)_{i \in S; S \subset N; S \neq \emptyset}$ satisfies (3.1) and (3.2). Then we subsequently have:

- $v^Q(\{1,3\}) = 11$ and $v^Q(\{1,3,4\})$ imply $x_4^{\{1,3,4\}} \leqslant 16 - 11 = 5$.
- $x_4^{\{1,3,4\}} \leqslant 5$ implies $x_4^{\{3,4\}} \leqslant 5$.
- $x_4^{\{3,4\}} \leqslant 5$ and $v^Q(\{3,4\}) = 8$ imply $x_3^{\{3,4\}} \geqslant 3$.
- $x_3^{\{3,4\}} \geqslant 3$ and $v^Q(\{2,4\}) = 6$ imply $\sum_{i \in \{2,3,4\}} x_i^{\{2,3,4\}} \geqslant 9$.

The last statement contradicts (3.1) and hence, the exact game v^Q possesses no pmas. ◁

4 The Run-to-the-Bank Rule

A multi-issue allocation solution α is a function assigning to every multi-issue allocation situation $(N;E;C)$ a vector $\alpha(N;E;C) \in \mathbb{R}^N$ such that $\sum_{i\in N} \alpha_i(N;E;C) = E$ (efficiency) and $0 \leq \alpha_i(N;E;C) \leq c_{Ri}$ for all $i \in N$ (reasonability). We define two solutions, called run-to-the-bank rules, based on Assumption 2.1 and 2.2. The proportional run-to-the bank rule is defined as

$$\rho^P = \frac{1}{|N|!} \sum_{\sigma\in\Pi(N)} \rho^P(\sigma);$$

where for all $\sigma \in \Pi(N)$, $\rho^P(\sigma) \in \mathbb{R}^N$ is defined recursively by

$$\rho^P_{\sigma(p)}(\sigma) = \max_{\tau\in\Pi(R)} \left[f^P_{\sigma(p)}(\tau) - \sum_{q=1}^{p-1} \left(\rho^P_{\sigma(q)}(\sigma) - f^P_{\sigma(q)}(\tau) \right) \right] \tag{4.1}$$

for all $p \in \{1;\ldots;n\}$. The vector $\rho^P(\sigma)$ is interpreted as follows. To divide the estate, a "race" is held between the players and they arrive at the person or institution administering the estate in the order given by σ. The first player that arrives, $\sigma(1)$, can choose the order in which the issues are dealt with and receives his payoff accordingly. Of course, he will choose that order $\tau \in \Pi(R)$ for which his payoff $f^P_{\sigma(1)}(\tau)$ is maximal. Next, player $\sigma(2)$ arrives and he is asked to do the same. However, if he chooses an order different from the first one, he has to compensate player $\sigma(1)$ for the difference between his settled payoff $\rho^P_{\sigma(1)}(\sigma)$ and his payoff according to the new order. Taking this into account, the second player will pick that order that maximises his own payoff minus corresponding compensation payments. The same procedure is applied to each subsequent player, each having to compensate all his predecessors.

The queue run-to-the-bank rule is defined as

$$\rho^Q = \frac{1}{|N|!} \sum_{\sigma\in\Pi(N)} \rho^Q(\sigma);$$

where for all $\sigma \in \Pi(N)$, $\rho^Q(\sigma) \in \mathbb{R}^N$ is defined recursively by

$$\rho^Q_{\sigma(p)}(\sigma) = \max_{\tau\in\Pi(R)} \max_{\gamma\in\Pi(N)} \left[f^Q_{\sigma(p)}(\gamma;\tau) - \sum_{q=1}^{p-1} \left(\rho^Q_{\sigma(q)}(\sigma) - f^Q_{\sigma(q)}(\gamma;\tau) \right) \right] \tag{4.2}$$

for all $p \in \{1;\ldots;n\}$. The interpretation is similar to the proportional case. The only difference is that the queue payoff function f^Q is used rather than the proportional function f^P and that in accordance with Assumption 2.2, players also have to specify an order γ on the players. It is immediately clear that it is optimal for player $\sigma(p)$, who arrives at the administrator at position p, to choose γ in such a way that he himself and all preceding players, $\sigma(1),\ldots,\sigma(p-1)$, whom he has to compensate, are in front of the queue. This can be done by setting $\gamma=\sigma$.

Proposition 4.1 The optimal γ in (4.2) equals σ.

As a result of Proposition 4.1, (4.2) can be rewritten as

$$\rho^Q_{\sigma(p)}(\sigma) = \max_{\tau\in\Pi(R)} \left[f^Q_{\sigma(p)}(\sigma;\tau) - \sum_{q=1}^{p-1}\left(\rho^Q_{\sigma(q)}(\sigma) - f^Q_{\sigma(q)}(\sigma;\tau)\right)\right] \tag{4.3}$$

In order to prove that both run-to-the-bank rules equal the Shapley values of their respective corresponding games, we first relate them to the marginal vectors. For this, we define for any order $\sigma\in\Pi(N)$ the reverse order $\sigma^*\in\Pi(N)$ by $\sigma^*(p)=\sigma(n-p+1)$ for all $p\in\{1,\ldots,n\}$.

Lemma 4.2 Let $(N;E;C)$ be a multi-issue allocation situation with corresponding games $(N;v^P)$ and $(N;v^Q)$. Then $\rho^P(\sigma)=m^{\sigma^*}(v^P)$ and $\rho^Q(\sigma)=m^{\sigma^*}(v^Q)$ for all $\sigma\in\Pi(N)$.

Proof: We only prove the statement for the proportional game; the proof for the queue game is similar. Let $\sigma\in\Pi(N)$. Then for all $p\in\{1,\ldots,n\}$ we have

$$\begin{aligned}
\rho^P_{\sigma(p)}(\sigma) &= \max_{\tau\in\Pi(R)} \left[f^P_{\sigma(p)}(\tau) - \sum_{q=1}^{p-1}\left(\rho^P_{\sigma(q)}(\sigma) - f^P_{\sigma(q)}(\tau)\right)\right] \\
&= \max_{\tau\in\Pi(R)} f^P_{\{\sigma(1),\ldots,\sigma(p)\}}(\tau) - \sum_{q=1}^{p-1}\rho^P_{\sigma(q)}(\sigma) \\
&= \max_{\tau\in\Pi(R)} f^P_{\{\sigma(1),\ldots,\sigma(p)\}}(\tau) - \max_{\tau\in\Pi(R)} f^P_{\{\sigma(1),\ldots,\sigma(p-1)\}}(\tau) \\
&= E - \min_{\tau\in\Pi(R)} f^P_{\{\sigma(p+1),\ldots,\sigma(n)\}}(\tau) - E + \min_{\tau\in\Pi(R)} f^P_{\{\sigma(p),\ldots,\sigma(n)\}}(\tau) \\
&= -\min_{\tau\in\Pi(R)} f^P_{\{\sigma^*(1),\ldots,\sigma^*(n-p)\}}(\tau) + \min_{\tau\in\Pi(R)} f^P_{\{\sigma^*(1),\ldots,\sigma^*(n-p+1)\}}(\tau) \\
&= -v^P(\{\sigma^*(1),\ldots,\sigma^*(n-p)\}) + v^P(\{\sigma^*(1),\ldots,\sigma^*(n-p+1)\}) \\
&= m^{\sigma^*}_{\sigma^*(n-p+1)}(v^P) \\
&= m^{\sigma^*}_{\sigma(p)}(v^P);
\end{aligned}$$

where the third equality follows from recursively substituting the formulas for $\rho^P_{\sigma(q)}(\sigma)$. ■

Theorem 4.3 Let $(N;E;C)$ be a multi-issue allocation situation with corresponding games $(N;v^P)$ and $(N;v^Q)$. Then $\rho^P=\Phi(v^P)$ and $\rho^Q=\Phi(v^Q)$.

Proof: This result follows immediately from Lemma 4.2 and from the observation that $\{\sigma^* \mid \sigma \in \Pi(N)\} = \Pi(N)$. ∎

5 Consistency

In this section we characterise the run-to-the-bank rules by means of consistency. For this, we broaden the domain of these rules to a larger class of situations, namely multi-issue allocation situations with awards. We should stress, that although this new class has a nice interpretation in itself, it is not directly intended as an extension of multi-issue allocation situations, but as a (technical) framework in which consistency arises as a natural property.

A multi-issue allocation situation with awards is a 4-tuple $(N; E; C; \beta)$, where $\beta \in \mathbb{R}^F$ represents some award vector to a coalition $F \subset N$, which has already been agreed upon. The sum of these awards cannot exceed the estate, so $\sum_{i \in F} \beta_i \leqq E$. Furthermore, $\sum_{i \in F} \beta_i = E$ if $F = N$. Note that a multi-issue allocation situation without awards is a special case with $F = \emptyset$.

A solution ψ is a function assigning to every multi-issue allocation situation with awards $(N; E; C; \beta)$ a vector $\psi(N; E; C; \beta) \in \mathbb{R}^N$ such that $\sum_{i \in N} \psi_i(N; E; C; \beta) = E$ and $\psi_F(N; E; C; \beta) = \beta$. That is, for a solution in this environment it should hold that every player in F gets exactly his award. Note that contrary to the situation without awards, we do not impose reasonability [2] on ψ. On this new class of situations we also define two run-to-the-bank rules. For this, we first fix an order on the players in F, so let $\tau \in \Pi(F)$. The proportional run-to-the-bank rule with awards is defined as:

$$\rho^P(\beta) = \frac{1}{|N \backslash F|!} \sum_{\sigma \in \Pi^\tau(N)} \rho^P(\sigma; \beta);$$

where

$$\Pi^\tau(N) = \left\{ \sigma \in \Pi(N) \mid \forall_{q \in \{1, \ldots, |F|\}} : \sigma(q) = \tau(q) \right\}$$

and for all $\sigma \in \Pi^\tau(N)$, $\rho^P(\sigma; \beta) \in \mathbb{R}^N$ is defined recursively by

$$\rho^P_{\sigma(p)}(\sigma; \beta) = \beta_{\sigma(p)}$$

for all $p \in \{1, \ldots, n\}$ such that $\sigma(p) \in F$ and

[2] To guarantee reasonability of the run-to-the-bank rules with awards as defined below, we would have to make some unnecessary diverting assumptions. We just note that for the specific multi-issue allocation situations with awards that are derived from a standard multi-issue allocation situation using either run-to-the-bank rule, reasonability is satisfied.

$$\rho^P_{\sigma(p)}(\sigma;\mu) = \max_{\tau\in\Pi(R)} \left\{ f^P_{\sigma(p)}(\tau) - \sum_{q=1}^{p-1} \left[\rho^P_{\sigma(q)}(\sigma;\mu) - f^P_{\sigma(q)}(\tau) \right] \right\}$$

for all $p \in \{1,\ldots,n\}$ such that $\sigma(p) \notin F$.

Note that the run-to-the-bank rule does not depend on the actual choice of γ. This definition differs from the run-to-the-bank rule without awards (4.1) in two respects: every player $i \in F$ gets μ_i rather than the maximum expression in (4.1) and the players in F have to be compensated (which is accomplished in an order $\sigma \in \Pi^{\gamma}(N)$ by putting them at the front). Note that for $F = \emptyset$, the two definitions coincide.

In a similar fashion, we define the queue run-to-the-bank rule with awards:

$$\rho^Q(\mu) = \frac{1}{|N\setminus F|!} \sum_{\sigma\in\Pi^{\gamma}(N)} \rho^Q(\sigma;\mu);$$

where for all $\sigma \in \Pi^{\gamma}(N)$, $\rho^Q(\sigma;\mu) \in \mathbb{R}^N$ is defined recursively by

$$\rho^Q_{\sigma(p)}(\sigma;\mu) = \mu_{\sigma(p)}$$

for all $p \in \{1,\ldots,n\}$ such that $\sigma(p) \in F$ and

$$\rho^Q_{\sigma(p)}(\sigma;\mu) = \max_{\tau\in\Pi(R)} \left\{ f^Q_{\sigma(p)}(\sigma;\tau) - \sum_{q=1}^{p-1} \left[\rho^Q_{\sigma(q)}(\sigma;\mu) - f^Q_{\sigma(q)}(\sigma;\tau) \right] \right\}$$

for all $p \in \{1,\ldots,n\}$ such that $\sigma(p) \notin F$. Note that this definition generalises (4.3) rather than (4.2). Proposition 4.1 can easily be extended to the situation with awards, so letting each player choose an order on the players would result in an equivalent definition.

For all $i \in N\setminus F$ and $\tau \in \Pi(R)$ we define the remainder functions

$$r^P_i(\tau) = f^P_{F\cup\{i\}}(\tau) - \sum_{j\in F}\mu_j \qquad \left(= f^P_i(\tau) + \sum_{j\in F}\left[f^P_j(\tau) - \mu_j\right]\right)$$

and

$$r^Q_i(\tau) = f^Q_{F\cup\{i\}}(\sigma;\tau) - \sum_{j\in F}\mu_j;$$

where $\sigma \in \Pi^{\gamma}(N)$ is such that $\sigma(|F|+1) = i$. These remainder functions represent the amount of money player i gets according to order τ, when he has to ensure that every player $j \in F$ gets μ_j. A rule Ψ is called P-consistent if for all multi-issue allocation situations with awards $(N;E;C;\mu)$ and all $i \in N\setminus F$ we have

$$\Psi_i(N;E;C;\mu) = \frac{1}{|N\setminus F|}\left(\max_{\tau\in\Pi(R)} r^P_i(\tau) + \sum_{\substack{j\in N\setminus F\\ j\neq i}} \Psi_i(N;E;C;\mu^j) \right); \tag{5.1}$$

where $\mu^j \in \mathbb{R}^{F \cup \{j\}}$ is such that $\mu^j_F = \mu$ and $\mu^j_j = \max_{\tau \in \Pi(R)} r^P_j(\tau)$. Ψ is Q-consistent if for all $(N;E;C;\mu)$ and all $i \in N \backslash F$

$$\Psi_i(N;E;C;\mu) = \frac{1}{|N \backslash F|}\left(\max_{\tau \in \Pi(R)} r^Q_i(\tau) + \sum_{\substack{j \in N \backslash F \\ j \neq i}} \Psi_i(N;E;C;\mu^j) \right)$$

with $\mu^j_j = \max_{\tau \in \Pi(R)} r^Q_j(\tau)$. The idea behind consistency is as follows (cf. O'Neill (1982)). Let i be a player in $N \backslash F$. Then the first term between parentheses is the amount of money player i gets when he maximises his own payoff by choosing an order on the issues, keeping in mind the players in F have to receive their awards. Next, let $j \in N \backslash F$; $j \neq i$. Now suppose that player j receives his maximal remainder. Then a new situation arises where player j has been awarded some fixed amount. The amount of money player i receives in this new situation is given by applying rule Ψ on the old μ extended with the fixed award to player j. A rule is called consistent if applying it directly yields the same outcome as averaging over all $|N \backslash F|$ situations where one of the non-fixed player get their maximum.

Theorem 5.1 The proportional run-to-the-bank rule ρ^P is the unique P-consistent rule and the queue run-to-the-bank rule ρ^Q is the unique Q-consistent rule.

Proof: We only give the proof for ρ^P. The proof for ρ^Q goes along similar lines. First, we prove that ρ^P satisfies P-consistency. Let $i \in N \backslash F$. Then

$$\begin{aligned}
&\rho^P_i(\mu) = \\
&= \frac{1}{|N \backslash F|!} \sum_{\sigma \in \Pi^{\gamma}(N)} \rho^P_i(\sigma;\mu) \\
&= \frac{1}{|N \backslash F|!} \sum_{j \in N \backslash F} \sum_{\substack{\sigma \in \Pi^{\gamma}(N) \\ \sigma(|F|+1)=j}} \rho^P_i(\sigma;\mu) \\
&= \frac{1}{|N \backslash F|!} \sum_{\substack{\sigma \in \Pi^{\gamma}(N) \\ \sigma(|F|+1)=i}} \max_{\tau \in \Pi(R)} \left\{ f^P_i(\tau) - \sum_{q=1}^{\sigma^{-1}(i)-1} \left[\rho^P_{\sigma(q)}(\sigma;\mu) - f^P_{\sigma(q)}(\tau) \right] \right\} + \\
&\quad \frac{1}{|N \backslash F|!} \sum_{\substack{j \in N \backslash F \\ j \neq i}} \sum_{\substack{\sigma \in \Pi^{\gamma}(N) \\ \sigma(|F|+1)=j}} \max_{\tau \in \Pi(R)} \left\{ f^P_i(\tau) - \sum_{q=1}^{\sigma^{-1}(i)-1} \left[\rho^P_{\sigma(q)}(\sigma;\mu) - f^P_{\sigma(q)}(\tau) \right] \right\} \\
&= \frac{1}{|N \backslash F|!} \sum_{\sigma \in \Pi^{\gamma,i}(N)} \max_{\tau \in \Pi(R)} \left\{ f^P_{F \cup \{i\}}(\tau) - \sum_{j \in F} \mu_j \right\} +
\end{aligned}$$

$$
\begin{aligned}
&\frac{1}{|N\setminus F|!}\sum_{\substack{j\in N\setminus F\\ j\neq i}}\;\sum_{\sigma\in\Pi^{\gamma;j}(N)}\max_{\tau\in\Pi(R)}\left\{f_i^P(\tau)-\sum_{q=1}^{\sigma^{-1}(i)-1}\left[\rho^P_{\sigma(q)}(\sigma;\mu)-f^P_{\sigma(q)}(\tau)\right]\right\}\\
=\;&\frac{1}{|N\setminus F|!}(|N\setminus F|-1)!\max_{\tau\in\Pi(R)} r_i^P(\tau)+\\
&\frac{1}{|N\setminus F|}\sum_{\substack{j\in N\setminus F\\ j\neq i}}\frac{1}{(|N\setminus F|-1)!}\sum_{\sigma\in\Pi^{\gamma;j}(N)}\max_{\tau\in\Pi(R)}\left\{f_i^P(\tau)-\sum_{q=1}^{\sigma^{-1}(i)-1}\left[\rho^P_{\sigma(q)}(\sigma;\mu^j)-f^P_{\sigma(q)}(\tau)\right]\right\}\\
=\;&\frac{1}{|N\setminus F|}\max_{\tau\in\Pi(R)} r_i^P(\tau)+\frac{1}{|N\setminus F|}\sum_{\substack{j\in N\setminus F\\ j\neq i}}\rho_i^P(\mu^j)\\
=\;&\frac{1}{|N\setminus F|}\left(\max_{\tau\in\Pi(R)} r_i^P(\tau)+\sum_{\substack{j\in N\setminus F\\ j\neq i}}\rho_i^P(\mu^j)\right);
\end{aligned}
$$

where $\Pi^{\gamma;i}(N)=\{\sigma\in\Pi^{\gamma}(N)\,|\,\sigma(|F|+1)=i\}$.

Uniqueness of the P-consistent rule is proved by induction on the size of F. Assume that rule φ is P-consistent. For $F=N$, $\varphi(N;E;C;\mu)=\mu$ by definition. Next, let $F\subset N, F\neq\emptyset$ and assume that $\varphi(N;E;C;\mu)$ is uniquely determined. Let $i\in F$. Then P-consistency (5.1) implies that $\varphi(N;E;C;\mu^{-i})$ must be uniquely determined as well, where $\mu^{-i}\in\mathbb{R}^{F\setminus\{i\}}$ is such that $\mu^{-i}_j=\mu_j$ for all $j\in F\setminus\{i\}$. Repeating this procedure until $F=\emptyset$, we conclude that there is a unique P-consistent rule, which is the proportional run-to-the-bank rule. □

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A Algorithms

In this appendix we present two algorithms to compute the proportional game v^P and queue game v^Q that correspond to any multi-issue allocation situation (N, E, C).

A.1 Proportional Game

Let (N, E, C) be a multi-issue allocation situation and let $S \subseteq N$ be a coalition of players. The value of S, $v^P(S)$, is computed in a number of steps:

1. Compute for every issue $k \in R$ the proportion of the total of claims corresponding to issue k that is claimed by coalition S:

   $p_k = \frac{c_{kS}}{c_k}$.

2. Take $\sigma \in \Pi(R)$ such that $\sigma^{-1}(k) \leq \sigma^{-1}(\ell)$ whenever $p_k \leq p_\ell$.

3. $v^P(S) = f_S^P(\sigma)$, where $f_S^P(\sigma)$ is defined in (2.1).

A.2 Queue Game

Let (N, E, C) be a multi-issue allocation situation and let $S \subseteq N$ be a coalition of players. The value of S, $v^Q(S)$, is computed in a number of steps:

1. For all $I \subseteq R$ calculate

$$
\begin{aligned}
x_I &= \sum_{k \in I} c_{kS}, \\
y_I &= \sum_{k \in I} c_{k,N \setminus S} + \max_{k \in R \setminus I} c_{k,N \setminus S}.
\end{aligned}
$$

2. If $y_\emptyset \geq E$ then $v^Q(S) = 0$, otherwise proceed.

3. Find $\overline{I} \subseteq R$ such that

   (a) $x_{\overline{I}} + y_{\overline{I}} \geq E$,

   (b) $x_I \geq x_{\overline{I}}$ for all $I \subseteq R$ such that $x_I + y_I \geq E$.

   Next, find $\underline{I} \subseteq R$ such that

   (a) $x_{\underline{I}} + y_{\underline{I}} \leq E$,

(b) $y_{\underline{I}} \geq y_I$ for all $I \subseteq R$ such that $x_I + y_I \leq E$.

4. Compute

$$v^Q(S) = \min\{x_{\overline{I}}; E - y_{\underline{I}}\}.$$

Proof: First of all, note that it follows from the de¯nition of f^Q that $v^Q(S)$ depends only on the aggregate claim of coalition S within each issue and not on the distribution of claims between the members of S.

The idea behind the proof is to represent all possible payo® pro¯les $(x; y)$ for all possible estates by paths in the payo® space ($\mathbb{R}^2_+$), where x (on the horizontal axis) is the payo® to S and y (on the vertical axis) the payo® to $N \setminus S$. The aim is to ¯nd the minimum possible payo® to S given the fact that the estate equals E. The estate E is represented by the line $x + y = E$.

Coalition $N \setminus S$ has the freedom to choose an order on the issues. Now, forget the actual amount of the estate for a moment and suppose that $N \setminus S$ choose to address issues $I \subseteq R$ fully and one other issue in $R \subseteq I$ that gives $N \setminus S$ their maximal payo® (without paying the claim of S according to that last issue). This action leads to a payo® pro¯le of $(x_I; y_I)$. If the estate were to equal $x_I + y_I$, the point $(x_I; y_I)$ would represent a payo® pro¯le that according to Assumption 2.2 would be feasible for $N \setminus S$ to reach.

With each order on I we associate a path connecting $(x_I; y_I)$ to the origin. Starting with an estate of 0 (and hence, a $(0; 0)$ payo®), we are going to increase the estate to $x_I + y_I$, plotting the payo® pro¯les associated with all intermediate estates (determined by the order on I) in the picture. From the origin, we start paying out money to $N \setminus S$ according to the ¯rst issue in I, represented by a vertical line segment. When the estate reaches the total claim of $N \setminus S$ corresponding to the ¯rst issue, we start paying out to coalition S, represented by a horizontal line segment. After the total claim associated with the ¯rst issue has been paid out, we continue with the second issue in the order, and so on. When all issues I have been addressed, we end with a vertical line segment representing the claim of $N \setminus S$ according the last issue. Typically, such a path looks as depicted in Figure 1. Note that some horizontal or vertical line segments may be absent because of zero claims. We draw such a path for every order on I. These paths represent all possible payo® pro¯les that coalition $N \setminus S$ can reach for estates smaller than $x_I + y_I$ if they choose to address the issues in I ¯rst and put themselves in front within each issue.

Doing this for all $I \subseteq R$ yields all feasible payo® pro¯les (provided $N \setminus S$ acts optimally within each issue) for any order on the issues for all estates smaller than the total of all claims.

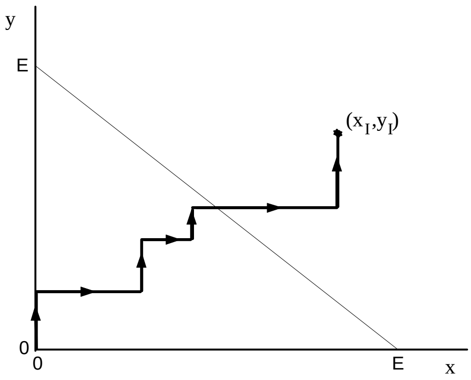

Figure 1

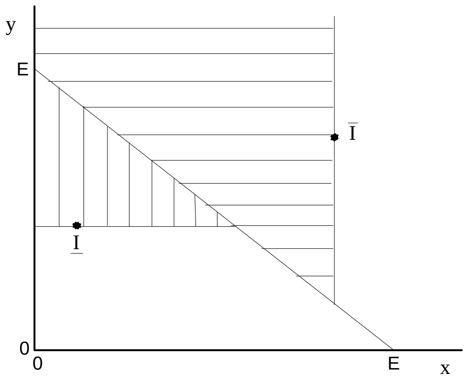

Figure 2

Note that every path associated with some set I \$ R of issues is part of a path connecting $(x_R; y_R)$ to the origin.

The value of coalition S is the x-coordinate of the leftmost intersection between some path and the line $x + y = E$. It is immediately clear that $v^Q(S) = 0$ if $y_{;} > E$. Otherwise, take $\overline{I}$ and $\underline{I}$ as stated (which is always possible because of R and ;, resp.).

Typically, $\overline{I}$ and $\underline{I}$ are situated as depicted in Figure 2. By construction, there is no I ½ R giving rise to a payo® pro¯le $(x_I; y_I)$ in either shaded are. Note also that whereas $\overline{I}$ and $\underline{I}$ need not be uniquely determined, $x_{\overline{I}}$ and $y_{\underline{I}}$ are.

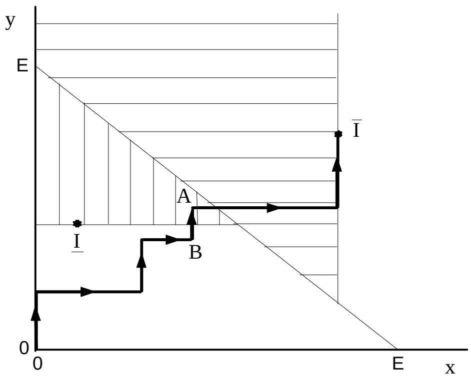

Figure 3

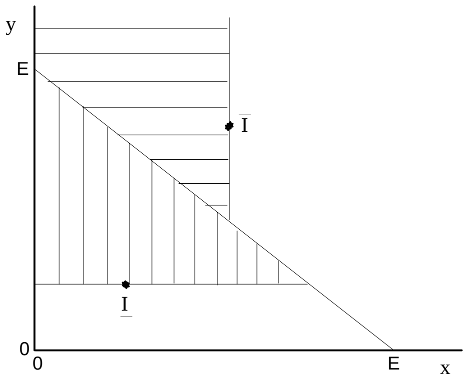

Figure 4

Now consider the paths associated with $\overline{I}$. We claim that there can be no path with a ⎾ kink in the shaded area. Suppose that such a path exists, as indicated in Figure 3, with a kink at A. Consider all issues $I^{¤}$ that are fully dealt with up to point B. [3] Then by construction,

[3] In fact, we need the last point below A where all issues up to that point have been fully addressed. If A is preceded by an issue in which S has a zero claim, this point may be between A and B.

(x_{I^*}, y_{I^*}) lies at or above point A. This contradicts the fact that there is no $I \subset R$ giving rise to a payoff profile in the shaded area.

As a consequence, every path connecting $(x_{\overline{I}}, y_{\overline{I}})$ to the origin must cross the line $x + y = E$ to the right of $(E - y_{\underline{I}}, y_{\underline{I}})$ if $x_{\overline{I}} + y_{\underline{I}} > E$ (the case depicted in Figure 3). The same holds for every path connecting any point above the line $x + y = E$ to the origin. Hence, $v^Q(S) \geq E - y_{\underline{I}}$. Furthermore, there is a path going through $(E - y_{\underline{I}}, y_{\underline{I}})$, because $N \setminus S$ can guarantee themselves $y_{\underline{I}}$ by addressing issues $\underline{I}$ first. Therefore, $v^Q(S) = E - y_{\underline{I}}$ if $x_{\overline{I}} + y_{\underline{I}} > E$.

Similarly, if $x_{\overline{I}} + y_{\underline{I}} < E$, as depicted in Figure 4, every path intersecting the line $x + y = E$ must do so to the right of $(x_{\overline{I}}, E - x_{\overline{I}})$ and there is a path going through this point. Hence, in this case $v^Q(S) = x_{\overline{I}}$.

If $x_{\overline{I}} + y_{\underline{I}} = E$, both sets of arguments can be used. One should also note that all these arguments still hold in case $(x_{\underline{I}}, y_{\underline{I}})$ or $(x_{\overline{I}}, y_{\overline{I}})$ lie on the line $x + y = E$ rather than below or above.

Summarising these cases, we obtain

$$v^Q(S) = \min\{x_{\overline{I}}, E - y_{\underline{I}}\},$$

as stated in the algorithm. ¤