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## **A RESULT ON OUTPUT FEEDBACK LINEAR QUADRATIC CONTROL**

By Jacob Engwerda, Arie Weeren

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# A result on output feedback linear quadratic control.

Jacob Engwerda\* and Arie Weeren†

**Abstract** In this note we consider the static output feedback linear quadratic control problem. We present both necessary and sufficient conditions under which this problem has a solution in case the involved cost depend only on the output and control variables.

This result is used to present both necessary and sufficient conditions under which the corresponding linear quadratic differential game has a Nash equilibrium in case the players use static output feedback control.

**Keywords:** LQ theory; Algebraic Riccati equations; Differential games.

**Jel-codes:** C61, C72, C73.

## 1 Introduction

The so-called indefinite, regular, zero-endpoint, infinite-horizon LQ (IRZILQ) problem of finding a control function  $u \in U_s(x_0)$  for each  $x_0 \in \mathbb{R}^n$  that minimizes the cost functional

$$J(x_0, u) := \int_0^{\infty} (x^T Q x + u^T R u) dt, \quad (1)$$

with  $Q = Q^T$ ,  $R > 0$ , and where the state variable  $x$  is the solution of  $\dot{x} = Ax + Bu$ ,  $x(0) = x_0$  has been studied by many authors (see e.g. [12], [8] and [11] or in a more general context [4]). Here the class of control functions  $U_s(x_0)$  is defined by:

$$U_s(x_0) = \left\{ u \in L_{2,loc} \mid J(x_0, u) \text{ exists in } \mathbb{R} \cup \{-\infty, \infty\}, \lim_{t \rightarrow \infty} x(t) = 0 \right\}.$$

In [2] the same problem was studied under the assumption that the class of control functions  $U_s(x_0)$  consists of the set of stabilizing, time-invariant state feedbacks, i.e.  $u = Fx$  where

$$F \in \mathcal{F} := \{F \mid A + BF \text{ is stable}\}.$$

Both necessary and sufficient conditions for the existence of a solution were studied in that paper and the relationship with the (IRZILQ) problem elaborated.

However, frequently it is not possible or economically feasible to measure all state variables in applications. The designer wants to control the system based on the directly observed output of the

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\*Corresponding author: Tilburg University; Dept. of Econometrics and O.R.; P.O. Box: 90153; 5000 LE Tilburg; The Netherlands; e-mail: engwerda@uvt.nl

†University of Antwerp, Dept. of ICT.

system. In this note we will consider the with the above problem corresponding output feedback problem. That is, we consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), x(0) = x_0, y(t) = Cx(t),$$

where  $x(t)$  is the unknown state variable,  $y(t)$  the observations on the system,  $(A, B)$  is stabilizable,  $(C, A)$  is detectable and matrix  $C$  has full row rank. Assuming that the matrices  $Q$  and  $R$  are symmetric and  $R$  is positive definite, we want to minimize

$$J(F) = \int_0^\infty y^T(t)Qy(t) + u^T(t)Ru(t)dt, \quad (2)$$

where  $u(t) = Fy(t)$  and  $F \in \mathcal{F} := \{F \mid \sigma(A + BFC) \subset \mathbb{C}^-\}$ .

The more general case where  $J(F)$  in (2) is replaced by  $J(F) := \int_0^\infty x^T(t)Qx(t) + u^T(t)Ru(t)dt$  has been studied in literature already by various authors (see e.g. [9], [6] and [13]). However, both necessary and sufficient conditions for solving this output feedback control problem in its original form were not provided. In section 2, below, we show that for the restricted problem it is possible to provide both necessary and sufficient conditions. This result can then immediately be used to formulate also conditions under which the corresponding game problem has a solution, which will be done in section 3. Finally, section 4 contains some concluding remarks.

## 2 The one-player case

Before we present the main result of this note, we first introduce some notation.

If  $\mathcal{X}$  and  $\mathcal{Y}$  are finite-dimensional vector spaces and  $\mathcal{D}$  is an open subset of  $\mathcal{X}$ , we denote the derivative of a differentiable map  $T : \mathcal{D} \rightarrow \mathcal{Y}$  by  $\partial T$  and the differential of  $T$  at  $x \in \mathcal{D}$  in the direction  $h$  by  $\delta T(x; h)$ . We have  $\delta T(x; h) = \partial T(x)h$  (see e.g. [7, Chapter 7]). Partial derivatives and differentials are denoted by  $\partial_i$  and  $\delta_i$  where the index refers to the corresponding argument.

Furthermore, the next algebraic Riccati equation will play an important role in the analysis below:

$$A^T P + PA - PBR^{-1}B^T P + C^T QC = 0. \quad (3)$$

We have the following result:

**Theorem 2.1** *Assume there exist  $P$  and  $F^*$  such that  $X := P$  solves (3) with  $\sigma(A - BR^{-1}B^T X) \subset \mathbb{C}^-$  and  $R^{-1}B^T X = -F^*C$ . Then  $\min_F J(F)$  exists for all  $x_0$  and is attained by  $F = F^*$ . Moreover,  $J(F^*) = x_0^T X x_0$ .*

*Conversely, if  $\min_{F \in \mathcal{F}} J(F)$  exists for all  $x_0$ , then there exist matrices  $X$  and  $F^*$  such that  $P := X$  satisfies (3),  $\sigma(A - BR^{-1}B^T X) \subset \mathbb{C}^-$  and  $R^{-1}B^T X = -F^*C$ .*

Proof:

" $\Rightarrow$  **part**" Let  $F \in \mathcal{F}$ . Then, with  $x_F(t) = e^{(A+BFC)t}x_0$ ,

$$\begin{aligned}
J &= \int_0^\infty \{x_F^T(t)[C^TQC + C^TF^TRFC]x_F(t) + \dot{x}_F^T(t)Xx_F(t) + x_F^T(t)X\dot{x}_F(t)\}dt + x_0^T X x_0 \\
&= \int_0^\infty \{x_F^T(t)[C^TQC + C^TF^TRFC]x_F(t) + x_F^T(t)(A + BFC)^T X x_F(t) + \\
&\quad x_F^T(t)X(A + BFC)x_F(t)\}dt + x_0^T X x_0 \\
&= \int_0^\infty \{x_F^T(t)[A^T X + XA + C^TQC + C^TF^TRFC + C^TF^TB^T X + XBFC]x_F(t)\}dt + x_0^T X x_0 \\
&= \int_0^\infty \{x_F^T(t)[XBR^{-1}B^T X + C^TF^TRFC + C^TF^TB^T X + XBFC]x_F(t)\}dt + x_0^T X x_0 \\
&= \int_0^\infty \{x_F^T(t)[C^TF^{*T}RF^*C + C^TF^TRFC - C^TF^TRF^*C - C^TF^{*T}RFC]x_F(t)\}dt + x_0^T X x_0 \\
&= \int_0^\infty \{x_F^T(t)C^T(F^{*T} - F^T)R(F^* - F)Cx_F(t)\}dt + x_0^T X x_0.
\end{aligned}$$

From this it is clear that  $J(F) \geq x_0^T X x_0$  and that equality is obtained by choosing  $F = F^*$ . Note that since  $C$  is full row rank this choice is uniquely determined.

" $\Leftarrow$  **part**" This part of the proof is based on a variational argument. First note that the set  $\mathcal{F}$  is a nonempty open set. Second note that the smoothness of the coefficients in a Lyapunov equation is preserved by the solution of this equation (see e.g. [5, Section 5.4]), which implies that  $J$  is differentiable with respect to  $F$ . Now, let  $F^* \in \mathcal{F}$  be a minimum of  $J$  for each  $x_0$ . Then,  $\delta_2 J(x_0, F^*; \Delta F) = 0$  for each  $\Delta F$  and for each  $x_0$ . Notice that for all  $F \in \mathcal{F}$ ,

$$J(F) = x_0^T \int_0^\infty e^{(A+BFC)^T t} [C^TQC + C^TF^TRFC] e^{(A+BFC)t} dt x_0 \quad (4)$$

$$=: x_0^T \varphi(F) x_0. \quad (5)$$

So in particular it follows that  $\delta\varphi(F^*; \Delta F) = 0$  for all increments  $\Delta F$ . Hence

$$\partial\varphi(F^*) = 0. \quad (6)$$

Next, introduce the map  $\Phi : \mathcal{F} \times \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$  by

$$\Phi(F, P) = (A + BFC)^T P + P(A + BFC) + C^TQC + C^TF^TRFC.$$

Using (4,5) we have that

$$\begin{aligned}
-(C^TQC + C^TF^TRFC) &= \int_0^\infty \left\{ \frac{d}{dt} [e^{(A+BFC)^T t} [C^TQC + C^TF^TRFC] e^{(A+BFC)t}] \right\} dt \\
&= (A + BFC)^T \varphi(F) + \varphi(F)(A + BFC).
\end{aligned}$$

So,  $\Phi(F, \varphi(F)) = 0$  for all  $F \in \mathcal{F}$ . Taking the derivative of this equality and applying the chain rule yields

$$\partial_1 \Phi(F, \varphi(F)) + \partial_2 \Phi(F, \varphi(F)) \partial\varphi(F) = 0 \text{ for all } F \in \mathcal{F}.$$

Substituting  $F = F^*$  in this equality, and using (6), shows that  $\partial_1 \Phi(F^*, \varphi(F^*)) = 0$ , or, equivalently,

$$\delta_1 \Phi(F^*, \varphi(F^*); \Delta F) = 0 \text{ for all } \Delta F. \quad (7)$$

The differential of  $\Phi$  with respect to its first argument with increment  $\Delta F$  is

$$\delta_1 \Phi(F, P; \Delta F) = C^T \Delta F^T (B^T P + R F C) + (P B + C^T F^T R) \Delta F C.$$

Combining this result with (7) produces

$$C^T \Delta F^T (B^T \varphi(F^*) + R F^* C) + (\varphi(F^*) B + C^T F^{*T} R) \Delta F C = 0 \text{ for all } \Delta F.$$

Since  $C$  is full row rank, the above equality implies that  $B^T \varphi(F^*) + R F^* C = 0$ , or, equivalently,  $F^* C = -R^{-1} B^T \varphi(F^*)$ . Now, since  $\Phi(F^*, \varphi(F^*)) = 0$  and  $F \in \mathcal{F}$ , we conclude that  $X := \varphi(F^*)$  is the stabilizing solution of the ARE (3).  $\square$

### Remark 2.2

- 1) Since matrix  $C$  is full row rank it follows directly from Theorem 2.1 that in case the optimization problem has a solution, the optimal feedback is unique and given by  $F^* = -R^{-1} B^T X C^T (C C^T)^{-1}$ .
- 2) If matrix  $C$  is invertible the optimal feedback coincides with the one that is obtained from the IRZILQ problem. For if  $C$  is invertible, the observed variable  $y$  satisfies the differential equation

$$\dot{y}(t) = C A C^{-1} y(t) + C B u(t)$$

So the optimal control minimizing  $J$  is:  $u^*(t) = -R^{-1} B^T C^T X y(t)$ , where  $X$  is the stabilizing solution of the algebraic Riccati equation

$$(C A C^{-1})^T X + X C A C^{-1} - X C B R^{-1} B^T C^T X + Q.$$

With  $\tilde{X} := C^T X C$  it is then easily verified that  $X$  satisfies the above equation if and only if  $\tilde{X}$  satisfies the algebraic Riccati equation (3). Furthermore it is obvious that  $u^*(t)$  can be rewritten as  $u^*(t) = -R^{-1} B^T C^T \tilde{X} C^{-1} y(t)$ , which is precisely the optimal control from Theorem 2.1 (see item 1) of this remark).  $\square$

### Remark 2.3

- 1) In applications usually the solvability condition for the output feedback problem has to be verified numerically. So the question arises how one can verify in a numerical reliable way the condition that some part of the column space of the stabilizing solution  $X$  of the Riccati equation is contained in some subspace. In general this is a difficult problem. For that reason the question remains whether it is possible to find more directly verifiable conditions under which the output feedback problem has a solution.

We will briefly indicate two ways how one might proceed numerically. However, it remains a topic for future research to see whether there exist numerically more reliable procedures to test the presented solvability condition.

The most simple and direct way is to calculate first the stabilizing solution  $X$  of (3), using e.g. Matlab (this step might be preceded by a step in which one verifies whether a stabilizing solution exists for this equation by verifying whether the corresponding Hamiltonian matrix has a stable invariant graph subspace). The next step is then to verify whether the set of linear equations

$$R^{-1} B^T X = -F C \quad (8)$$

has a solution  $F$ . One may either verify this directly (using e.g. Matlab again) or by noting that, with  $G := I - C^T(CC^T)^{-1}C$ , (8) has a solution if and only if

$$GXBR^{-1} = 0 \text{ or } GXB = 0. \quad (9)$$

Another way one might proceed is by including the restriction (8) directly in the with (3) associated LMI. All solutions  $X$  of (9) are:  $\{L - G^+GLBB^+ \mid L \in \mathbb{R}^{n \times n}\}$ . Here  $H^+$  is the Moore-Penrose inverse of matrix  $H$  (see e.g. [1]).

Therefore the output feedback problem has a solution if and only if the next LMI has a solution  $L$  such that  $X := L - G^+GLBB^+$  is symmetric and  $\sigma(A - BR^{-1}B^T X) \subset \mathbb{C}^-$  (see e.g. [12] for more details)

$$\begin{bmatrix} A^T(L - G^+GLBB^+) + (L - G^+GLBB^+) + C^TQC & (L - G^+GLBB^+)B \\ B^T(L - G^+GLBB^+) & R \end{bmatrix} \geq 0.$$

2) If matrix  $B$  is invertible the computational efforts in verifying the existence of a solution can be reduced. In that case the problem has a solution if and only if  $X = C^T P C$  for some symmetric matrix  $P$ . Consequently the problem has a solution if and only if there exists a symmetric matrix  $P$  such that

$$A^T C^T P C + C^T P C A - C^T P C B R^{-1} B^T C^T P C + C^T Q C = 0, \text{ with } \sigma(A - B R^{-1} B^T C^T P C) \subset \mathbb{C}^-. \quad (10)$$

Since by assumption  $C$  is full row rank, this equation only has a solution if the next ordinary Riccati equation (which dimension is smaller) has the stabilizing solution  $P$

$$(C C^T)^{-1} C A^T C^T P + P C A C^T (C C^T)^{-1} - P C B R^{-1} B^T C^T P + Q = 0. \quad (11)$$

So, by first determining the stabilizing solution  $P$  from (11) and next verifying whether this solution satisfies (10) calculation speed can be improved.  $\square$

Before we proceed with some examples we first notice that the output feedback problem has a solution if and only if using any state and/or input transformation the corresponding output feedback problem has a solution w.r.t. these new coordinates. So, without loss of generality, one may assume e.g. (see Example 2.5 item i) below) that the system is posed in terms of its observable canonical form.

**Example 2.4** Consider the optimal control problem to find the minimum w.r.t.  $f$  for all  $x_0 \in \mathbb{R}^n$  of

$$J(f) = \int_0^\infty y^2(t) + u^2(t) dt,$$

where  $u(t) = f y(t)$  with  $f \in \mathcal{F}$ , subject to the system

$$\dot{x}(t) = \begin{bmatrix} 0 & -a_n & 0 & \cdots & \cdots & 0 \\ 1 & 0 & -a_{n-1} & 0 & \cdots & 0 \\ 0 & 1 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 & -a_2 \\ 0 & \cdots & \cdots & 0 & 1 & -a_1 \end{bmatrix} x(t) + b u(t), \quad x(0) = x_0, \quad y(t) = [0 \cdots 0 \ 1] x(t), \quad (12)$$

where  $a_1 \neq 0$  and  $(A, b)$  is stabilizable.

According to Theorem 2.1, with  $P := [p_{ij}]$  the stabilizing solution of the algebraic Riccati equation (3), this problem has a solution only if the next condition is satisfied

$$b^T P = -f[0 \cdots 0 \ 1].$$

Using this, the algebraic Riccati equation (3) reduces to

$$A^T P + P A - \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & (b^T p_n)^2 - 1 \end{bmatrix} = 0.$$

Since the system is both stabilizable and observable this algebraic Riccati equation has a positive definite solution (see, e.g., [5, prop.16.2.8]). Using this, straightforward (but lengthy) induction arguments show that the above equation has a solution only if  $a_i > 0$ ,  $i > 1$  and  $p_{ij} = 0$ ,  $i \neq j$  and  $p_{ii} = a_{n-i+2}p_{i-1i-1}$ ,  $i = 2, \dots, n$ . Consequently, with  $b^T := [b_n, \dots, b_1]$  and  $p := p_{11}$ ,

$$b^T P = [b_n p \quad b_{n-1} a_n p \quad \cdots \quad b_1 a_2 \cdots a_n p].$$

From this it is then easily verified that the output feedback problem has a solution if and only if  $a_i > 0$  and  $b_i = 0$  for  $i > 1$ . Furthermore the appropriate output feedback gain  $f$  is in that case

$$f = -b_1 a_2 \cdots a_n p, \text{ with } p = \frac{-a_1 + \sqrt{a_1^2 + b_1^2}}{b_1^2 a_2 \cdots a_n}.$$

Finally notice that not every SISO observable system can be rewritten into the form (12). However, using a state transformation  $S$  (where  $S$  is upper triangular with on its main diagonal entries  $-a_1$  and all even subdiagonal entries zeros) generically one can transform the standard state canonical observable system with

$$A := \begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & -v_n \\ 1 & 0 & 0 & 0 & \cdots & -v_{n-1} \\ 0 & 1 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 & -v_2 \\ 0 & \cdots & \cdots & 0 & 1 & -a_1 \end{bmatrix} \text{ and } C := [0 \cdots 0 \ 1],$$

into the form (12) under the assumption that  $a_1 \neq 0$ . □

**Example 2.5** Assume  $Q > 0$ . The output feedback problem has a solution in the next two cases.

i)  $A = \begin{bmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{bmatrix}$ ,  $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ ,  $C = [C_1 \ 0]$ , with  $C_1$  invertible,  $A_{22}$  stable and  $(A_{11}, B_1)$  stabilizable.

By choosing  $X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}$ , where  $X_1$  is the stabilizing solution of the algebraic Riccati equation

$$A_{11}^T X_1 + X_1 A_{11} - X_1 B_1 R^{-1} B_1^T X_1 + C_1^T Q C_1 = 0, \tag{13}$$

it is easily verified that both (3) and the equation  $R^{-1}B^T X = -FC$  have an appropriate solution.

ii)  $A = \begin{bmatrix} A_{11} & A_2 \\ -\alpha A_2^{-1} & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$ ,  $C = [C_1 \ 0]$ , with  $C_1$  invertible,  $A_2$  invertible,  $\alpha > 0$  and  $(A_{11}, B_1)$  stabilizable.

Straightforward calculations show that with  $X = \begin{bmatrix} X_1 & 0 \\ 0 & X_3 \end{bmatrix}$ , where  $X_1$  is the stabilizing solution of (13) again and  $X_3 = \alpha A_2^T X_1 A_2$ , both (3) and the equation  $R^{-1}B^T X = -FC$  have a solution. Furthermore, the property that  $A_{cl} := A - BR^{-1}B^T X$  is a stable matrix follows by noticing that the inertia of matrix  $GA_{cl}G^T$ , with  $G := \begin{bmatrix} \alpha A_2^{-1} A_{cl}^{-1} & I \\ -I & 0 \end{bmatrix}$ , coincide with those of  $\alpha A_2^{-1} A_{cl}^{-1} A_2 \cup A_{cl}$ .

From which the claim is obvious.

iii) The output feedback problem has no solution if  $(A, B)$  is stabilizable,  $(C\sqrt{Q}, A)$  observable and  $CB = 0$ .

This follows since the stabilizing solution,  $X$ , of (3) is positive definite under these assumptions (see [5, prop.16.2.8]). Therefore  $B^T X B > 0$ . However, if an output feedback solution would exist, then it follows that  $-R^{-1}B^T X B = FCB = 0$ . That is,  $B^T X B = 0$ .  $\square$

**Example 2.6** This example provides a case where there does not exist a feedback which is simultaneously optimal for all initial states, though for every initial state there exists an optimal feedback (depending on the initial state).

Consider the optimal control problem to find for a fixed initial state the minimum w.r.t.  $f$  for

$$J(f) = \int_0^\infty y^2(t) + u^2(t) dt,$$

where  $u(t) = fy(t)$  with  $f \in \mathcal{F}$ , subject to the system

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \quad x(0) = x_0, \quad y(t) = [1 \ 0]x(t).$$

Notice that this system is not controllable. Now, let  $P = [p_{ij}]$  be a with this problem corresponding solution of (3). Since  $[1 \ 0]P = -f[1 \ 0]$  should hold it follows that  $p_{12} = 0$ . Using this it follows then by an elementary spelling of (3) that it does not have a solution. So this control problem does not have an output feedback which is optimal for every initial state.

Let  $x(t) =: [x_1(t) \ x_2(t)]^T$  and  $x_0 =: [\alpha \ \beta]$ . Elementary calculations show then that with  $f \in \mathcal{F}$ ,

$$x_1(t) = \begin{cases} (\alpha + \frac{\beta}{f+1})e^{(f-1)t} - \frac{\beta}{f+1}e^{-2t}, & f \neq -1; \\ \alpha e^{-2t} + \beta t e^{-2t}, & f = -1. \end{cases}$$

Therefore, for  $f \neq -1$ ,

$$\begin{aligned} J(f) &= (1 + f^2) \int_0^\infty x_1^2(t) dt = -(1 + f^2) \left[ \frac{(\alpha(f+1) + \beta)^2}{2(f+1)^2(f-1)} - \frac{2\beta(\alpha(f+1) + \beta)}{(f+1)^2(f-3)} - \frac{\beta^2}{4(f+1)^2} \right] \\ &= \frac{-(1 + f^2)}{4(f-1)(f-3)} [2\alpha^2(f-1) - (2\alpha + \beta)^2]. \end{aligned}$$

Whereas  $J(-1) = 2[\frac{1}{4}\alpha^2 + \frac{1}{8}\alpha\beta + \frac{1}{32}\beta^2]$ .

From this it follows in particular that  $J(f)$  is differentiable on  $(-\infty, 1)$ . Straightforward differentiation yields then that, with  $\gamma := 2 + \frac{\beta}{\alpha}$ ,

$$J'(f) = \frac{-\alpha^2[f^4 - 8f^3 + (2\gamma^2 + 12)f^2 - (2\gamma^2 + 4)f - (2\gamma^2 + 1)]}{(f-1)^2(f-3)^2}.$$



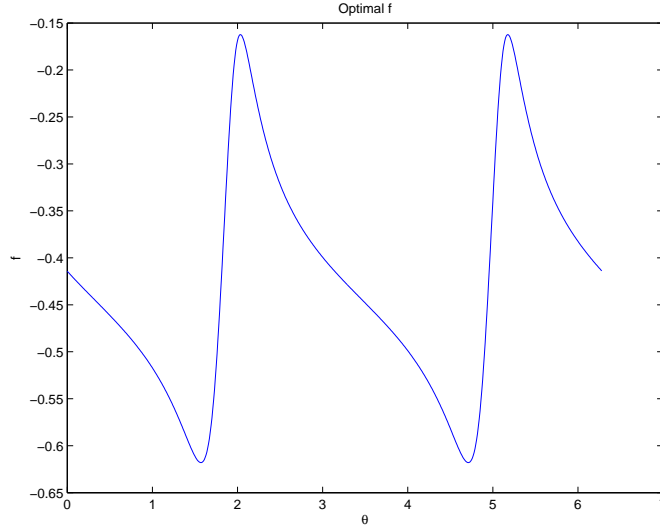


Figure 1: Optimal  $f$  as a function of the angle  $\theta$  for Example 2.4

Obviously,  $J'(f)$  is continuous on  $(-\infty, 1)$ . Furthermore it is clear that  $\lim_{f \rightarrow -\infty} J'(f) < 0$ , whereas  $\lim_{f \uparrow 1} J'(f) > 0$ . So,  $J(f)$  has a global minimum on  $(-\infty, 1)$ . Furthermore it can be verified, using e.g. a numerical symbolic toolbox, that the location where this minimum is attained depends on the choice of  $\gamma$ , that is on  $x_0$ . Using polar coordinates we plotted in Figure 1 the optimal  $f$  as a function of the angle  $\theta$ . Since  $J'(f)$  only depends on  $\gamma$  it is clear that the optimal  $f$  does not depend on the radius  $r$  in this example.  $\square$

**Remark 2.7** The fact that in the previous example the optimal  $f$  is the same for every initial state on a line through the origin is not a coincidence. This property holds in general. For, if  $F$  minimizes  $J(F, x_0) = x_0^T \int_0^\infty e^{(A+BFC)^T t} (Q + F^T R F) e^{(A+BFC)t} dt x_0$  then, obviously, it also minimizes  $J(F, \lambda x_0) = \lambda^2 J(F, x_0)$ .  $\square$

Finally, the next example shows (see also [10]) that there also exist cases where the problem has no solution for every initial state (even though  $J(f)$  has an infimum).

**Example 2.8** Consider the optimal control problem to find for a fixed initial state the minimum w.r.t.  $f \in \mathcal{F}$  for

$$J(f) = \int_0^\infty -y^2(t) + u^2(t) dt,$$

where  $u(t) = fy(t)$  with  $f \in \mathcal{F}$ , subject to the system

$$\dot{x}(t) = -x(t) + u(t), \quad x(0) = x_0, \quad y(t) = x(t).$$

The outcome of this example seems obvious since the cost function promotes large values of  $y$  whereas one the other hand the controller is constrained to be stabilizing one may expect that the problem does not have a solution. Straightforward calculations show that

$$\begin{aligned} J(f) &= (-1 + f^2) \int_0^\infty x^2(t) dt = (-1 + f^2) x_0^2 \int_0^\infty e^{2(-1+f)t} dt \\ &= \frac{1 - f^2}{2(f - 1)} x_0^2 = \frac{-1}{2} (1 + f) x_0^2. \end{aligned}$$

Since  $\mathcal{F} = \{f \mid f < 1\}$  it is clear from the last equality that, for  $x_0 \neq 0$ ,  $J(f)$  has not a minimum in  $\mathcal{F}$ , although the infimum exists for any such initial state and is  $-x_0^2$ .  $\square$

### 3 An application to LQ differential games

In this section we use the equivalence result from Theorem 2.1 to characterize output feedback Nash equilibria in infinite-horizon LQ differential games.

The following notation will be used. For an  $N$ -tuple  $\hat{F} = (\hat{F}_1, \dots, \hat{F}_N) \in \Gamma_1 \times \dots \times \Gamma_N$  for given sets  $\Gamma_i$ , we shall write  $\hat{F}_{-i}(\alpha) = (\hat{F}_1, \dots, \hat{F}_{i-1}, \alpha, \hat{F}_{i+1}, \dots, \hat{F}_N)$  with  $\alpha \in \Gamma_i$ .

Consider the cost function of player  $i$  defined by

$$J_i(x_0, F_1, \dots, F_N) = \int_0^\infty \left( y_i^T Q_i y_i + \sum_{j=1}^N u_j^T R_{ij} u_j \right) dt$$

with  $u_j = F_j y_j$  for  $j = 1, \dots, N$ , and where  $x$  is generated by

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^N B_j u_j(t), \quad x(0) = x_0, \quad \text{and } y_i(t) = C_i x(t).$$

Assume that  $Q_i$  is symmetric,  $R_{ii}$  is positive definite,  $C_i$  is full row rank and  $(F_1, \dots, F_N) \in \mathcal{F}_N$ , where

$$\mathcal{F}_N = \left\{ (F_1, \dots, F_N) \left| A + \sum_{j=1}^N B_j F_j C_j \text{ is stable} \right. \right\}. \quad (14)$$

This last assumption spoils the rectangular structure of the strategy spaces, i.e. choices of feedback matrices cannot be made independently. However, such a restriction is motivated by the fact that closed-loop stability is usually a common objective.

In our setting the concept of an output feedback Nash equilibrium is defined as follows.

**Definition 3.1** An  $N$ -tuple  $\hat{F} = (\hat{F}_1, \dots, \hat{F}_N) \in \mathcal{F}_N$  is called an *output feedback Nash equilibrium* if for all  $i$  the following inequality holds:

$$J_i(x_0, \hat{F}) \leq J_i(x_0, \hat{F}_{-i}(\alpha))$$

for each  $x_0$  and for each matrix  $\alpha$  such that  $\hat{F}_{-i}(\alpha) \in \mathcal{F}_N$ .  $\square$

Next, consider the set of coupled algebraic Riccati equations:

$$\begin{aligned} A^T X_i + X_i A - \sum_{j=1, j \neq i}^N X_i B_j R_{jj}^{-1} B_j^T X_j - \sum_{j=1, j \neq i}^N X_j B_j R_{jj}^{-1} B_j^T X_i + \\ - X_i B_i R_{ii}^{-1} B_i^T X_i + \sum_{j=1, j \neq i}^N X_j B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T X_j + C_i^T Q_i C_i = 0, \quad i = 1, \dots, N. \end{aligned} \quad (15)$$

A *stabilizing solution* of (15) is an  $N$ -tuple  $(X_1, \dots, X_N)$  of real symmetric  $n \times n$  matrices satisfying (15) such that  $A - \sum_{j=1}^N B_j R_{jj}^{-1} B_j^T X_j$  is stable. In contrast to the stabilizing solution of (3), stabilizing

solutions of (15) are not necessarily unique (see e.g. [3]). The next theorem states that output feedback Nash equilibria are completely characterized by stabilizing solutions of (15) satisfying some important constraint. Its proof follows directly from Theorem 2.1 (see also the proof of [3, Theorem 8.5]).

**Theorem 3.2** *Let  $(X_1, \dots, X_N)$  be a stabilizing solution of (15) and  $(F_1, \dots, F_N)$  be such that  $R_i^{-1} B_i^T X_i = -F_i C_i$  for  $i = 1, \dots, N$ . Then  $(F_1, \dots, F_N)$  is an output feedback Nash equilibrium. Conversely, if  $(F_1, \dots, F_N)$  is an output feedback Nash equilibrium, there exists a stabilizing solution  $(X_1, \dots, X_N)$  of (15) and  $(F_1, \dots, F_N)$  such that  $R_i^{-1} B_i^T X_i = -F_i C_i$ . Furthermore, if the game has an output feedback Nash equilibrium then with this equilibrium corresponding cost for player  $i$  is  $J_i(x_0, \hat{F}) = x_0^T X_i x_0$ .  $\square$*

## 4 Concluding Remarks

In this note we considered the LQ static output feedback problem for a special type of cost functional. The cost function is special in the sense that apart from the controls just the observed output variables are taken into account. Following the analysis of [2] (see also [3]) we derived both necessary and sufficient conditions for this problem under which there exists a solution.

The conditions under which a solution exists are in general rather strict. That is, in general there will not exist a static output feedback control which will be optimal for every initial state. We illustrated this in some examples. Moreover, we showed in an example that in case not for all initial states the same output feedback is optimal it may still be possible that for every initial state there exists an optimal (initial state dependent) output feedback.

The existence conditions for a solution of the problem are stated in terms of the stabilizing solution of an algebraic Riccati equation. An open problem remains to find conditions on the system parameters under which one can directly verify (without first calculating the solution of this Riccati equation) whether the problem will have a solution.

Finally, the one-player result was used to find both necessary and sufficient conditions for existence of an output feedback Nash equilibrium. The presented conditions are phrased in terms of existence of a stabilizing solution of a set of coupled algebraic Riccati equation. Like in the full state observation case it remains in particular an open question under which conditions this set of equations will have a unique stabilizing solution.

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