CentER

Discussion Paper

No. 2003–29

ON A COMPROMISE SOCIAL CHOICE CORRESPONDENCE

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March 2003

ISSN 0924-7815

TILBURG UNIVERSITY

On a compromise social choice correspondence

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Abstract

This paper analyzes the compromise social choice correspondence derived from the τ-value of digraph games. Among other things monotonicity of this correspondence is shown.

Keywords: τ-value, compromise social choice correspondence.

JEL Classification Number: C71, D71.

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1 Introduction

A social choice correspondence assigns to each social choice situation a non-empty subset of alternatives. Elements of this subset can be interpreted as most preferred alternatives by society as a whole. There is an abundant literature on different types of social choice correspondences, cf. Fishburn (1977) and Laslier (1997) for surveys. Van den Brink and Borm (1994) and Borm, Van den Brink, Levínsky, and Slikker (2000) use tools of cooperative game theory to define two new social choice correspondences. These correspondences are based on the Shapley value.

Quant, Borm, Reijnierse, and Voorneveld (2002) introduce and characterize the τ-value (cf. Tijs (1981) and Tijs (1987)) of digraph games. This paper analyzes the compromise social choice correspondence derived from the τ-value of digraph games. We show that if the compromise correspondence selects some alternative, it remains selected when some agent changes his profile by ranking this alternative higher (monotonicity). Moreover we establish a connection between properties of social choice correspondences based on game theoretical solutions and monotonicity of these solutions. This result can not only be applied to the compromise correspondence, but also to for instance the one derived from the Shapley value.

2 Social choice situations and digraph games

A social choice situation can be represented by a triple (N, A, p), in which N is a finite group of individuals or agents, A a finite set of alternatives and p a vector of preference relations of the group of individuals on the set A. The class of all social choice situations is denoted by $\mathcal{S}$. A social choice correspondence C assigns to each social choice situation $(N, A, p) \in \mathcal{S}$ a non-empty subset $C(N, A, p)$ of A. This set can be interpreted as the set of most preferred alternatives by the group of individuals.

The preference relation of individual i will be denoted by p_i, xp_iy means that i prefers alternative x to alternative y. We assume that preference relations are linear order preferences. This means that p_i is reflexive [1], complete [2], transitive [3] and antisymmetric [4]. A profile $p = (p_i)_{i \in N}$ contains

[1] A preference relation p_i on A is reflexive if for all $x \in A$ it holds that xp_ix

[2] A preference relation p_i on A is complete if for all $x, y \in A$ it holds that xp_iy or yp_ix (or both).

[3] A preference relation p_i on A is transitive if for all $x, y, z \in A$ it holds that: if xp_iy and yp_iz, then xp_iz.

[4] A preference relation p_i on A is antisymmetric if for all $x, y \in A, x \neq y$, xp_iy implies

the preferences of the individuals in N.

With $(N, A, p) \in \mathcal{S}$, one can associate a simple majority win digraph $D_p \subset A \times A$ in the following way. For $x, y \in A$, $x \neq y$:

$$(x, y) \in D_p \iff n_p(x, y) > n_p(y, x),$$

where $n_p(x, y) := |\{i \in N \mid xp_iy \text{ and } \neg yp_ix\}|$ denotes the number of agents who strictly prefer x to y in profile p. Moreover we assume that $(x, x) \in D_p$ for each $x \in A$. So for each social choice situation, D_p is a reflexive digraph. Here our approach differs from the approach chosen in Borm et al. (2000), where no loops are present in the digraph D_p.

The following example illustrates that the digraph D_p can contain a cycle.

Example 2.1 *Consider the following social choice situation in which we have a group of three agents $N = \{1, 2, 3\}$ and three alternatives $\{a_1, a_2, a_3\}$. Suppose that (with obvious notation) linear preferences are given by $p_1 = a_1a_2a_3$, $p_2 = a_2a_3a_1$, $p_3 = a_3a_1a_2$ (we assume that p_i puts most preferred alternatives to the left and least preferred alternatives to the right). Then D_p is the graph drawn in figure 1. For example, the arc (a_2, a_3) is drawn, because two of three agents prefer a_2 to a_3.*

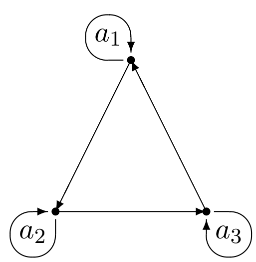

Figure 1: The digraph D_p of example 2.1.

For a digraph $D \subset A \times A$ and $x \in A$ the set $P_D(x) = \{y \in A \mid (y, x) \in D\}$ is the set of predecessors of x in D_p. The set $S_D(x) = \{y \in A \mid (x, y) \in D\}$ consists of all successors of x. We denote the set of successors of x, for which x is the only predecessor by $\tilde{S}_D(x)$: $\tilde{S}_D(x) = \{y \in A \mid P_D(y) = \{x\}\}$. For a reflexive digraph D, $\tilde{S}_D(x)$ is either empty or consists only of the node x. The set I_D consists of all nodes with a non-zero indegree, the subset $\tilde{I}_D$

$\neg yp_ix$.

contains all nodes with exactly one predecessor. If D is reflexive, the set I_D contains all nodes and $\tilde{I}_D$ is the set of all nodes with no other predecessors than itself.

Following Van den Brink and Borm (1994) and Quant et al. (2002) we have:

Definition 2.1 *The score game corresponding to a digraph $D \subset A \times A$ is the game $v_D : 2^A \to \mathbb{R}$ given by:*

$$v_D(T) = |\{x \in A \mid P_D(x) \subset T, P_D(x) \neq \emptyset\}| \quad \textit{for all } T \in 2^A \backslash \{\emptyset\}.$$

As usual we take $v_D(\emptyset) = 0$.

There is a natural way to associate a social choice correspondence with a (one-point) game theoretical solution. If γ is a game theoretical solution, then the corresponding social choice correspondence C_γ is given by:

$$C_\gamma(N, A, p) \quad = \quad \{x \in A \mid \gamma_x(v_{D_p}) \geq \gamma_y(v_{D_p}),\ \forall y \in A\},$$

the set of alternatives to which γ assigns the highest payoff in the corresponding score game.

3 Monotonicity of the compromise social choice correspondence

Borm et al. (2000) examines the social choice correspondence C_ϕ corresponding to the Shapley value ϕ. The τ-value or compromise value, introduced in Tijs (1981), is a an alternative to the Shapley value based on the minimum right of players and their utopia demand. In this paper we will concentrate on the corresponding so called compromise correspondence C_τ. The τ-value of an arbitrary (quasi-balanced) TU-game (A, v) is given by:

$$\tau(v) \quad = \quad \lambda m(v) + (1 - \lambda) M(v),$$

in which λ is chosen in $[0, 1]$ such that τ is efficient. The vectors $m(v)$ and $M(v)$ are, for $x \in A$ defined by:

$$\begin{aligned} M_x(v) \quad &= \quad v(A) - v(A \backslash \{x\}) \\ m_x(v) \quad &= \quad \max_{T : x \in T} \left\{ v(T) - \sum_{y \in T \backslash \{x\}} M_y(v) \right\}. \end{aligned}$$

Quant et al. (2002) proved that for an arbitrary digraph $D \subset A \times A$ and for $x \in A$:

$$\tau_x(v_D) \quad = |\tilde{S}_D(x)| + (|S_D(x)| - |\tilde{S}_D(x)|) * \frac{|I_D| - |\tilde{I}_D|}{|D| - |\tilde{I}_D|}.$$

This formula holds if $|D| \neq |\tilde{I}_D|$. If $|D| = |\tilde{I}_D|$ each alternative gets exactly the amount $|\tilde{S}_D(x)|$ (its minimum right). This case is not very challenging, especially if the digraph D is reflexive, since then the graph consists of all loops only.

For a reflexive digraph D, with $|D| \neq |A|$, the formula above becomes:

$$\tau_x(v_D) \quad = \quad \begin{cases} 1 + (|S_D(x)| - 1) * \dfrac{|A| - |\tilde{I}_D|}{|D| - |\tilde{I}_D|} & \text{if } x \in \tilde{I}_D, \\ |S_D(x)| * \dfrac{|A| - |\tilde{I}_D|}{|D| - |\tilde{I}_D|} & \text{if } x \notin \tilde{I}_D. \end{cases} \tag{1}$$

For $(N, A, p) \in \mathcal{S}$ the compromise social choice correspondence C_τ is given by:

$$C_\tau(N, A, p) \quad = \{x \in A \mid \tau_x(D_p) \geq \tau_y(D_p),\ \forall y \in A\}.$$

Note that for a social choice situation $(N, A, p) \in \mathcal{S}$ and $x \in C_\tau(N, A, p)$, with $x \notin \tilde{I}_{D_p}$, it holds that for all $y \in A$, $|S_{D_p}(x)| \geq |S_{D_p}(y)|$. If $y \notin \tilde{I}_{D_p}$ this is immediately clear from formula (1). For $y \in \tilde{I}_{D_p}$ this readily follows from the fact that, since $|D_p| > |A|$:

$$\frac{|A| - |\tilde{I}_{D_p}|}{|D_p| - |\tilde{I}_{D_p}|} \quad < \quad 1.$$

An important property of social choice correspondences is monotonicity. Monotonicity implies that if in a situation (N, A, p) alternative x is a best alternative according to C, then x should be a best alternative according to C in the situation (N, A, p'), where p' is a preference relation obtained from p by moving x to the left.

Property 3.1 Monotonicity: *C is monotonic if for all $(N, A, p), (N, A, p') \in \mathcal{S}$, $x \in A$, such that for all $i \in N$:*

(i) $yp_iz \Rightarrow yp'_iz$ for all $y, z \in A \backslash \{x\}$, $y \neq z$,

(ii) $xp_iy \Rightarrow xp_i'y$ *for all* $y \in A\backslash\{x\}$*,*

(iii) $x \in C(N, A, p)$*,*

it holds that $x \in C(N, A, p')$*.*

The following theorem states that C_τ is monotonic as social choice correspondence.

Theorem 3.1 *The compromise social choice correspondence* C_τ *satisfies monotonicity.*

Proof: Let $(N, A, p), (N, A, p') \in \mathcal{S}$ and $x \in A$ be such that the conditions (i)-(iii) in property 3.1 hold. We assume that p' differs only slightly from p, in the sense that there is only one agent $i \in N$ for which $p_i' \neq p_i$ and in p_i' alternative x has moved one step to the left compared to p_i. If one can prove monotonicity for this case, one can prove monotonicity for all cases, just by changing preference profile p into p' in small steps and applying the above result at each step. Let $y \in A$ be the alternative which is preferred to x in p_i, but not to x in p_i'. The digraphs D_p and $D_{p'}$ can only differ in three different ways (it is also possible that they are the same, but then there is nothing to prove):

- An arc (y, x) that is present in D_p, is removed in $D_{p'}$.
- A new arc (x, y) arises in $D_{p'}$, where in D_p no arc between x and y is present.
- An arc (y, x) present in D_p is reversed in $D_{p'}$.

Let $D_1, D_2 \subset A \times A$ be such that D_2 only differs from D_1 by deleting the arc (y, x) or adding the arc (x, y), or reversing the arc (y, x). Clearly it suffices to prove the following: if $\tau_x(v_{D_1}) \geq \tau_z(v_{D_1})$, for all $z \in A$, then $\tau_x(v_{D_2}) \geq \tau_z(v_{D_2})$. Note that the last case (reversing an edge (y, x) in D_1) is a combination of the first two cases and hence we only need to consider the first two cases. The trivial cases that $|D_1| = |A|$ or $|D_2| = |A|$ are left to the reader.

To simplify notations, we denote e.g. $\tau_x(D_1)$ instead of $\tau_x(v_{D_1})$ and define for $i \in \{1, 2\}$:

$$\begin{aligned} n_i &= |A| - |\tilde{I}_{D_i}|, \\ d_i &= |D_i| - |\tilde{I}_{D_i}|, \\ c_i &= \frac{n_i}{d_i}. \end{aligned}$$

Case 1: Assume that $(y, x) \in D_1$, $D_2 = D_1 \backslash \{(y, x)\}$ and $\tau_x(D_1) \geq \tau_z(D_1)$ for all $z \in A$.

We first consider the case $x \notin \tilde{I}_{D_2}$. From $x \notin \tilde{I}_{D_1}$ and $\tau_x(D_1) \geq \tau_z(D_1)$, for all $z \in A$, we can conclude that $|S_{D_1}(x)| \geq |S_{D_1}(z)|$ for all $z \in A$. It holds that for $z \in A \backslash \{y\}$: $S_{D_2}(z) = S_{D_1}(z)$, and $S_{D_2}(y) = S_{D_1}(y) \backslash \{x\}$, furthermore $|D_2| = |D_1| - 1$ and $\tilde{I}_{D_2} = \tilde{I}_{D_1}$. Hence $c_2 > c_1$ and we can conclude that for $z \in A$:

$$\tau_x(D_2) - \tau_x(D_1) \geq \tau_z(D_2) - \tau_z(D_1).$$

Consequently $\tau_x(D_2) \geq \tau_z(D_2)$.

Secondly assume that $x \in \tilde{I}_{D_2}$, then the following is true: $|D_2| = |D_1| - 1$, $\tilde{I}_{D_2} = \tilde{I}_{D_1} \cup \{x\}$ and for $z \in A \backslash \{y\}$, $S_{D_2}(z) = S_{D_1}(z)$ and $S_{D_2}(y) = S_{D_1}(y) \backslash \{x\}$. Since the indegree of a node not in $\tilde{I}_{D_1}$ is at leat two, it holds that:

$$2(|A| - |\tilde{I}_{D_1}|) \leq |D_1| - |\tilde{I}_{D_1}|,$$

it follows that:

$$c_2 - c_1 = \frac{n_1 - 1}{d_1 - 2} - \frac{n_1}{d_1} = \frac{2n_1 - d_1}{d_1 * (d_1 - 2)} \leq 0.$$

One can conclude that for $z \in A \backslash \{x\}$, $\tau_z(D_2) \leq \tau_z(D_1)$. Because τ is efficient, it then holds that:

$$\tau_x(D_2) \geq \tau_x(D_1).$$

Hence $\tau_x(D_2) \geq \tau_z(D_2)$.

Case 2: Let D_1 and D_2 be such that $(y, x), (x, y) \notin D_1$, $D_2 = D_1 \cup \{(x, y)\}$ and $\tau_x(D_1) \geq \tau_z(D_1)$, for all $z \in A$.

We first consider the case that $x, y \notin \tilde{I}_{D_1}$. Then $|S_{D_1}(x)| \geq |S_{D_1}(z)|$ for all $z \in A$. The following equations hold: $S_{D_2}(z) = S_{D_1}(z)$ for $z \in A \backslash \{x\}$, $S_{D_2}(x) = S_{D_1}(x) \cup \{y\}$, $|D_2| = |D_1| + 1$, $\tilde{I}_{D_2} = \tilde{I}_{D_1}$. Hence $c_2 \leq c_1$. Consequently for each $z \in A \backslash \{x\}$, it holds that $\tau_z(D_2) \leq \tau_z(D_1)$. Using efficiency we have:

$$\tau_x(D_2) \geq \tau_x(D_1).$$

It then directly follows that $\tau_x(D_2) \geq \tau_z(D_2)$.

Secondly assume that $x \notin \tilde{I}_{D_1}$ and $y \in \tilde{I}_{D_1}$. One can deduce that $\tilde{I}_{D_2} = \tilde{I}_{D_1} \backslash \{y\}$ and $|D_2| = |D_1| + 1$. Together with $2(|A| - \tilde{I}_{D_1}) \leq |D_1| - |\tilde{I}_{D_1}|$, this implies that:

$$c_2 - c_1 = \frac{n_1+1}{d_1+2} - \frac{n_1}{d_1} = \frac{d_1 - 2n_1}{d_2(d_1+2)} \geq 0.$$

Because $|S_{D_1}(x)| \geq |S_{D_1}(z)|$, for all $z \in A$, one can conclude that:

$$\tau_x(D_2) - \tau_x(D_1) \geq \tau_z(D_2) - \tau_z(D_1),$$

and hence $\tau_x(D_2) \geq \tau_z(D_2)$.

In the third case we assume that $x \in \tilde{I}_{D_1}$ and $y \notin \tilde{I}_{D_1}$. It holds that: $\tilde{I}_{D_2} = \tilde{I}_{D_1}$, $|D_2| = |D_1| + 1$, from which we can conclude that $c_2 \leq c_1$. Because $S_{D_2}(z) = S_{D_1}(z)$ for all $z \in A \backslash \{x\}$, it holds that $\tau_z(D_2) \leq \tau_z(D_1)$. Efficiency yields that:

$$\tau_x(D_2) \geq \tau_x(D_1).$$

It follows that $\tau_x(D_2) \geq \tau_z(D_2)$.

In the final case we assume that $x, y \in \tilde{I}_{D_1}$. It holds that $\tilde{I}_{D_2} = \tilde{I}_{D_1} \backslash \{y\}$, $|D_2| = |D_1| + 1$. One can conclude that $|S_{D_1}(x)| \geq |S_{D_1}(y)|$ and $S_{D_2}(x) = S_{D_1}(x) \cup \{y\}$, $S_{D_2}(y) = S_{D_1}(y)$. It immediately follows that $\tau_x(D_2) \geq \tau_y(D_2)$.

Let $z \in A \backslash \{y\}$. There are two possible cases. First let $z \in \tilde{I}_{D_1}$. It holds that $|S_{D_1}(x)| \geq |S_{D_1}(z)|$ and hence $\tau_x(D_2) \geq \tau_z(D_2)$ according to formula (1).

Secondly let $z \notin \tilde{I}_{D_1}$. Suppose that after adding the arc (y, x) the value of τ-measure of z is larger than the value assigned to x:

$$\begin{aligned}
\tau_z(D_2) &= |S_{D_1}(z)| * \frac{|A| - |\tilde{I}_{D_1}| + 1}{|D_1| - |\tilde{I}_{D_1}| + 2} \\
&> 1 + (|S_{D_1}(x)|) * \frac{|A| - |\tilde{I}_{D_1}| + 1}{|D_1| - |\tilde{I}_{D_1}| + 2} \\
&= \tau_x(D_2)
\end{aligned} \tag{2}$$

Since x achieves the highest payoff in D_1 according to τ, it holds that:

$$\begin{aligned}
\tau_z(D_1) &= |S_{D_1}(z)| * \frac{|A| - |\tilde{I}_{D_1}|}{|D_1| - |\tilde{I}_{D_1}|} \\
&\leq 1 + (|S_{D_1}(x)| - 1) * \frac{|A| - |\tilde{I}_{D_1}|}{|D_1| - |\tilde{I}_{D_1}|} \\
&= \tau_x(D_1)
\end{aligned} \tag{3}$$

From (2) and (3) it follows that:

$$\begin{aligned} |S_{D_1}(z)| * (|A| - |\tilde{I}_{D_1}| + 1) &> |D_1| - |\tilde{I}_{D_1}| + 2 + \\ &\quad + |S_{D_1}(x)| * (|A| - |\tilde{I}_{D_1}| + 1) \quad (4) \\ -(|S_{D_1}(z)| * (|A| - |\tilde{I}_{D_1}|)) &\geq -(|D_1| - |\tilde{I}_{D_1}| + \\ &\quad + (|S_{D_1}(x)| - 1) * (|A| - |\tilde{I}_{D_1}|)). \quad (5) \end{aligned}$$

Adding (5) and (4) results in:

$$|S_{D_1}(z)| > 2 + |A| - |\tilde{I}_{D_1}| + |S_{D_1}(x)|. \quad (6)$$

However, since $z \notin \tilde{I}_{D_1}$ we have that $S_{D_1}(z) \cap \tilde{I}_{D_1} = \emptyset$. So $|S_{D_1}(z)| + |\tilde{I}_{D_1}| \leq |A|$, which contradicts formula (6). □

4 Pareto optimality and Smith's Condorcet principle

There are several other interesting properties of social choice correspondences besides monotonicity. We will mention some of them shortly. An alternative $x \in A$ is a *Condorcet winner* in a social choice situation (N, A, p) if for all $y \in A \backslash \{x\}$, $(x, y) \in D_p$. C is a *Condorcet social choice correspondence* if if for this type of social choice situations $C(N, A, p) = \{x\}$.

The next property states that if all agents prefer alternative x to alternative y, then y is not a most preferred alternative.

Property 4.1 Pareto optimality: *C is Pareto optimal if for all $(N, A, p) \in \mathcal{S}$ and for all $x, y \in A$: if xp_iy for all $i \in N$, then $y \notin C(N, A, p)$.*

Suppose that the set of alternatives can be partitioned in A_1 and A_2 such that the majority of the group prefers an arbitrary alternative from A_1 to an arbitrary alternative of A_2. The following property states that no alternative of A_2 is a most preferred alternative.

Property 4.2 Smith's Condorcet principle: *C satisfies Smith's Condorcet principle if for all $(N, A, p) \in \mathcal{S}$: if $A = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$, $A_1 \neq \emptyset$, $A_2 \neq \emptyset$, and $(x, y) \in D_p$, for all $x \in A_1$, $y \in A_2$, then $A_2 \cap C(N, A, p) = \emptyset$.*

Borm, Van den Brink, Levínsky, and Slikker (2000) proved that C_ϕ is a Condorcet social choice correspondence satisfying Pareto optimality and Smith's Concorcet principle. One can prove that C_τ is also a Condorcet social choice correspondence satisfying Pareto optimality and Smith's Concorcet principle. This gives rise to the thought that there exists a game theoretical property which implies some properties of social choice correspondences based on a game theoretical solution.

We will define a special type of monotonicity in TU-games. It will turn out to imply various properties in social choice theory.

Property 4.3 Monotonicity: *A game theoretic (one-point) solution γ satisfies monotonicity if for each game (A, v), it holds that $\gamma_x(v) > \gamma_y(v)$, if for all $T \in 2^{A\backslash\{x,y\}}$ the following holds: $v(T \cup \{x\}) \geq v(T \cup \{y\})$, with a strict inequality if $T = A\backslash\{x, y\}$.*

The following proposition states a relation between monotonicity in TU-games and properties of social choice correspondences.

Proposition 4.1 *If a game theoretic solution γ satisfies monotonicity on the subclass of convex games* [5]*, then C_γ is a Condorcet social choice correspondence satisfying Pareto optimality and Smith's Condorcet principle* [6]*.*

Proof: Let $(N, A, p) \in \mathcal{S}$ be a social choice situation. Then the game v_{D_p} is convex (cf. Van den Brink and Borm (1994)). Let γ be a game theoretical solution satisfying monotonicity on the class of convex games. First of all we prove that C_γ is a Condorcet social choice correspondence.

Suppose $x \in A$ is a Condorcet winner. This means that for each $y \in A$ it holds that $(x, y) \in D_p$ and hence for each $y \in A$ and each $T \in 2^{A\backslash\{x,y\}}$:

$$v_{D_p}(T \cup \{x\}) > v_{D_p}(T \cup \{y\}) = 0.$$

The solution γ is monotonic and hence $\gamma_x(v_{D_p}) > \gamma_y(v_{D_p})$ for all $y \in A$, which implies $C_\gamma(N, A, p) = \{x\}$.

We proceed with Pareto optimality. Let $x, y \in A$, such that for all $i \in N$, xp_iy. The digraph D_p contains the arc (x, y) and if an arc (y, z), $z \in A$, is present, then arc (x, z) is present too. This structure of D_p implies that $S_{D_p}(y) \subsetneq S_{D_p}(x)$, from which we can conclude that:

[5] A game v is convex if for all $x \in A$, for all $T_1, T_2 \subset A\backslash\{x\}$, $T_1 \subset T_2$ it holds that $v(T_1 \cup \{x\}) - v(T_1) \leq v(T_2 \cup \{x\}) - v(T_2)$.

[6] One can also prove that if γ is efficient and individual rational on the subclass of convex games, then C_γ satisfies the so called subset condition 2, dicussed in Fishburn (1977).

$$
\begin{aligned}
v_{D_p}(A\backslash\{x\}) &= v_{D_p}(A) - |S_{D_p}(x)| \\
&< v_{D_p}(A) - |S_{D_p}(y)| \\
&= v_{D_p}(A\backslash\{y\}).
\end{aligned}
$$

For all $T \in 2^{A\backslash\{x,y\}}$ the inequality $v_{D_p}(T \cup \{x\}) \geq v_{D_p}(T \cup \{y\})$ holds and for $T = A\backslash\{x,y\}$ this inequality is strict. Monotonicity of γ yields $\gamma_x(v_{D_p}) > \gamma_y(v_{D_p})$ and hence $y \notin C_\gamma(N,A,p)$.

Finally, we show that monotonicity of γ implies Smith's Condorcet Principle. Let $(N,A,p) \in \mathcal{S}$ be such that $A = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$ and $(x,y) \in D_p$ for all $x \in A_1$, $y \in A_2$. Let $x \in A_1$ and $y \in A_2$, then $S_{D_p}(y) \subsetneq S_{D_p}(x)$. In a similar way as above it follows that for each $T \in 2^{A\backslash\{x,y\}}$: $v_{D_p}(T \cup \{x\}) \geq v_{D_p}(T \cup \{y\})$, and for $T = A\backslash\{x,y\}$ this inequality is strict. Monotonicity of γ implies $\gamma_x(v_{D_p}) > \gamma_y(v_{D_p})$ and $y \notin C_\gamma(N,A,p)$, hence $A_2 \cap C_\gamma(N,A,p) = \emptyset$. □

As a consequence of theorem 4.1 we can prove that the social choice correspondences based on the τ-value and the Shapley value are Condorcet social choice correspondences satisfying Pareto optimality, Smith's Condorcet principle.

Corollary 4.1 *C_τ is a Condorcet social choice correspondence satisfying Pareto optimality and Smith's Condorcet principle.*

Proof: According to theorem 4.1 we only need to prove that the τ-value is monotonic on the class of convex games. Let (A,v) be a convex TU-game and let $x,y \in A$ be such that for all $T \in 2^{A\backslash\{x,y\}}$ it holds that:

$$v(T \cup \{x\}) \geq v(T \cup \{y\}), \tag{7}$$

with a strict inequality for $T = A\backslash\{x,y\}$. Then:

$$
\begin{aligned}
M_x(v) &= v(A) - v(A\backslash\{x\}) \\
&> v(A) - v(A\backslash\{y\}) = M_y(v), \\
m_x(v) &= v(\{x\}) \\
&\geq v(\{y\}) = m_y(v).
\end{aligned}
$$

This yields $\tau_x(v) > \tau_y(v)$ [7]. □

[7] It is not possible that $\sum_{x\in A} m_x(v) = v(A)$ and $m_x(v) = m_y(v)$, because together with convexity this implies that v is an additive game with $v(\{x\}) = v(\{y\})$. This contradicts the fact that $v(A\backslash\{x\}) < v(A\backslash\{y\})$.

Corollary 4.2 *C_ϕ is a Condorcet social choice correspondence satisfying Pareto optimality and Smith's Condorcet principle.*

Proof: According to theorem 4.1 we only need to prove that the Shapley value is monotonic on the class of convex games. Let (A, v) be a convex TU-game and $x, y \in A$. Suppose that for each coalition $T \in 2^{A \backslash \{x,y\}}$ it holds that:

$$v(T \cup \{x\}) \geq v(T \cup \{y\}), \tag{8}$$

with strict inequality if $T = A \backslash \{x, y\}$. Then it holds that:

$$\begin{aligned}
\phi_x(v) &= \sum_{T:x \notin T} \frac{|T|!(|A|-1-|T|)!}{|A|!}(v(T \cup \{x\}) - v(T)) \\
&= \sum_{T:x,y \notin T} \frac{|T|!(|A|-1-|T|)!}{|A|!}(v(T \cup \{x\}) - v(T)) + \\
&\quad + \sum_{T:x \notin T,\ y \in T} \frac{|T|!(|A|-1-|T|)!}{|A|!}(v(T \cup \{x\}) - v(T)) \\
&> \sum_{T:x,y \notin T} \frac{|T|!(|A|-1-|T|)!}{|A|!}(v(T \cup \{y\}) - v(T)) + \\
&\quad + \sum_{T':x,y \notin T'} \frac{(|T'|+1)!(|A|-|T'|)!}{|A|!}(v(T' \cup \{x,y\}) - v(T' \cup \{y\})) \\
&\geq \sum_{T:x,y \notin T} \frac{|T|!(|A|-1-|T|)!}{|A|!}(v(T \cup \{y\}) - v(T)) + \\
&\quad + \sum_{T:x,y \notin T'} \frac{|T'|+1!(|A|-|T'|)!}{|A|!}(v(T' \cup \{x,y\}) - v(T' \cup \{x\})) \\
&= \sum_{T:y \notin T} \frac{|T|!(|A|-1-|T|)!}{|A|!}(v(T \cup \{y\}) - v(T)) \\
&= \phi_y(v).
\end{aligned}$$

□

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