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MULTI-CRITERIA GAMES**

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Approximate Equilibria for Bayesian Multi-criteria Games¹

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Abstract

In this paper the existence of Bayesian multi-criteria equilibria in mixed strategies is established for finite Bayesian multi-criteria games. Sufficient conditions are given under which approximate equilibria exist for non-finite Bayesian multi-criteria games.

Key words: non-cooperative games, Bayesian multi-criteria equilibria.

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1 Introduction

The theory of strategic games with complete information starts with the works of J. von Neumann [10] and J. Nash [5]. Here the payoff functions assign real numbers to strategy profiles. Extensions to multicriteria games (MC-games for short) are given in Shapley [6] and Borm et al. [1]. The payoff functions here assign to strategy profiles vectors of length equal to the number of criteria of a player. For more details about multi-criteria games the reader can see also [7] and [12].

Harsanyi in [3] introduces games with incomplete information for a player about the real valued payoff functions of the other players.

A natural follow up is taken in this paper where we study interactive situations with incomplete information and where each player may have various objectives.

As far as we know, these Bayesian MC-games are not studied till now. It is easy to imagine practical interactive situations demanding to be modeled as Bayesian-MC-games (BM-games for short).

Natural solution concepts will be that of BM-equilibrium and of approximate BM-equilibrium.

In this paper we first concentrate on the existence of such equilibria in mixed strategies for situations where each player has a finite number of pure strategies, a finite number of criteria and a finite number of types.

Then we consider situations where one of the players may have an infinite set of pure strategies. The work of A. Wald [13] indicates already that it is difficult to obtain general equilibrium results if more than one player has a large strategy space.

The outline of the paper is as follows. In section 2 we introduce the Bayesian Multi-Criteria games (BM-games) and the natural equilibrium concept of Bayesian Multi-Criteria Equilibrium (BM-equilibrium). We prove that in case the type spaces and action spaces are finite there exists a BM-equilibrium in mixed strategies. In section 3 we study BM-games with finite type spaces and all but one finite action spaces. An upper boundedness condition guarantees then the existence of ϵ -BM-equilibria. Section 4 concludes.

2 Bayesian multi-criteria games

A Bayesian multicriteria game (BM-game for short) models a conflict situation with incomplete information where possibly players have multiple objectives. The incomplete information is described with the aid of a type space for each player and a probability distribution on the set of type space profiles. In this paper we restrict our attention to finite type spaces. Further the criteria spaces

for each player will be supposed to be finite. To be more concrete an n -person BM-game is a tuple:

$$\Gamma = \langle N, A_1, A_2, \dots, A_n, T_1, T_2, \dots, T_n, p, C_1, C_2, \dots, C_n, u_1, u_2, \dots, u_n \rangle,$$

for short: $\Gamma = \langle N, A, T, p, C, u \rangle$. Here

- $N = \{1, 2, \dots, n\}$ is the set of players;
- for each $i \in N$ the action space is A_i , and $A = \prod_{i \in N} A_i$, the type space is T_i and C_i is the criteria space;
- p is a probability distribution on the set $T = \prod_{i \in N} T_i$ of type profiles;
- $u_i : C_i \times A \times T \rightarrow \mathbb{R}$ is the payoff function which assigns to player i the payoff $u_i(c_i, a_1, a_2, \dots, a_n, t_1, t_2, \dots, t_n)$ to his criterium c_i given that the players $1, 2, \dots, n$ have type t_1, t_2, \dots, t_n and choose actions a_1, a_2, \dots, a_n respectively.

A play of such a game proceeds as follows: before the types are announced each player i chooses a strategy $x_i \in A_i^{T_i}$ (where then action $x_i(t)$ is chosen if t_i turns out to be the type of player i). Each type profile t results then for player i in a payoff $u_i(c_i, x_1(t_1), \dots, x_n(t_n), t_1, \dots, t_n)$ w.r.t. his criterion c_i .

The a-priori expected payoff for player i w.r.t. criterion $c_i \in C_i$, if the players use strategies x_1, \dots, x_n equals:

$$U_i(c_i, x_1, \dots, x_n) = \sum_{t \in T} p(t) u_i(c_i, x_1(t_1), x_2(t_2), \dots, x_n(t_n), t).$$

Let $\epsilon > 0$.

An ϵ -BM equilibrium of $\langle N, A, T, p, C, u \rangle$ is a strategy profile $(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n) \in A_1^{T_1} \times A_2^{T_2} \times \dots \times A_n^{T_n}$ such that for each $i \in N$, each $c_i \in C_i$, each $x_i \in A_i^{T_i}$:

$$U_i(c_i, \hat{x}_1, \hat{x}_2, \dots, \hat{x}_n) \geq U_i(c_i, \hat{x}_{-i}, x_i) - \epsilon.$$

Here $(\hat{x}_{-i}, x_i) := (\hat{x}_1, \dots, \hat{x}_{i-1}, x_i, \hat{x}_{i+1}, \dots, \hat{x}_n)$, the profile which we obtain when player i deviates from \hat{x}_i to x_i .

Remarks 2.1

(i) If in Γ the criteria spaces are trivial i.e. $|C_1| = |C_2| = \dots = |C_n| = 1$, then we can write $\langle N, A, T, p, u \rangle$ and we obtain a classical Bayesian game (B-game)

and then an approximate Bayesian Pareto equilibrium ($\epsilon - BPE$ for short) boils down to an approximate Bayesian equilibrium ($\epsilon - BE$ for short).

(ii) If in Γ the type spaces are trivial i.e. $|T_1| = |T_2| = \dots |T_n| = 1$ then we can write $\langle N, A, C, u \rangle$ and we obtain a classical multi-objective game (M-game) and an $\epsilon - BPE$ boils down to an $\epsilon - PE$ (approximate Pareto equilibrium).

(iii) If the type spaces and the criteria spaces are trivial then we obtain a classical game in strategic form with complete information (C-game for short).

We will need the notion of *f-mixed extension* of a BM-game for existence results of equilibria.

The letter "f" in *f-mixed extension* stands for finite because we allow only finite mixtures of pure strategies to avoid convergence problems.

Let $\Gamma = \langle N, A_1, A_2, \dots, A_n, T_1, T_2, \dots, T_n, p, C_1, C_2, \dots, C_n, u_1, u_2, \dots, u_n \rangle$.

Then the *f-mixed extension* of Γ is the BM-game

$$\tilde{\Gamma} = \langle N, \tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n, T_1, T_2, \dots, T_n, p, C_1, C_2, \dots, C_n, \tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_n \rangle$$

Here \tilde{A}_i is the family of probability measures (on the σ -algebra of all subsets of A_i) with finite support. Such probability measures are of the form $\mu_i = \sum_{k=1}^s p_k e_{a_k}$ where $a_1, a_2, \dots, a_s \in A_i$, $p_k \geq 0$ for all $k \in \{1, 2, \dots, s\}$ and $\sum_{k=1}^s p_k = 1$, where

$$e_{a_k}(B) = \begin{cases} 1 & \text{if } B \subset A_i, a_k \in B \\ 0 & \text{if } B \subset A_i, a_k \notin B \end{cases}$$

Further $\tilde{u}_i(c, \mu_1, \mu_2, \dots, \mu_n, t) = \int u_i(c, a_1, a_2, \dots, a_n, t) d\mu_1(a_1) d\mu_2(a_2) \dots d\mu_n(a_n)$ for all $i \in N$ and $(\mu_1, \mu_2, \dots, \mu_n) \in \tilde{A} = \prod_{i \in N} \tilde{A}_i$

A BM-game is called a finite game if the action spaces A_1, A_2, \dots, A_n are finite sets. For finite C-games Γ , Nash in [5] proves that the f-mixed extension $\tilde{\Gamma}$ possesses an equilibrium point and for finite B-game Γ , Harsanyi in [3] proves the existence of Bayesian equilibrium for the mixed extension $\tilde{\Gamma}$. In both proofs fixed point theorems play a role applied on the aggregate best response multifunction.

In [1] the existence of Pareto equilibria is proved for mixed extensions of finite M-games by transforming such a game to a C-game with the aid of weight

vectors for each player on the various objectives and then using the classical existence result of J. Nash. We do not know literature where BM-games are studied. But inspired by the proof in [1] it is not difficult to obtain the following existence result.

Theorem 2.2 *Mixed extensions of finite BM-games have a BM-equilibrium.*

Proof Transform with the aid of weight vectors the BM-game into a B-game and apply the existence theorem of Bayesian equilibria of Harsanyi. It is easy to show that each Bayesian equilibrium in the B-game is a Bayesian multi-criteria equilibrium in the BM-game. \square

3 Almost finite BM-games and approximate equilibria

We will call a BM-game

$$\Gamma = \langle N, A_1, A_2, \dots, A_n, T_1, T_2, \dots, T_n, p, C_1, C_2, \dots, C_n, u_1, u_2, \dots, u_n \rangle$$

almost finite if

- (i) $N, A_1, A_2, \dots, A_{n-1}, T_1, T_2, \dots, T_n, C_1, C_2, \dots, C_n$ are finite sets and A_n is infinite,
- ii) $u_n : C_n \times A \times T \rightarrow \mathbb{R}$ is an upper bounded function.

Let $\epsilon > 0$. We are interested in the existence of ϵ -BM equilibria for $\tilde{\Gamma}$. Our main result in this section is:

Theorem 3.1 *Let Γ be an almost finite game. Then for each $\epsilon > 0$ there is an ϵ -BM equilibrium for $\tilde{\Gamma}$.*

Proof (i) Take $\epsilon > 0$. The proof is based on the following claim which we prove at the end of this section.

CLAIM. Given Γ , there is a finite BM-subgame $\Gamma(\epsilon)$ of Γ with

$$\Gamma(\epsilon) = \langle N, A_1, \dots, A_{n-1}, A_n(\epsilon), T_1, T_2, \dots, T_n, p, C_1, C_2, \dots, C_n, u_1, \dots, u_n \rangle$$

and $A_n(\epsilon)$ is a finite subset of A_n such that for each $a_n \in A_n$ there is an $a_n(\epsilon) \in A_n(\epsilon)$ such that for all $c_n \in C_n, t \in T, a_{-n} \in A_{-n}$ we have

$$u_n(c_n, (a_{-n}, a_n(\epsilon)), t) \geq u_n(c_n, (a_{-n}, a_n), t) - \epsilon \tag{3.1}$$

[So $a_n(\epsilon)$ is ϵ -almost as good as a_n for player n].
We call $\Gamma(\epsilon)$ an ϵ -approximation of Γ .

(ii) Using the above claim we prove that the f -mixed extension $\tilde{\Gamma}(\epsilon)$ of $\Gamma(\epsilon)$ is an ϵ -approximation of $\tilde{\Gamma}$ i.e. for each $\nu = (\nu_{t_n})_{t_n \in T_n} \in \tilde{A}_n^{T_n}$ there is a $\nu^\epsilon = (\nu_{t_n}^\epsilon)_{t_n \in T_n} \in (\tilde{A}_n(\epsilon))^{T_n}$ such that for all $c_n \in C_n$, $t \in T$, $\mu_{-n} \in \tilde{A}_{-n}$, we have

$$\tilde{u}_n(c_n, \mu_{-n}, \nu_{t_n}^\epsilon, t) \geq \tilde{u}_n(c_n, \mu_{-n}, \nu_{t_n}, t) - \epsilon \quad (3.2)$$

For the proof note that the CLAIM implies that we can define a selection

$\beta : A_n \rightarrow A_n(\epsilon)$ such that $\beta(a) = a$ for each $a \in A_n(\epsilon)$ and

$$u_n(c_n, (a_{-n}, \beta(a_n)), t) \geq u_n(c_n, (a_{-n}, a_n), t) - \epsilon \quad (3.3)$$

for each $c_n \in C_n$, $t \in T$, $a_{-n} \in A_{-n}$, $a_n \in A_n \setminus A_n(\epsilon)$. Given $\nu \in \tilde{A}_n^{T_n}$ and $t \in T$, ν_t is a finite sum of the form

$$\sum_{a \in A_n} \nu_{t_n}(\{a\}) e_a.$$

Take

$$\nu_{t_n}^\epsilon = \sum_{b \in A_n(\epsilon)} \left(\sum_{a \in A_n} \{\nu(a) : \beta(a) = b\} \right) e_b.$$

Then $(\nu_{t_n}^\epsilon)_{t_n \in T_n} \in (\tilde{A}_n(\epsilon))^{T_n}$, and (3.2) follows from (3.3).

(iii) By theorem 2.2 we can find a mixed BM-equilibrium

$(\hat{\nu}_1, \hat{\nu}_2, \dots, \hat{\nu}_n) \in \tilde{A}_1^{T_1} \times \dots \times \tilde{A}_{n-1}^{T_{n-1}} \times \dots \times \tilde{A}_n(\epsilon)^{T_n}$ of the mixed extension $\tilde{\Gamma}(\epsilon)$ of the finite game $\Gamma(\epsilon)$.

Define for each $\mu \in \tilde{A}_n(\epsilon)$ the mixed strategy $\alpha(\mu) \in \tilde{A}_n$ by

$$\alpha(\mu)(C) = \mu(C \cap A_n(\epsilon)) \text{ for each finite } C \subset A_n.$$

We prove that $(\hat{\nu}_1, \hat{\nu}_2, \dots, \hat{\nu}_{(n-1)}, (\alpha((\hat{\nu}_n)_{t_n}))_{t_n \in T_n})$ is an ϵ -BM equilibrium of Γ .

First, note that for each $i \in N \setminus \{n\}$, $c_i \in C_i$, $\mu_i \in \tilde{A}_i$ we have

$$\begin{aligned}
U_i(c_i, (\hat{\nu}_k)_{k \in N \setminus \{i, n\}}, \mu_i, (\alpha((\hat{\nu}_n)_{t_n}))_{t_n \in T_n}) &= \sum_{t \in T} p(t) \tilde{u}_i(c_i, (\hat{\nu}_k(t))_{k \in N \setminus \{i, n\}}, \mu_i(t), \alpha((\hat{\nu}_n)_{t_n}, t)) = \\
&= \sum_{t \in T} p(t) \tilde{u}_i(c_i, (\hat{\nu}_k(t))_{k \in N \setminus \{i, n\}}, \mu_i(t), \hat{\nu}_n(t), t) = \\
&= U_i(c_i, (\hat{\nu}_k)_{k \in N \setminus \{i, n\}}, \mu_i, \hat{\nu}_n) \leq U_i(c_i, (\hat{\nu}_k)_{k \in N}),
\end{aligned}$$

where the last inequality follows from the fact that $\hat{\nu}$ is a mixed

BM-equilibrium of $\tilde{\Gamma}(\epsilon)$. So deviation from $\hat{\nu}_i$ to μ_i does not pay for player $i \in N \setminus \{n\}$.

Secondly, note that for player $i = n$ deviation from $\alpha((\hat{\nu}_n)_t)_{t \in T}$ to $\mu_n \in \tilde{A}_n$ pays at most ϵ because

$$\begin{aligned}
U_n(c_n, (\hat{\nu}_k)_{k \in N \setminus \{n\}}, \mu_n) &= \sum_{t \in T} p(t) \tilde{u}_n(c_n, (\hat{\nu}_k(t))_{k \in N \setminus \{n\}}, \mu_n(t), t) \leq \\
&\leq \sum_{t \in T} p(t) \tilde{u}_n(c_n, (\hat{\nu}_k(t))_{k \in N \setminus \{n\}}, (\mu_n^\epsilon(t))_{t \in T}, t) + \epsilon \leq \\
&\leq \sum_{t \in T} p(t) \tilde{u}_n(c_n, (\hat{\nu}_k(t))_{k \in N}, t) + \epsilon = \\
&= U_n(c_n, (\hat{\nu}_k)_{k \in N \setminus \{n\}}, \alpha((\hat{\nu}_n)_t)_{t \in T}) + \epsilon
\end{aligned}$$

where the first inequality follows from (3.2) and the second inequality from the fact that $(\hat{\nu}_1, \hat{\nu}_2, \dots, \hat{\nu}_{n-1}, \hat{\nu}_n)$ is an ϵ -BM equilibrium of $\tilde{\Gamma}(\epsilon)$. \square

It remains to prove the CLAIM in the proof of theorem 3.1. For this objective the following two theorems are helpful.

A proof of theorem 3.2 can be found in [9].

Theorem 3.2 (Finite covering property by orthants for upper bounded sets in an Euclidean space).

For each upper bounded set V in \mathbb{R}^n and each $\epsilon > 0$, there is a finite subset W of V such that

$$V \subset \bigcup \{O(w, \epsilon) \text{ s.t. } w \in W\}$$

where $O(w, \epsilon)$ is the orthant

$\{x \in \mathbb{R}^m \text{ s. t. } x_i \leq w_i + \epsilon \quad \forall i \in \{1, 2, \dots, m\}\}$. □

A direct consequence of this theorem is the next theorem 3.3, which one can also find in [8] as lemma 4.3.

For convenience of the reader we give also the proof.

Theorem 3.3 *Let E be a finite set, $\epsilon > 0$ and let \mathcal{F} be an upper bounded family of real valued function on E . Then there exists a finite subfamily \mathcal{G} of \mathcal{F} which ϵ -dominates \mathcal{F} i.e.*

$$\forall f \in \mathcal{F} \exists g \in \mathcal{G} : \forall x \in E [f(x) \leq g(x) + \epsilon]$$

Proof. Let $E = \{a_1, a_2, \dots, a_m\}$. For each $f \in \mathcal{F}$ let $\alpha(f)$ be the vector $(f(a_1), f(a_2), \dots, f(a_m))$ in \mathbb{R}^m . Since \mathcal{F} is an upper bounded family of functions, the set $V = \{\alpha(f) : f \in \mathcal{F}\}$ is an upper bounded subset of \mathbb{R}^m . In view of theorem 3.2 we can find a finite subset W of V , which ϵ dominates V . But then $\mathcal{G} = \{f \in \mathcal{F} : \alpha(f) \in W\}$ is a finite subfamily of \mathcal{F} which ϵ -dominates \mathcal{F} . □

Now we are able to give the

Proof of the CLAIM in theorem 3.1. Given the almost finite game Γ , we consider the upper bounded family

$\mathcal{F} = \{\hat{a}_n : E \rightarrow \mathbb{R} : a_n \in A\}$ of functions on the finite set

$E = C^n \times A^{-n} \times T$, where \hat{a}_n is defined by

$\hat{a}_n(c_n, a_{-n}, t) = u_n(c_n, (a_{-n}, a_n), t)$ for each $(c_n, a_{-n}, t) \in E$.

According to theorem 3.3 there is a finite subfamily \mathcal{G} of \mathcal{F} such that \mathcal{G} ϵ -dominates \mathcal{F} .

Note that \mathcal{G} is of the form $\{\hat{b}_n : E \rightarrow \mathbb{R} : b_n \in A_n(\epsilon)\}$ where $A_n(\epsilon)$ is a suitable finite subset of A_n . The ϵ -dominance of \mathcal{F} by \mathcal{G} implies that for all $(c, a_{-n}, t) \in E$ we have:

$$\begin{aligned} \hat{b}_n(c, a_{-n}, t) &\geq \hat{a}_n(c, a_{-n}, t) - \epsilon \quad \text{or} \\ u_n(c, (a_{-n}, b_n), t) &\geq u_n(c, (a_{-n}, a_n), t) - \epsilon \end{aligned} \tag{3.4}$$

Let $\Gamma(\epsilon)$ be the finite game with

$\Gamma(\epsilon) = \langle N, A_1, A_2, \dots, A_{n-1}, A_n(\epsilon), T_1, T_2, \dots, T_n, p, C_1, C_2, \dots, C_n, u_1, u_2, \dots, u_n \rangle$.

By taking $a_n(\epsilon) = b_n$ in (3.4) we obtain (3.1). □

4 Concluding remarks

In theorem 2.2 we established the existence of BM-equilibria in mixed strategies for finite BM-games. Using the covering theorem and then the approximation with finite games, the existence of ϵ -BM equilibria is established for semi-infinite BM-games with upper bounded payoff functions.

A topic for further research could be the study of BM-games where all (or all but one) strategy spaces are compact sets and the payoff functions satisfy suitable continuity and concavity properties guaranteeing BM-equilibria (or ϵ -BM-equilibria).

For the subclass of strategic games such a research was done in [8].

Another topic for further research could be the study of BM-games with a potential ([4], [2], [11]).

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