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CHAOTIC PLANNING SOLUTIONS IN THE TEXTBOOK MODEL OF LABOR MARKET SEARCH AND MATCHING

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Chaotic Planning Solutions in the Textbook Model of Labor Market Search and Matching*

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Abstract

This paper demonstrates that cyclical and chaotic planning solutions are possible in the standard "textbook model" of search and matching in labor markets. More specifically, it takes a discrete-time adaptation of the continuous-time matching economy described in Pissarides (1990, 2001), and computes the solution to the dynamic planning problem. The solution is shown to be completely characterized by a first-order, non-linear map with a unique stationary solution. Additionally, the existence of a large number of periodic and even *aperiodic* non-stationary solutions is shown. Even when the well-known Li-Yorke and three-period cycle conditions for chaos are violated, we are able to verify the new Mitra (2001) sufficient condition for topological chaos. The implication is that even in a simple economy characterized by search and matching frictions, an omniscient social planner may have to contend with a fairly robust and "bewildering" variety of possible dynamic paths.

JEL Classification: J64
Keywords: search, chaos, cycles

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1 Introduction

This paper takes the planning solution to the standard "textbook model" of search and matching in labor markets, and shows that chaotic behavior is possible in that framework. More specifically, it takes as a starting point, Ljungqvist and Sargent's (2000; Chapter 19) discrete-time adaptation of the continuous-time matching economy described in Pissarides (1990, 2001) and computes the solution to the dynamic planning problem. The solution is completely characterized by a first-order, non-linear scalar difference equation. There is a unique stationary solution as was shown by Ljungqvist and Sargent (2000). The main contribution of this paper is to show that, additionally, there are a large number of periodic and even aperiodic dynamical solutions that may exist.

The implication is strong and clear: in an economy characterized by search and matching frictions, even an omniscient social planner may have to contend with a "bewildering" (in the prose of Azariadis, 1993) variety of possible dynamic solutions. In a sense, this result is reminiscent of the characterization of chaotic planning solutions to the Ramsey growth model as enunciated in Boldrin and Montruchhio (1986), and more recently in Mitra, Majumdar, and Nishimura (2000), and further exposited in Mitra and Nishimura (2001).[1] There is however one main difference. In the standard aggregative optimal growth model, the decentralized equilibrium is efficient; in the Pissarides model, the decentralized equilibrium is generically *not* optimal.

Several papers in the literature have investigated the possibility of endogenous cycles in search models of the labor market. The seminal papers in this area are Drazen (1988) and Diamond and Fudenberg (1989); both build on Diamond (1982) and prove the existence of stable limit cycles in a model where there are frictions in coordinating trade, and the matching technology is subject to increasing returns. More recently, Mortensen (1999) revisits the standard textbook model of search and matching in the labor market as described in Pissarides (1990) but introduces an increasing returns to scale production technology to generate multiple long-run unemployment equilibria and stable limit cycles.[2] Shimer and Smith (2001) explore optimal matching policies in constant returns to scale search economies with *heterogenous* agents and find the possibility of non-stationarity.

The current endeavour is different from the previous literature in three important ways. First, the focus here (as in Shimer and Smith, 2001) is on the planning solution as opposed to the "decentralized"

[1] Here, as in the exposition of Mitra and Nishimura (2001), periodic and aperiodic behavior is not an outcome of agents' expectations about the actual realization of a certain random variable. For a insightful treatment of such a expectations-driven model of cycles, see Hommes and Sorger (1998), and the discussion in Dechert and Hommes (2000).

[2] Mortensen (1999) assumes that match productivity is an increasing function of the aggregate number of matches. This generates the needed increasing returns in production.

solution (the focus of the other aforementioned papers). In fact, typically in these papers, the market solution is representable by a *system* of *differential* equations, rather than a single first-order non-linear difference equation as is the case here. Second, neither the production nor the matching technologies in our model exhibit any increasing returns. Finally, unlike the continuous-time framework used by Mortensen (1999) and Shimer and Smith (2001), we use the discrete-time adaptation.[3] In a sense, our results suggest that merely delegating the job of coordinating labor market search activity to a planner may not necessarily render an economy immune to endogenous fluctuations; in fact, a planner may introduce fluctuations in an economy that was decentralized and otherwise possibly immune to cyclical variations!

The plan for the rest of the paper is as follows. Section 2 outlines the Pissarides (1990) model of search and matching in labor markets. Section 3 contains a detailed analysis of the main difference equation alluded to earlier. It establishes both analytically and through examples, that following either the Li-Yorke route or the three-period cycles route, or the Mitra condition route, it is possible to demonstrate the existence of topological chaos. Section 4 concludes. Proofs of some central results are to be found in the appendices.

2 The Model

The model (and the notation) is based entirely on Ljungqvist and Sargent's (2000) discrete-time adaptation of the continuous-time matching economy described in Pissarides (1990, 2001). Let $t = 0, 1, 2, 3, ...$ index time. There are two types of agents: workers and firms. There is a continuum of identical workers of unit measure. These workers are all infinitely-lived, they discount the future at the rate β, and are risk-neutral. Workers potentially get matched with a firm; the result of such a match is output y.[4] Each firm may employ at most one worker. A firm incurs a vacancy cost of c in each period when looking for a worker. A match between a worker and a firm gets dissolved with an exogenously-specified probability s. An unmatched worker is an unemployed worker; such a worker enjoys the current utility from leisure of amount z.

Matches are brought together by a standard matching technology connecting only unemployed job

[3]The discrete-time version of the Pissarides (1990) search-and-matching story has also been employed by Merz (1995), Andolfatto (1996), Shi and Quan (1999), Cooley and Quadrini (1999), Cole and Rogerson (1999), Yuan and Li (2000), and Yashiv (2000), among others.

[4]In the decentralized equilibria, the match surplus is divided between the worker and the firm according to some bargaining protocol. Below we assume that the planner cares only about the match output y, and not the division of the match surplus.

seekers with open vacancies. The number of successful matches in a period is given by $M(u_t, v_t)$ where u_t is the total measure of unemployed workers looking for jobs, and v_t is the number of vacancies or firms looking for employees. The matching function is increasing in both arguments, concave, and homogenous of degree one. Let $\theta_t \equiv v_t/u_t$ indicate the measure of labor market tightness, or the ratio of vacancies to unemployed workers. Then define $q(\theta_t) \equiv M(u_t, v_t)/v_t$ as the probability of a vacancy being filled at date t. For all that we present below, we will assume a standard constant returns to scale formulation,

$$M(u_t, v_t) = Au_t^{\alpha} v_t^{1-\alpha} \qquad A > 0, \quad \alpha \in (0,1) \tag{1}$$

where A is a scale parameter. The parameter α is the elasticity of the matching function with respect to the measure of unemployed workers. It follows that $q(\theta) = A\theta^{-\alpha}$. Finally, define n_{t+1} as the total number of employed workers at the start of $t+1$. Then, it follows that

$$n_{t+1} = (1-s)n_t + q(\theta_t) \cdot v_t, \text{ where } \theta_t \equiv \frac{v_t}{u_t} = \frac{v_t}{1-n_t}. \tag{2}$$

The number of undissolved matches (which were formed at the start of t that survived onto the start of $t+1$) is given by $(1-s)n_t$. The term $q\left(\frac{v_t}{1-n_t}\right) \cdot v_t$ measures the number of new matches formed at t between the unemployed workers $(1-n_t)$, and the vacancies created at t.

A planner's problem could then be outlined as follows. Assume that the planner chooses an allocation that maximizes the discounted value of output and leisure net of vacancy costs. The principal tensions are as follows. An extra vacancy adds a cost, makes it easier for unemployed workers to find jobs, but makes it harder for firms to find workers. Employed workers "lose" leisure utility. More output is produced if the extra vacancy creates more matches. The planner takes all this into account when choosing the number of vacancies. Following Ljungqvist and Sargent (2000; p. 578), the planner chooses v_t and next period's employment level, n_{t+1} by solving (P) where (P) is defined by

$$\max_{\{v_t, n_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[yn_t + z(1-n_t) - cv_t \right] \tag{P}$$

subject to (2), given a n_0. The Lagrangian can be written as

$$\pounds = \sum_{t=0}^{\infty} \left\{ \beta^t \left[yn_t + z(1-n_t) - cv_t \right] + \lambda_t \left[(1-s)n_t + q\left(\frac{v_t}{1-n_t}\right) \cdot v_t - n_{t+1} \right] \right\}$$

where λ_t is the Lagrange multiplier on (2). Then, the first-order conditions with respect to v_t and n_{t+1} for an interior solution are given by

$$-\beta^t c + \lambda_t \left[q'(\theta_t)\,\theta_t + q(\theta_t) \right] = 0 \tag{3}$$

and

$$-\lambda_t + \beta^{t+1}(y-z) + \lambda_{t+1}\left[(1-s) + q'(\theta_{t+1})\,\theta_{t+1}^2\right] = 0 \tag{4}$$

In Appendix A, we show that (3) and (4) reduce to the following first-order difference equation in θ :

$$a\theta_{t+1}^{\alpha} - b\theta_{t+1} = \theta_t^{\alpha} - d \tag{5}$$

where

$$a \equiv \beta(1-s) \in (0,1), \tag{6}$$

$$b \equiv A\alpha\beta > 0, \tag{7}$$

and

$$d \equiv \frac{A(1-\alpha)\beta\,(y-z)}{c} > 0. \tag{8}$$

Equation (5) is the law of motion for the index of labor market tightness in the economy under the planner's solution. For future reference, define

$$\gamma \equiv \frac{(y-z)}{c},$$

the ratio of per worker match output (net of lost leisure) to hiring and vacancy posting costs.

Given an initial n_0, eq. (5) completely characterizes the trajectory of θ.[5] In other words, the backwards dynamics of this model can be characterized by the continuous four-parameter family of maps $g : [0, \theta_{\max}] \to [0, g_{\max}]$, where

$$g(\theta) = (a\theta^{\alpha} - b\theta + d)^{\frac{1}{\alpha}}, \qquad (\alpha, a, b, d) \in (0,1) \times (0,1) \times (0,\infty) \times (0,\infty), \tag{9}$$

and $\theta_{\max} = \left(\frac{\alpha a}{b}\right)^{\frac{1}{1-\alpha}}$ and $g_{\max}$ is implicitly defined as the lowest positive root of the following equation:

$$ag_{\max}^{\alpha} - bg_{\max} + d = 0.$$

The first derivative of the map can be calculated as

$$g'(\theta) = (a\theta^{\alpha} - b\theta + d)^{\frac{1-\alpha}{\alpha}}\left(a\theta^{\alpha-1} - \frac{b}{\alpha}\right), \quad \theta \in [0, \theta_{\max}],$$

which implies that g is unimodal with a unique maximum at $\theta_{\max} \equiv \left(\frac{b}{a\alpha}\right)^{\frac{1}{\alpha-1}}$. Note however that sufficiently high values of α make it impossible for $g'(.)$ to ever become negative, the latter being a

[5] Ljungqvist and Sargent (2000; p. 578) assume that the planner knows n_0. From the first order conditions to (P), it is possible to compute v_0. Since $u_0 = 1 - n_0$ is known, then θ_0 becomes known. Using eq. (5), the optimal solution sequence for θ, and (2), the definition of the employment level at various dates, it is then possible to compute the optimal sequences $\{v_t\}_{t=0}^{\infty}$, $\{u_t\}_{t=1}^{\infty}$, and $\{n_t\}_{t=1}^{\infty}$.

necessary condition for the map g to exhibit any periodic behavior.[6] In addition, g has a unique fixed point located to the right of $\theta_{\max}$ if $g(\theta_{\max}) > \theta_{\max}$. This last condition simplifies to the following parametric restriction

$$d > \left(\frac{\alpha a}{b}\right)^{\frac{\alpha}{1-\alpha}} [1 - a + a\alpha]. \tag{10}$$

Henceforth we will maintain this as an assumption on the parameters.

The unique fixed point of g is *implicitly* given by

$$a\theta_{ss}^{\alpha} - b\theta_{ss} = \theta_{ss}^{\alpha} - d.$$

The fixed point is an attractor in the forward dynamics, if $g'(\theta_{ss}) < -1$. Since it is not possible to obtain a closed form expression for θ_{ss}, this condition cannot be checked in general, but will have to be verified for each set of parameters separately. As will be evident shortly, the fact that θ_{ss} is implicitly defined, makes it a non-trivial problem to check the standard sufficient conditions for complex dynamics.

3 Cycles and Complex Dynamics

3.1 Two Period Cycles

Before proceeding to establish the possibility of chaotic behavior for the map g, we undertake a quick study of the existence of two period cycles. This is useful because an analysis of the underlying economic intuition driving any kind of periodic behavior is best undertaken via a study of two-period cycles. For future reference, let us formally define a two-period cycle. Let (X, g) be a dynamical system where X is a subset of $\Re$ and let $g^2(m)$ denote the second iterate of the point m. A two-period cycle is a periodic point m of order 2 if $g^2(m) = m$ (and m is such that $g(m) \neq m$). We now present a numerical example of a two period cycle, and use it to discuss the economic intuition behind the fluctuations in the various variables.

Example 1 *Suppose the set of parameters is given by* [7]

A	α	β	s	γ
2.40	0.348	0.99	0.0348	0.423

. Then, the map g admits a stationary solution, $\theta_{ss} = 0.743$, *and a two period cycle with* $\theta_1 = 0.724$ *and* $\theta_2 = 0.761$.

[6] The condition, $g'(\theta) > 0$ for all θ, is sufficient to rule out any kind of periodic behavior in our setup. See Azariadis (1993; Chapter 8) for details. This condition may be thought of as the unidimensional, discrete-time analog, of Bendixson's criterion which is a sufficient condition to rule out cyclical behavior in two-dimensional continuous time systems. See Lorenz (1993; Chapter 2) and Mortensen (1999).

[7] These correspond to $a = 0.956$, $b = 0.827$, and $d = 0.654$.

The following set of figures (Figure 1) demonstrate the simultaneous movement of the central economic variables in the two period cycle, compared with the steady state. Recall that the timing of the model is such that, faced with the employment rate n_t (or the unemployment rate $u_t = 1 - n_t$), the planner selects how many vacancies v_t to post. Since u_t is already known, the number of vacancies also directly determines the labor market tightness, θ_t and, together with the matching function, the tightness determines the (un)employment rate next period.

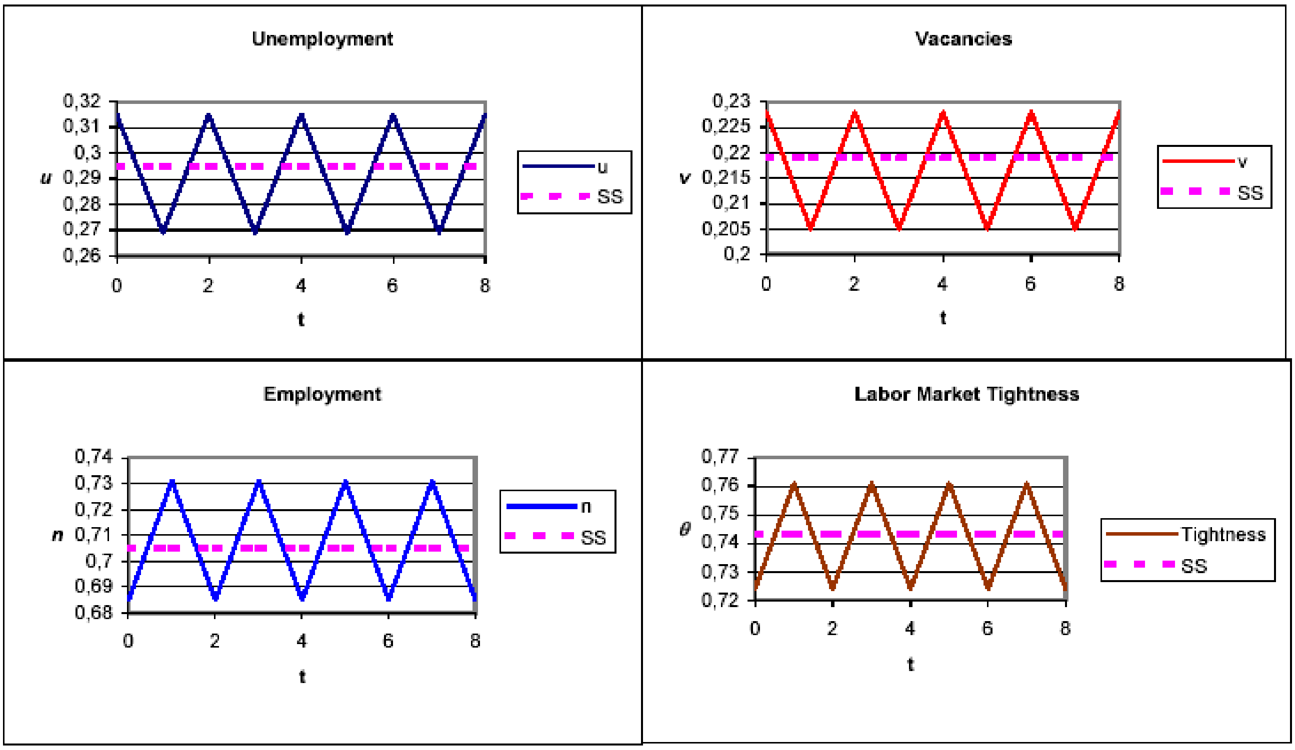

Figure 1: The two-period cycle in Example 1

In the illustration of Example 1, the difference in the unemployment level between the high and the low unemployment periods varies by about five percentage points. Since there is no variation in the size of the working population, the movements in employment are exactly the opposite of the movements in the unemployment rate. In periods with low unemployment, a low number of vacancies are posted, but in spite of this, the overall labor market tightness is still above the level of labor market tightness in high unemployment periods. This is driven by the low number of job seekers in the low unemployment periods.

To understand the economic forces driving the two period cycle, recall that in periods when unemployment is high, the marginal benefit of posting a vacancy is high, because there are more unemployed workers to potentially match with the vacancy. The constant returns to scale matching function ensures this. This high marginal benefit implies that it will be optimal for the planner to create a high number of vacancies. This action, however, will increase the number of matches, causing the unemployment rate to be lower in the next period. This lower level of unemployment maps onto a reduced marginal benefit of posting a vacancy, as there are fewer job seekers available to match with the vacancy. Since the cost of creating vacancies is a constant, the implication of this lower marginal benefit to vacancy posting is that the optimal number of new vacancies must be reduced. This will then bring down the number of matches, causing higher unemployment in the following period.

3.2 Li-Yorke route

We now investigate the possibility for the time-map g to exhibit complex dynamics, i.e., periodic, aperiodic, and chaotic behavior. The strategy will be to write down a set of conditions under which the Li-Yorke "overshooting" inequalities hold. We start by restating the Li-Yorke theorem (as stated in Benhabib and Day, 1981).

Theorem 1 *Li-Yorke (1975) Let J be an interval in $\Re$ and $\theta_{t+1} = g(\theta_t)$ be a difference equation in which g is a continuous mapping of $J \to J$. Suppose there exists a point $\theta \in J$ such that*

$$g^3(\theta) \leq \theta < g(\theta) < g^2(\theta). \tag{11}$$

Then,
a) for every $k = 1, 2, 3, ...$ there is a $k-$ periodic solution of $\theta_{t+1} = g(\theta_t)$ in J; and
b) there is an uncountable set $S \in J$, which contains no periodic points, such that for every initial condition in S, the solution of $\theta_{t+1} = g(\theta_t)$ is aperiodic, and remains in S.

Below, we write down general conditions on parameters under which these Li-Yorke inequalities are satisfied for our map g defined in (9). Recall $\theta_{\max} \equiv \left(\frac{\alpha a}{b}\right)^{\frac{1}{1-\alpha}}$.

Theorem 2 *Suppose the following three parametric conditions hold:*

$$d < \theta_{\max}^{\alpha}, \tag{12}$$

$$a\left(a\theta_{\max}^{\alpha} - b\theta_{\max} + d\right) - b\left(a\theta_{\max}^{\alpha} - b\theta_{\max} + d\right)^{\frac{1}{\alpha}} + d < \theta_{\max}^{\alpha}, \tag{13}$$

and

$$\begin{aligned} &\left(a^3 - 1\right)\theta_{\max}^{\alpha} - a^2 b\theta_{\max} - ab\left(a\theta_{\max}^{\alpha} - b\theta_{\max} + d\right)^{\frac{1}{\alpha}} + d\left(1 + a + a^2\right) \\ < \; & b\left(a^2\theta_{\max}^{\alpha} - ab\theta_{\max} + (1+a)\,d - b\left(a\theta_{\max}^{\alpha} - b\theta_{\max} + d\right)^{\frac{1}{\alpha}}\right)^{\frac{1}{\alpha}}. \end{aligned} \tag{14}$$

Then the condition (11) of the Li-Yorke Theorem is satisfied.

The set of three conditions constitute merely a set of *sufficient* conditions. A quick look at condition (11) of the Li-Yorke Theorem suggests that a natural choice for the first iterate of θ, i.e., $g(\theta)$ is $\theta_{\max}$. This is because it is known that if θ is defined such that $g(\theta) = \theta_{\max}$, then $0 < \theta < g(\theta) < g^2(\theta)$ will always hold as long as (10) and (12) are satisfied. Then all that would remain would be to compare $g^3(\theta)$ with θ. Assumption (13) corresponds to $g^2(\theta_{\max}) < \theta_{\max}$, which for unimodal maps is a *necessary* condition for chaos and hence is an assumption which always has to be made in some form whichever route is chosen to prove the existence of chaos; see Mitra (2001) for further details. Assumption (13) turns out to be especially useful for us because we have chosen θ such that $g(\theta) = \theta_{\max}$. In that case, Assumption (13) simply states that $g^3(\theta) < g(\theta)$. This last condition is clearly necessary, but not sufficient, to show that $g^3(\theta) < \theta$. In our case, even if it were straightforward to compute $g^3(\theta)$, it is not possible to directly compare it with θ, as the equation $g(\theta) = \theta_{\max}$ does not have an explicit solution. This is why condition (14) required. In Appendix B, we prove that this final condition ensures that $g^3(\theta) < \theta$ holds.

While these conditions appear somewhat complicated, they are simple to numerically verify for any given set of parameters. We now provide slightly simpler conditions in the special case of $\theta_{\max} = 1$. This does greatly simplify the calculations, but the final conditions are still non-trivially daunting.

Corollary 1 *Suppose the following three parametric conditions hold:*

$$\frac{\alpha a}{b} = 1,$$

$$a(1-\alpha)(a + a\gamma + \gamma) - \alpha(1-\alpha)^{\frac{1}{\alpha}} a^{1+\frac{1}{\alpha}} (1+\gamma)^{\frac{1}{\alpha}} < 1,$$

and

$$a\left(a(1-\alpha)(a + a\gamma + \gamma) - \alpha(1-\alpha)^{\frac{1}{\alpha}} a^{1+\frac{1}{\alpha}} (1+\gamma)^{\frac{1}{\alpha}}\right)^{\frac{1}{\alpha}}$$
$$-\alpha a\left(a(1-\alpha)(a + a\gamma + \gamma) - \alpha(1-\alpha)^{\frac{1}{\alpha}} a^{1+\frac{1}{\alpha}} (1+\gamma)^{\frac{1}{\alpha}}\right) < 1 - a(1-\alpha)\gamma.$$

Then condition (11) of the Li-Yorke Theorem is satisfied.

We now proceed to provide an example of a set of parameters that satisfy the conditions of the Li-Yorke inequalities as stated in (11).

Example 2 *The set of parameters* [8]

A	α	β	s	γ
0.2536636	0.2251169	0.99	0.004390207	7.610282

are associated with $\theta_{\max}^{\alpha} = 1.487710$ *implying that condition (12) is satisfied, the left hand side of (13) is*

[8] These correspond to $a = 0.9856537$, $b = 0.05653293$, and $d = 1.480915$.

$0.001971010 < \theta^{\alpha}_{\max} = 1.48771$, *implying that condition (13) is satisfied, and the left and right hand sides of (14) are* -0.004851456 *and* 0.431773×10^{-14} *respectively, implying that the final condition of Theorem (2), (14), is satisfied.*

Remarks: A few remarks about the realism of these numbers is in order. To begin with, A is just a scale variable; Blanchard and Diamond (1989) and many others, use $A \approx 0.4$. The use of $\beta = 0.99$ is standard in the literature and follows Garibaldi and Wasmer (2001) who report using $s = 0.02$, while Fonseca and Muñoz (1999) and many others, depending on the frequency of the data used, use a smaller number for s. Yashiv (2000) reports that the mean value of c lies between 12-22% of average match output while our choice of γ is consistent with c near 13% of match output (net of leisure). Yashiv (2000) suggests that, at least during 1975-79, estimates of α for the United States ranged between 0.2-0.25, even though it was somewhat higher during other periods. Van Ours (1995) reports, using annual Dutch data, that α is near 0.27. Mumford and Smith (1999) using Australian gross flows data report estimates of α near 0.28 while Anderson and Burgess (2001) find it to be also near 0.3 using annual (panel) U.S data. While some of the reported estimates of α in Petrongolo and Pissarides (2001) are quite a bit higher than ours, there are some estimates, like the one using Israeli data which place α near 0.29, the one using English and Wales data that report an α near 0.3, and even one using Spanish data that find α to be near 0.12. It is well-known that these estimates of α depend crucially on the frequency of the data, whether search intensity is modeled, on the definitions of the terms "unemployed" and "job seekers" among other factors (see Mumford and Smith (1999) for details).[9]

3.3 Three-Period Cycles

As is well-known, a sufficient condition for chaotic behavior is the existence of a three-period cycle, i.e., the existence of a point θ, different from the steady state, satisfying $\theta = g^3(\theta)$. This follows from the fact that a three-period cycle satisfies (11) of the Li-Yorke Theorem with the first " $\leq$ " holding with equality. One simple way of providing a graphical representation of three-period cycles is to graph g^3. The map g^3 will naturally cross the 45-degree line at the steady state. In addition each three-period cycle will generate three additional intersections, as each of the points in the three-period cycle, by definition, is a fixed point of the function g^3.[10] Below, we present two examples of three-period cycles.

[9] Recall that sufficiently high values of α make the map $g(.)$ a monotonic function of θ, thereby ruling out the possibility of any kind of cyclical behavior. It is however possible to generate two-period cycles using a value of $\alpha \approx 0.45$. Specifically, set $a = 0.990$, $b = 0.656$, $d = 1.19$, and $\alpha = 0.454$. This corresponds to $s = 0.000158$, $A = 1.46$, $\beta = 0.99$, and $\gamma = 1.52$. Then θ cycles between 1.82 and 1.79. Similar high even-period cycles are easy to generate with $\alpha > 0.4$.

[10] See Devaney (1986, Fig 13.1) for additional information on this type of graphical depiction.

Example 3 *Let the set of parameters be defined by* $A = 7.629, \alpha = 0.1241$, $s = 0.055$, *and* $\gamma = 0.06$. *The* g *locus in this case has the shape illustrated in Figure 1. There is a unique steady state* $\theta_{ss} = 0.367$, *and* $g'(\theta_{ss}) = -2.21$, *indicating that* θ_{ss} *is locally unstable in the backward dynamics and stable in the normal forward dynamics. There is a 3-cycle that starts from 0.0009965, goes to 0.1625529 to 0.9704193, and returns to 0.0009963.* [11]

Figures 2 and 3 below provide graphical representations of this cycle.

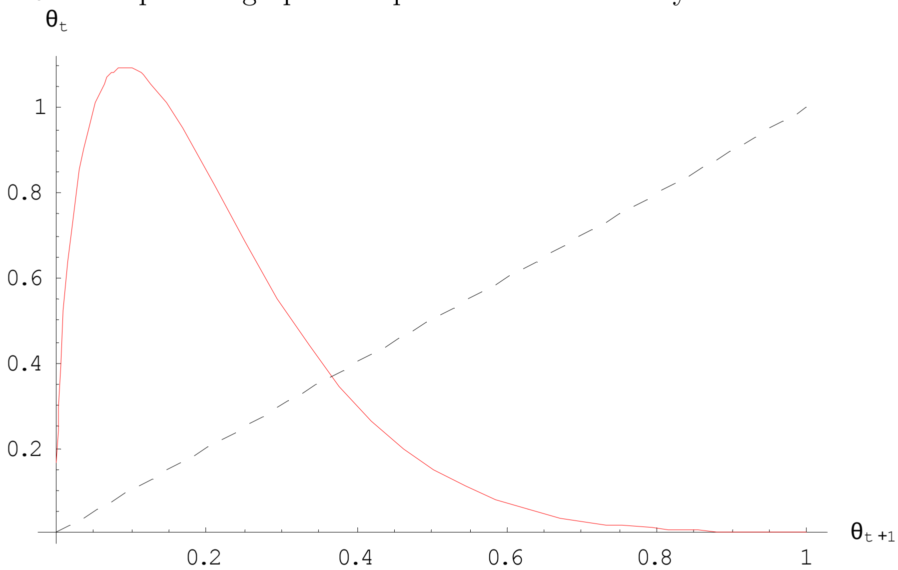

Figure 2: The map $g(.)$ for Example 3

[11] The deeper parameters correspond to $a = 0.9348154$, $b = 0.9377004$, and $d = 0.4026384$. Lorenz (1993, Appendix A.4) discusses how the "standard floating-point arithmetic" on computers that considers only a finite number of digits and truncates the rest is responsible for the divergence in the last (seventh) digit of the starting and ending point of the three-cycle in Example 3.

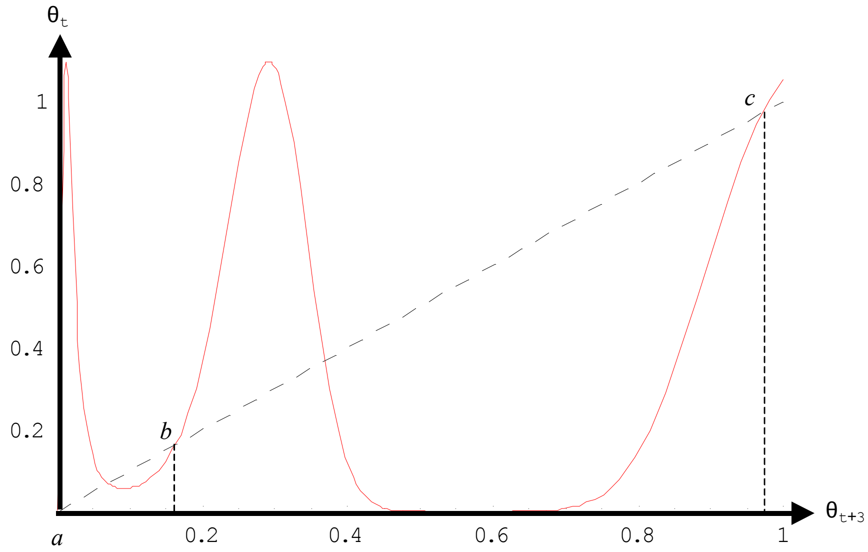

Figure 3: The map $g^3(.)$ in Example 3

Notice the intersections of the $g^3(.)$ locus with the 45^0 line; one of these corresponds to the stationary solution $\theta_{ss} = 0.367$, while the other intersections (points a, b, and c) correspond to points on a three-cycle outlined in Example 3 respectively.

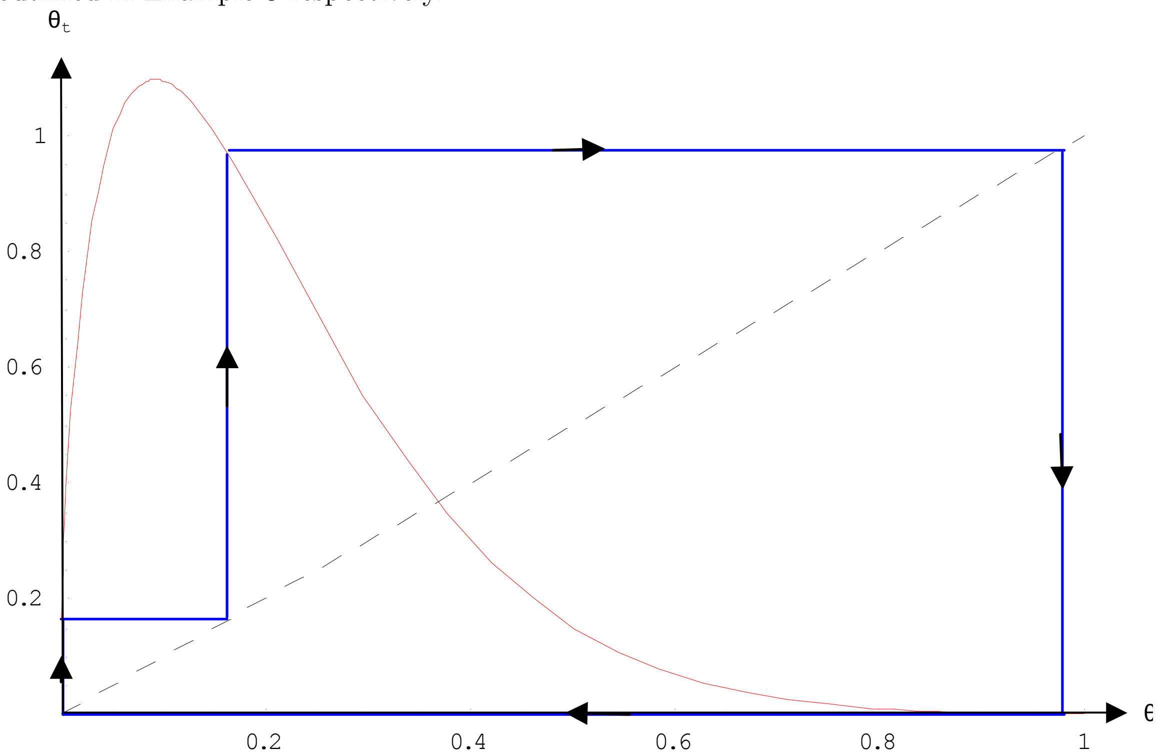

Figure 4: The map of $g(.)$ against θ for Example 3

We conclude this sub-section by presenting another example of a three-period cycle, one using a more "realistic" value of α.

Example 4 *Let the set of parameters be defined by*

A	α	β	s	γ
0.333	0.2430	0.99	0.013	6.623

. For this configuration, there is a unique steady state $\theta_{ss} = 20.02$, *and* $g'(\theta_{ss}) = -2.21$, *indicating that* θ_{ss} *is locally unstable in the backward dynamics and stable in the normal forward dynamics. There is a 3-cycle in* θ *that starts from* 0.00000015, *goes to 8.509655 to 52.20478 and returns to* 0.00000015.

3.4 When three-period cycles are ruled out

It is possible that depending on the choice of parameters, the map g may not admit a three-period cycle. Is it still possible to show the existence of chaotic planning solutions in this case? Mitra (2001) offers a sufficient condition for chaos in unimodal maps (like g) which do not admit three-period cycles. In this section we will verify that this model may display topological chaos, even for combinations of parameters which rule out three-period cycles.

Mitra (2001) focuses solely on dynamical systems (X, g), where the state space X is an interval on the non-negative part of the real line. The map, g, is required to be a continuous function from X to X, unimodal with a maximum at $\theta_{\max}$ with $g(\theta_{\max}) > \theta_{\max}$, and the unique steady state (θ_{ss}) must satisfy $\theta_{ss} > \theta_{\max}$. For such maps, Mitra (2001) states the following theorem (his Proposition 2.3, p. 142) which we restate for the sake of completeness.

Theorem 3 *(Mitra, 2001) Let* (X, g) *be a dynamical system. If* g *satisfies* $g^2(\theta_{\max}) < \theta_{\max}$ *and* $g^3(\theta_{\max}) < \theta_{ss}$, *then* (X, g) *exhibits topological chaos.*

It is well-known that $g^2(\theta_{\max}) < \theta_{\max}$ (corresponding to our condition (13)) is *necessary* for chaos (see Mitra, 2001). Below we present a parametric specification for which the dynamical system g does not admit any three-period cycles, and yet, the Mitra condition is satisfied.

Example 5 *Let the set of parameters be defined by*

A	α	β	s	γ
0.484	0.2414	0.99	0.004	3.445

. As is evident from Figure 5 [a plot of $g^3(.)$*], the map* g *does not admit three-period cycles. It is easy to check that* $\theta_{ss} = 10.6$, $\theta_{\max} = 2.58$, $g^2(\theta_{\max}) = 0.28$ *and* $g^3(\theta_{\max}) = 8.26$ *implying that the Mitra condition for topological chaos is satisfied.*

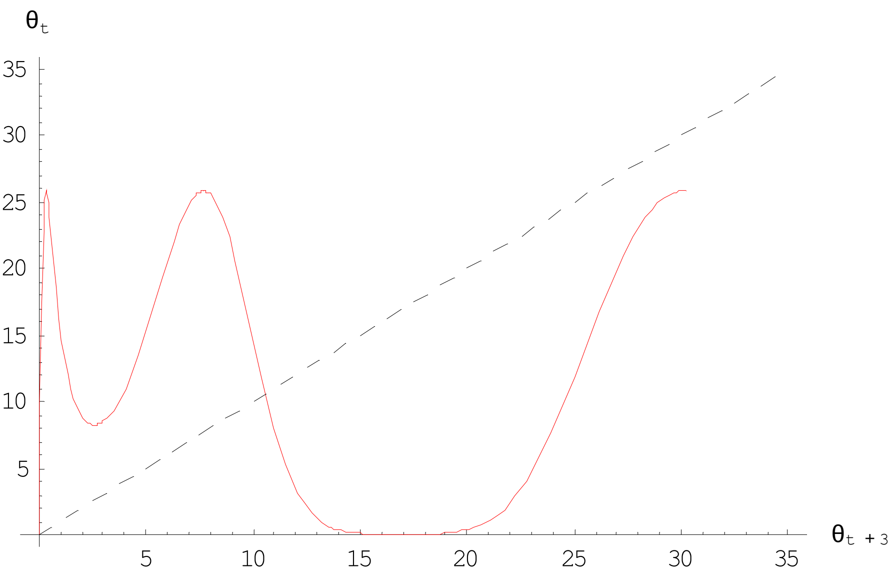

Figure 5: The map $g^3(.)$ for Example 5

Remarks: Using the new Mitra condition, we have thus verified that even for ranges of parameters where the Li-Yorke and three-period cycles routes to chaos cannot be taken, our map g may still exhibit topological chaos. In passing, it is useful to point out that neither the Mitra condition nor the Li-Yorke "overshooting" condition are *necessary* for the existence of topological chaos. In other words, for the map g, there may exist ranges of parameters for which we are not able to determine whether chaos is a possibility.

4 Concluding remarks

This paper takes the planning solution to the standard Pissarides (1990, 2001) "textbook model" of search and matching in labor markets, and shows that chaotic behavior may emerge in that framework. We show that the planning solution is completely characterized by a first-order, non-linear scalar difference equation. There is a unique stationary solution. Additionally, there are a large number of periodic and even aperiodic dynamical solutions that may exist. Unlike assumptions made in some recent work in this literature investigating the market solution, we do not require the production nor the matching technologies to exhibit any increasing returns. We go on to check the robustness of our claim of topological chaos by verifying the new sufficient condition due to Mitra (2001) in addition to the more standard Li-Yorke three-period cycle condition.

Appendix

A Derivation of Equation (5)

From (3), we obtain the following expression for λ_t :

$$\lambda_t = \frac{\beta^t c}{q'(\theta_t)\theta_t + q(\theta_t)} \tag{A1}$$

Using (A1) for period t and period $t+1$, we can insert this expression into (4):

$$-\frac{\beta^t c}{q'(\theta_t)\theta_t + q(\theta_t)} + \beta^{t+1}(y-z) + \frac{\beta^{t+1} c}{q'(\theta_{t+1})\theta_{t+1} + q(\theta_{t+1})}\left[(1-s) + q'(\theta_{t+1})\theta_{t+1}^2\right] = 0$$

Inserting the expressions for $q(\theta)$ and $q'(\theta)$ (recall that $q(\theta) = A\theta^{-\alpha}$) and re-arranging, we get:

$$\beta(y-z) + \frac{\beta c}{-\alpha A\theta_{t+1}^{-\alpha} + A\theta_{t+1}^{-\alpha}}\left[(1-s) - \alpha A\theta_{t+1}^{-\alpha+1}\right] = \frac{c}{-\alpha A\theta_t^{-\alpha} + A\theta_t^{-\alpha}} \tag{A2}$$

From here, straightforward manipulation yields

$$(1-s)\beta\theta_{t+1}^{\alpha} - \alpha A\beta\theta_{t+1} = \theta_t^{\alpha} - \frac{(1-\alpha)A\beta(y-z)}{c},$$

which immediately provides the desired expression in (5).■

B Proof of Theorem 2

To prove the theorem, we need to verify that there exists a point where the "overshooting" inequality of the Li-Yorke Theorem, equation (11), is satisfied. Specifically, we need to show that there exists a set of four points θ_1, θ_2, θ_3, and θ_4, where the variables are defined as $\theta_2 \equiv g(\theta_1)$, $\theta_3 \equiv g(\theta_2)$, and $\theta_4 \equiv g(\theta_3)$, and where the four points satisfy $\theta_4 \leq \theta_1 < \theta_2 < \theta_3$. We intend to do this by showing that equation (11) holds for the four points chosen such that $\theta_2 = \theta_{\max} = \left(\frac{\alpha a}{b}\right)^{\frac{1}{1-\alpha}}$. Specifically, we will show that if conditions (12), (13), and (14) hold, then the Li-Yorke conditions are satisfied for the points where θ_1 is implicitly defined by the equation $g(\theta_1) = \theta_{\max}$, θ_2 is chosen to be $\theta_{\max}$, $\theta_3 = g(\theta_{\max})$ and $\theta_4 = g^2(\theta_{\max})$. To ensure that the steady state is to the right of the maximum, we have already made the assumption that $g(\theta_{\max}) > \theta_{\max}$, which is equivalent to assuming $\theta_2 < \theta_3$. Similarly it is obvious that since $g(\theta_2)$ lies above the 45-degree line and g is unimodal, then if θ_1 is well-defined, it must be less than θ_2. To complete the proof we thus need to ensure that the equation $g(\theta_1) = \theta_{\max}$ has a solution with $\theta_1 > 0$ and we need to verify that $\theta_4 \leq \theta_1$.

First we will check that $g(\theta_1) = \theta_{\max}$ does indeed have a solution. The equation can be conveniently formulated as

$$ax_1^{\alpha} - bx_1 = \left(\frac{\alpha a}{b}\right)^{\frac{\alpha}{1-\alpha}} - d \tag{15}$$

Note that the left-hand side of (15) is a function of θ_1, while the right-hand side is a constant. Define $f(x) = ax^{\alpha} - bx$. To ensure that this equation has a solution, and provides a value $\theta_1 > 0$, let us examine the properties of f. It is simple to establish that f obtains it maximum at $\theta_{\max}$. First calculate the first derivative and set it equal to 0:

$$f'(x) = \alpha a x^{\alpha-1} - b = 0 \Leftrightarrow x = \left(\frac{b}{a\alpha}\right)^{\frac{1}{\alpha-1}} = \left(\frac{\alpha a}{b}\right)^{\frac{1}{1-\alpha}}$$

Checking the second derivative $f''(x) = \alpha a(\alpha-1)x^{\alpha-2} < 0$ verifies that the left-hand side of (15) is a concave function with a unique maximum at $\theta_{\max} = \theta_2$. Thus, to ensure that the equation indeed has a solution, it is necessary and sufficient that $f(\theta_{\max}) > \left(\frac{\alpha a}{b}\right)^{\frac{\alpha}{1-\alpha}} - d$. Rewriting this condition, simple algebra provides us with the following expression:

$$d > \left(\frac{\alpha a}{b}\right)^{\frac{\alpha}{1-\alpha}}[1 - a + a\alpha].$$

This exactly corresponds to (10) which was made to ensure that the steady state occurred to the right of the maximum.

Now that the existence of θ_1 is verified, we need to check that θ_1 is positive. Note that $f(0) = 0$, so θ_1 will be positive as long as the constant on the right hand side of (15) is greater than 0. Thus, to ensure that $\theta_1 > 0$, we require that

$$d < \left(\frac{\alpha a}{b}\right)^{\frac{\alpha}{1-\alpha}}.$$

This condition corresponds exactly to (12) and $\theta_1 > 0$ is ensured. Note that since $\alpha, a < 1$, the upper bound on d imposed by (12) is indeed greater than the lower bound provided by (10). We have thus satisfied ourselves that θ_1 is well-defined and greater than zero, and we can move on to the last part of the proof.

All that remains now is to show that $\theta_4 \leq \theta_1$. To this end, define $\Gamma(x) \equiv g(x)^\alpha = ax^\alpha - bx + d - \left(\frac{\alpha a}{b}\right)^{\frac{\alpha}{1-\alpha}}$, such that θ_1 is implicitly defined by the equation $\Gamma(\theta_1) = 0$. Note that Γ is strictly concave and attains its maximum at θ_2. This implies that $\Gamma(x)$ is strictly increasing for $x < \theta_2$. Therefore we know that *if* θ_4 is less than θ_2, a necessary and sufficient condition for $\theta_4 < \theta_1$ is $\Gamma(\theta_4) \leq \Gamma(\theta_1) = 0$. Before proceeding it is worth noting that $\theta_4 \leq \theta_2$ is a necessary but not sufficient condition for $\theta_4 \leq \theta_1$, since we have already established that $\theta_1 < \theta_2$. The rest of the proof will therefore proceed in two steps, the first is to verify that $\theta_4 < \theta_2$, and the second is to show that $\Gamma(\theta_4) \leq 0$.

To verify that $\theta_4 < \theta_2$, we will need to find the precise expression for θ_4. Recall that by definition, $\theta_4 = g^2(\theta_{\max}) = g(\theta_3)$. Now since $\theta_3 = g(\theta_{\max})$, we can calculate θ_3 directly as

$$\theta_3 = g(\theta_2) = \left(a\theta_{\max}^\alpha - b\theta_{\max} + d\right)^{\frac{1}{\alpha}}.$$

We can then proceed to calculate the expression for θ_4 :

$$\theta_4 = \left(a\left(a\theta_{\max}^\alpha - b\theta_{\max} + d\right) - b\left(a\theta_{\max}^\alpha - b\theta_{\max} + d\right)^{\frac{1}{\alpha}} + d\right)^{\frac{1}{\alpha}}. \tag{16}$$

The necessary condition $\theta_4 < \theta_2 = \theta_{\max}$ can then be written as

$$\begin{aligned} \left(a\left(a\theta_{\max}^\alpha - b\theta_{\max} + d\right) - b\left(a\theta_{\max}^\alpha - b\theta_{\max} + d\right)^{\frac{1}{\alpha}} + d\right)^{\frac{1}{\alpha}} &= \theta_{\max} \Leftrightarrow \\ a\left(a\theta_{\max}^\alpha - b\theta_{\max} + d\right) - b\left(a\theta_{\max}^\alpha - b\theta_{\max} + d\right)^{\frac{1}{\alpha}} + d &< \theta_{\max}^\alpha. \end{aligned}$$

This corresponds exactly to the condition (13) in the theorem, and we have therefore established that $\theta_4 < \theta_2$. Recall that when $\theta_4 < \theta_2$, $\Gamma(\theta_4) \leq 0$ is a necessary and sufficient condition for $\theta_4 \leq \theta_1$, so all that remains to be shown now is that $\Gamma(\theta_4) \leq 0$. By the definition of Γ, this inequality can be written as

$$ax_4^{\alpha} - bx_4 + d - \left(\frac{\alpha a}{b}\right)^{\frac{\alpha}{1-\alpha}} \leq 0.$$

Inserting the expression of θ_4 from (16), this becomes,

$$\begin{aligned} & a\left(a\left(a\theta_{\max}^{\alpha} - b\theta_{\max} + d\right) - b\left(a\theta_{\max}^{\alpha} - b\theta_{\max} + d\right)^{\frac{1}{\alpha}} + d\right) \\ -b & \left(a\left(a\theta_{\max}^{\alpha} - b\theta_{\max} + d\right) - b\left(a\theta_{\max}^{\alpha} - b\theta_{\max} + d\right)^{\frac{1}{\alpha}} + d\right)^{\frac{1}{\alpha}} + d - \left(\frac{\alpha a}{b}\right)^{\frac{\alpha}{1-\alpha}} \leq 0 \end{aligned}$$

Finally, using the definition of $\theta_{\max}$ and re-arranging, provides the following expression.

$$\begin{aligned} & \left(a^3 - 1\right)\theta_{\max}^{\alpha} - a^2 b\theta_{\max} - ab\left(a\theta_{\max}^{\alpha} - b\theta_{\max} + d\right)^{\frac{1}{\alpha}} + d\left(1 + a + a^2\right) \\ < \; & b\left(a^2\theta_{\max}^{\alpha} - ab\theta_{\max} + (1+a)\,d - b\left(a\theta_{\max}^{\alpha} - b\theta_{\max} + d\right)^{\frac{1}{\alpha}}\right)^{\frac{1}{\alpha}}, \end{aligned}$$

which exactly corresponds to the condition (14) in the theorem. We have thus established that $\theta_4 \leq \theta_1$ and the proof is complete.■

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