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Abstract

The class of maximin actions in general decision problems is characterized.

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1 Introduction

Choosing between alternatives according to the maximin criterion essentially involves associating with each alternative the worst possible consequence and then choosing the alternative(s) for which this worst-case scenario offers the best possible result. Different ways of modeling these actions, consequences (or states), and preferences/utilities over them yield an abundance of applications of this decision principle and its sibling, minimax behavior, in the social sciences:

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- Game theory: The minimax theorem of von Neumann (cf. [11]) is one of the corner stones of game theory. It establishes maximin behavior as an equilibrating device that assigns to every mixed extension of a finite two person zero-sum (or purely antagonistic) game a well-defined value.
- Experimental economics: [14] and [15] show that maximin behavior is the outcome of a natural and simple dynamic process of strategy adjustment and provides a good prediction of human behavior in several experimental settings.
- Statistical decision theory: Next to the Bayesian paradigm, the maximin approach is standard in statistical decision theory (cf. [3], [6]).
- Social choice and welfare: Rawlsian welfare aims for the maximization of the utility of the least "happy" member of a society; see [10] for a textbook treatment.
- Operations research: Problems like the optimal location of warehouses often involve the minimization of suitable distance functions. Among these distance functions, the Chebychev/supremum norm is a common one, transforming the problem in one of the minimax type (cf. [8]).
- Constrained optimization: The Lagrangean dual of a constrained minimization problem is of the maximin type (cf. [2, Ch. 6]).

Given the ubiquity of the maximin principle, it is hardly surprising that also its fundaments have been the subject of study. These studies tend to focus on one of two aspects: *(a)* characterizing the *order* induced by the maximin criterion, like in the classical study [9] and in [1], or *(b)* characterizing the maximin *value* associated with zero-sum games, like [17] and [16], or, more recently, [4] and [13] [1] .

To our knowledge, the current note is the first to characterize a third aspect, namely the solution that assigns to each decision problem its *set* of maximin actions. The purpose of our next section is to formally define decision problems, list the properties used in our characterization, and state the characterization theorem. The proof of our characterization is contained in the final section.

2 A characterization of the set of maximin actions

A *decision problem* is a tuple (A, Ω, u), where A is a nonempty set of actions, Ω is a nonempty set of states, and $u : A \times \Omega \to \mathbb{R}$ is a bounded function which represents the decision-maker's payoff/utility function. The set of all

[1] These authors take payoffs/utilities in the game as given. The authors of [7], go one step further by first deriving utilities from a number of properties on players' preferences and then making the step to evaluations using the value function.

decision problems is denoted by $\mathcal{D}$. A *solution* on $\mathcal{D}$ is a correspondence φ that assigns to every $(A, \Omega, u) \in \mathcal{D}$ a set $\varphi(A, \Omega, u) \subset A$ of actions. Our aim is to characterize the solution M that assigns to every decision problem $(A, \Omega, u) \in \mathcal{D}$ its set of *maximin actions*

$$M(A, \Omega, u) := \left\{ a \in A : \inf_{\omega \in \Omega} u(a, \omega) = \sup_{a' \in A} \inf_{\omega \in \Omega} u(a', \omega) \right\}.$$

Since only the order of the payoffs matters, order-preserving transformations do not affect the solution and the assumption that our payoffs are bounded entails no loss of generality:

$$M(A, \Omega, u) = M(A, \Omega, \arctan u).$$

In our general setting, some properties of simpler, finite problems no longer hold: (all) maximin actions can be strictly dominated (Example 1) and the set of maximin actions may be empty (Example 2). Recall that an action $a \in A$ in a decision problem $D = (A, \Omega, u) \in \mathcal{D}$ is *strictly dominated* if there is an action $a' \in A$ with $u(a', \omega) > u(a, \omega)$ for all $\omega \in \Omega$.

Example 1 *Consider a decision problem (A, Ω, u) with $A = \Omega = \mathbb{Z}$ and $u(a, \omega) = \arctan(a - \omega)$ for all $(a, \omega) \in \mathbb{Z} \times \mathbb{Z}$. Then $\inf_{\omega \in \mathbb{Z}} u(a, \omega) = -\pi/2$ for all $a \in \mathbb{Z}$: every $a \in \mathbb{Z}$ is maximin, yet also strictly dominated, for instance by $a + 1$.*

Example 2 *Consider a decision problem (A, Ω, u) with $A = \Omega = \mathbb{N}$ and $u(a, \omega) = a/(a + 1)$ for all $(a, \omega) \in A \times \Omega$. Then $\inf_{\omega \in \Omega} u(a, \omega) = a/(a + 1)$, a function which does not achieve a maximum: $M(A, \Omega, u) = \emptyset$.*

We introduce some properties for a solution φ on $\mathcal{D}$. They are standard and are mostly taken from earlier publications, particularly [9,1]. Anonymity requires that the solution does not depend on the way actions and states are labeled.

Anonymity (ANO). Let $(A, \Omega, u), (A', \Omega', u') \in \mathcal{D}$. If there are bijections $f : A \to A'$ and $g : \Omega \to \Omega'$ such that $u(a, \omega) = u'(f(a), g(\omega))$ for all $(a, \omega) \in A \times \Omega$, then $\varphi(A', \Omega', u') = f(\varphi(A, \Omega, u))$.

Independence of irrelevant actions states that if the action set of a decision problem is reduced, but some elements in the solution of the large problem remain feasible, then the solution of the small problem consists of the feasible elements in the solution of the original problem.

Independence of irrelevant actions (IIA). Let $(A, \Omega, u), (A', \Omega, u') \in \mathcal{D}$ be such that $A \subsetneq A'$ and $u'_{|A \times \Omega} = u$. If $\varphi(A', \Omega, u') \cap A \neq \emptyset$, then $\varphi(A', \Omega, u') \cap A = \varphi(A, \Omega, u)$.

Inheritance of nonemptiness states that adding finitely many actions to a decision problem with a nonempty solution yields a new decision problem whose solution is also nonempty.

Inheritance of nonemptiness (INH-NEM). Let $(A, \Omega, u), (A', \Omega, u') \in \mathcal{D}$ be such that $A \subsetneq A'$ and $u'_{|A\times\Omega} = u$. If $\varphi(A, \Omega, u) \neq \emptyset$ and $A' \setminus A$ is a finite set, then $\varphi(A', \Omega, u') \neq \emptyset$.

In a decision problem $(A, \Omega, u) \in \mathcal{D}$, action $a' \in A$ *weakly dominates* action $a \in A$ if $u(a', \omega) \geq u(a, \omega)$ for all $\omega \in \Omega$, with strict inequality for some $\omega \in \Omega$. The weak-domination property states that if an action weakly dominates an action in the solution of the problem, then also the weakly dominating action belongs to the solution.

Weak domination (WDOM). Let $(A, \Omega, u) \in \mathcal{D}$ and $a^*, a' \in A$. If $a^* \in \varphi(A, \Omega, u)$ and a' weakly dominates a^*, then $a' \in \varphi(A, \Omega, u)$.

The next property requires that duplicating states does not affect the solution.

Duplication of states (DOS). Let $(A, \Omega, u), (A, \Omega', u') \in \mathcal{D}$ with $\Omega \subsetneq \Omega'$. If there is a surjection $g : \Omega' \to \Omega$ such that $u'(a, \omega') = u(a, g(\omega'))$ for all $(a, \omega') \in A \times \Omega'$, then $(A, \Omega', u') \in \mathcal{D}$ and $\varphi(A, \Omega', u') = \varphi(A, \Omega, u)$.

Continuity states that if an action is always contained in the solution of a sequence of decision problems in $\mathcal{D}$ with fixed action and state spaces and pointwise convergent utility functions, then this action is also contained in the solution of the limiting problem.

Continuity (CONT). Let $(A, \Omega, u) \in \mathcal{D}$ and let $\{(A, \Omega, u_k)\}_{k\in\mathbb{N}}$ be a sequence in $\mathcal{D}$ such that $\lim_{k\to\infty} u_k(a, \omega) = u(a, \omega)$ for all $(a, \omega) \in A \times \Omega$. If there is an $a^* \in A$ with $a^* \in \varphi(A, \Omega, u_k)$ for all $k \in \mathbb{N}$, then $a^* \in \varphi(A, \Omega, u)$.

Restricted nonemptiness states that, for a given decision problem, if there exists some maximin action, then there also exists some element of the solution. This is not a new property in the literature, it is used in both cooperative games (cf. [18]) and noncooperative games (cf. [5], [12], [19]). In our context, it is related with the possibility of nonemptiness of the set of maximin actions.

Restricted Nonemptiness (r-NEM). Let $(A, \Omega, u) \in \mathcal{D}$. If $M(A, \Omega, u)$ is nonempty, then $\varphi(A, \Omega, u)$ is also nonempty.

Convexity states that if two actions belong to the solution of a decision problem and an action is added whose payoff is the $(\frac{1}{2}, \frac{1}{2})$-convex combination of the above actions' payoffs, then the new action belongs to the solution of the new problem. This is a standard risk neutrality property already present in [9]: if two actions belong to the problem's solution, the decision-maker does

not mind tossing a coin to decide between them.

Convexity (CONV). Let $(A, \Omega, u), (A', \Omega, u') \in \mathcal{D}$ be such that $A' = A \cup \{a'\}$ for some $a' \notin A$ and $u'_{|A\times\Omega} = u$. If there are $a^*, \tilde{a} \in \varphi(A, \Omega, u)$ such that

$$u'(a', \omega) = \frac{1}{2}u(a^*, \omega) + \frac{1}{2}u(\tilde{a}, \omega)$$

for all $\omega \in \Omega$, then $a' \in \varphi(A', \Omega, u')$.

Finally, if there is only one state, then the solution chooses the actions that maximize the payoff.

One state rationality (OSR). Take $(A, \Omega, u) \in \mathcal{D}$ with $|\Omega| = 1$; then, writing $\Omega = \{\omega\}$: $\varphi(A, \Omega, u) = \arg\max_{a\in A} u(a, \omega)$.

The former properties characterize the solution M on $\mathcal{D}$ which assigns to each decision problem its set of maximin actions:

Theorem 3 *The maximin solution M is the unique solution on $\mathcal{D}$ satisfying ANO, IIA, INH-NEM, WDOM, DOS, CONT, r-NEM, CONV, and OSR.*

Its proof is given in the next section.

3 Proof of characterization theorem

The purpose of this section is to prove our characterization theorem. The proof is based on a series of lemmas.

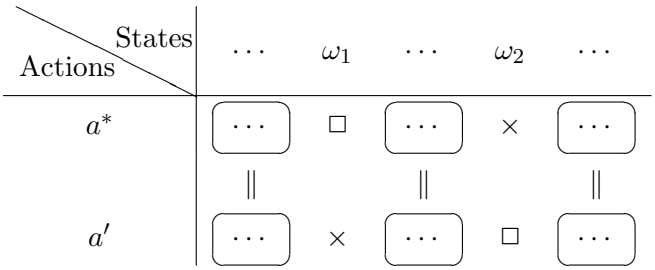
The properties ANO and IIA of a solution guarantee that if an action has the same payoff function as an element of the solution of the problem — up to relabeling of the states — then also the former action is part of the solution. We only use a simple version:

Lemma 4 *Let φ be a solution on $\mathcal{D}$ satisfying ANO and IIA, and let $D = (A, \Omega, u) \in \mathcal{D}$. If $a^* \in \varphi(D)$ and $a' \in A$ is such that, for some $\omega_1, \omega_2 \in \Omega$,*

(i) $u(a', \omega_1) = u(a^*, \omega_2)$ and $u(a', \omega_2) = u(a^*, \omega_1)$,
(ii) $u(a', \omega) = u(a^*, \omega)$ for all $\omega \in \Omega \setminus \{\omega_1, \omega_2\}$,

then $a' \in \varphi(D)$.

PROOF. Assume that $u(a^*, \omega_1) \neq u(a^*, \omega_2)$ (otherwise ANO concludes the result). The utility functions for actions a^* and a' are represented in the table below, where $\square$ and $\times$ represent two different values:



Consider decision problems

$$D_1 = (\{a^*, a'\}, \Omega, u_{|\{a^*,a'\}\times\Omega}),\ D_2 = (\{a^*, a'\}, \Omega, v),$$

where the utility for a^* and a' is interchanged, i.e.

$$v(a^*, \omega_1) = v(a', \omega_2) := u(a^*, \omega_2),$$
$$v(a^*, \omega_2) = v(a', \omega_1) := u(a^*, \omega_1),$$

and $v(b, \omega) := u(b, \omega)$ for all other $(b, \omega) \in \{a^*, a'\} \times (\Omega \setminus \{\omega_1, \omega_2\})$. By *(i)* and *(ii)*, D_2 is isomorphic to D_1, either via switching the labels of a^* and a' or via switching the labels of ω_1 and ω_2.

Note that D can be obtained from D_1 by adding actions and, moreover, $a^* \in \varphi(D) \cap \{a^*, a'\}$. Therefore, by IIA:

$$\varphi(D_1) = \varphi(D) \cap \{a^*, a'\}, \tag{1}$$

so that $a^* \in \varphi(D_1)$. It is shown that also $a' \in \varphi(D_1)$. Consider the bijection $f : \{a^*, a'\} \to \{a^*, a'\}$ with $f(a^*) = a', f(a') = a^*$ and let $g : \Omega \to \Omega$ be the identity function. Since $u(a, \omega) = v(f(a), g(\omega))$ for all $(a, \omega) \in \{a^*, a'\} \times \Omega$, ANO implies that $\varphi(D_2) = f(\varphi(D_1))$, so $a' = f(a^*) \in \varphi(D_2)$. Next, consider the bijection $\bar{g} : \Omega \to \Omega$ with $\bar{g}(\omega_1) = \omega_2$, $\bar{g}(\omega_2) = \omega_1$, keeping other states unchanged, and let $\bar{f} : \{a^*, a'\} \to \{a^*, a'\}$ be the identity function. Since $v(a, \omega) = u(\bar{f}(a), \bar{g}(\omega))$ for all $(a, \omega) \in \{a^*, a'\} \times \Omega$, ANO implies that $\varphi(D_1) = \bar{f}(\varphi(D_2)) = \varphi(D_2)$. Remember that $a' \in \varphi(D_2)$, so $a' \in \varphi(D_1)$. This shows that $\{a^*, a'\} = \varphi(D_1)$.

Finally, by (1), $a' \in \varphi(D)$. $\square$

With the INH-NEM property and Lemma 4 one can establish the following consequence. If we add an action to a decision problem with the same utility as an action in the solution of the original problem, except in two states where the utilities are interchanged, then both actions belong to the solution of the new problem:

Lemma 5 *Let φ be a solution on $\mathcal{D}$ satisfying ANO, IIA, and INH-NEM, and let $D = (A, \Omega, u) \in \mathcal{D}$. Take $D' = (A', \Omega, u') \in \mathcal{D}$ satisfying that $A' = A \cup \{a'\}$*

for some $a' \notin A$ and $u'_{|A\times\Omega} = u$. Suppose that there exist $a^* \in \varphi(A,\Omega,u)$ and $\omega_1, \omega_2 \in \Omega$ such that

(i) $u'(a',\omega_1) = u'(a^*,\omega_2)$ *and* $u'(a',\omega_2) = u'(a^*,\omega_1)$,
(ii) $u'(a',\omega) = u'(a^*,\omega)$ *for all* $\omega \in \Omega \setminus \{\omega_1,\omega_2\}$.

Then $\{a^*, a'\} \subseteq \varphi(D')$.

PROOF. Note that D' is well-defined. Suppose that $a' \notin \varphi(D')$. Since φ satisfies INH-NEM, $A' \setminus A = \{a'\}$ is a finite set, and $\varphi(D) \neq \emptyset$: $\varphi(D') \neq \emptyset$. So $\varphi(D') \cap A \neq \emptyset$ and IIA implies that $\varphi(D') \cap A = \varphi(D)$. Therefore $a^* \in \varphi(D')$. By Lemma 4, also $a' \in \varphi(D')$, a contradiction. Hence, $a' \in \varphi(D')$ and using Lemma 4 again it follows that $a^* \in \varphi(D')$. So $\{a^*, a'\} \subset \varphi(D')$. □

Consider the following modification of weak dominance. In a decision problem $(A,\Omega,u) \in \mathcal{D}$, action $a' \in A$ *quasi-dominates* action $a \in A$ if there exist $\omega_1, \omega_2 \in \Omega$ such that:

(i) $u(a',\omega) \geq u(a,\omega)$ for all $\omega \in \Omega \setminus \{\omega_1\}$, and
(ii) $u(a',\omega_2) \geq u(a,\omega_1) > u(a',\omega_1) \geq u(a,\omega_2)$.

Intuitively, a' quasi-dominates a if it is at least as good as a in all states except some ω_1, and the loss from choosing a' in state ω_1 is compensated for by a utility gain in another state ω_2.

The next Lemma shows that a solution satisfying ANO, IIA, INH-NEM, and WDOM, satisfies the following property: if an action quasi-dominates an action in the solution, then the former action also belongs to the solution.

Lemma 6 *Let φ be a solution on $\mathcal{D}$ satisfying ANO, IIA, INH-NEM, and WDOM, and let $D = (A,\Omega,u) \in \mathcal{D}$. If $a^* \in \varphi(D)$ and $a' \in A$ quasi-dominates a^*, then $a' \in \varphi(D)$.*

PROOF. Let $\omega_1, \omega_2 \in \Omega$ be as in the definition of quasi-dominance. Define the decision problem $D' = (A \cup \{\alpha\}, \Omega, u') \in \mathcal{D}$ with $\alpha \notin A$, $u'_{|A\times\Omega} = u$, $u'(\alpha,\omega) = u(a^*,\omega)$ for all $\omega \in \Omega \setminus \{\omega_1,\omega_2\}$, $u'(\alpha,\omega_1) = u(a^*,\omega_2)$, and $u'(\alpha,\omega_2) = u(a^*,\omega_1)$. By Lemma 5: $\{a^*,\alpha\} \subset \varphi(D')$. Now a' weakly dominates α unless $u(a',\omega) = u(\alpha,\omega)$ for all $\omega \in \Omega$ (in which case $a' \in \varphi(D')$ by ANO). So, by WDOM, $a' \in \varphi(D')$.

Hence, $\{a^*,\alpha,a'\} \subset \varphi(D')$. Now $\varphi(D) = \varphi(D') \cap A$ by IIA, so $a' \in \varphi(D)$. □

If a solution satisfies ANO, IIA, INH-NEM, WDOM, DOS, and CONT, then

whether or not an action belongs to the solution of a decision problem depends exclusively on the infimum and supremum of its payoffs.

Lemma 7 *Let φ be a solution on $\mathcal{D}$ satisfying ANO, IIA, INH-NEM, WDOM, DOS, and CONT, and let $D = (A, \Omega, u) \in \mathcal{D}$. If $a^* \in \varphi(D)$ and $a' \in A$ is such that*

$$\inf_{\omega\in\Omega} u(a',\omega) = \inf_{\omega\in\Omega} u(a^*,\omega) = m \text{ and } \sup_{\omega\in\Omega} u(a',\omega) = \sup_{\omega\in\Omega} u(a^*,\omega) = M,$$

then $a' \in \varphi(D)$.

PROOF. If $m = M$, then a^* and a' yield the same, constant payoff, regardless of ω, so ANO and $a^* \in \varphi(D)$ imply that $a' \in \varphi(D)$. So henceforth assume that $m < M$. This means that Ω has at least two elements. Let $\omega_1 \in \Omega$. Define for each $(\varepsilon, \delta) \in \mathbb{R}^2_+$ the decision problem $D_{\varepsilon,\delta} = (A \cup \{\alpha, \beta\}, \Omega, u_{\varepsilon,\delta})$ with $\alpha, \beta \notin A$ as follows. For all $(\tilde{a}, \omega) \in (A \cup \{\alpha, \beta\}) \times \Omega$,

$$u_{\varepsilon,\delta}(\tilde{a},\omega) = \begin{cases} u(a',\omega) + \delta & \text{if } \tilde{a} = a', \\ m + \varepsilon & \text{if } (\tilde{a},\omega) = (\alpha,\omega_1), \\ m & \text{if } \tilde{a} = \beta \text{ and } \omega \neq \omega_1, \\ M & \text{if } (\tilde{a},\omega) = (\beta,\omega_1) \text{ or } (\tilde{a} = \alpha \text{ and } \omega \neq \omega_1), \\ u(\tilde{a},\omega) & \text{otherwise.} \end{cases}$$

The table below summarizes the definition of $D_{\varepsilon,\delta}$.

Actions \ States	ω_1	$\omega \in \Omega \setminus \{\omega_1\}$
a^*	$u(a^*,\omega_1)$	$u(a^*,\omega)$
a'	$u(a',\omega_1) + \delta$	$u(a',\omega) + \delta$
α	$m + \varepsilon$	M
β	M	m
all other a	$u(a,\omega_1)$	$u(a,\omega)$

Let $D' = (A \setminus \{a'\}, \Omega, u_{|(A\setminus\{a'\})\times\Omega}) \in \mathcal{D}$. Since $a^* \in \varphi(D) \cap (A \setminus \{a'\})$, IIA implies that $\varphi(D') = \varphi(D) \cap (A \setminus \{a'\}) \neq \emptyset$. For all $(\varepsilon, \delta) \in \mathbb{R}^2_+$, $D_{\varepsilon,\delta}$ is obtained from D' by adding finitely many actions, so INH-NEM implies that $\varphi(D_{\varepsilon,\delta}) \neq \emptyset$.

Step 1: Let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be a sequence of strictly positive real numbers with

$\lim_{k\to\infty} \varepsilon_k = 0$. We show that $\alpha \in \varphi(D_{\varepsilon_k,0})$ for all $k \in \mathbb{N}$. By CONT, we then have $\alpha \in \varphi(D_{0,0})$.

Let $k \in \mathbb{N}$ and suppose, to the contrary, that $\alpha \notin \varphi(D_{\varepsilon_k,0})$. Since $\varphi(D_{\varepsilon_k,0}) \neq \emptyset$, we have two cases:

- $\beta \in \varphi(D_{\varepsilon_k,0})$. This is not possible, because α quasi-dominates β and applying Lemma 6 one obtains that $\alpha \in \varphi(D_{\varepsilon_k,0})$.
- $\beta \notin \varphi(D_{\varepsilon_k,0})$. Since $\varphi(D_{\varepsilon_k,0}) \neq \emptyset$ and $\alpha, \beta \notin \varphi(D_{\varepsilon_k,0})$ there is an $a \in \varphi(D_{\varepsilon_k,0}) \cap A$. By IIA: $\varphi(D_{\varepsilon_k,0}) \cap A = \varphi(D)$, so $a^* \in \varphi(D_{\varepsilon_k,0})$.
  - If $u(a^*, \omega_1) \leq m + \varepsilon_k$, then α weakly dominates a^*: $u(a^*, \omega) \leq u(\alpha, \omega)$ for all $\omega \in \Omega$, and there is an $\omega_0 \in \Omega$ such that $u(a^*, \omega_0) < u(\alpha, \omega_0)$, because otherwise $u(a^*, \omega) = u(\alpha, \omega)$ for all $\omega \in \Omega$, so that $m = \inf_{\omega\in\Omega} u(a^*, \omega) = \inf_{\omega\in\Omega} u(\alpha, \omega) = \min\{m + \varepsilon_k, M\} > m$, a contradiction. Using WDOM, it follows that $\alpha \in \varphi(D_{\varepsilon_k,0})$.
  - If $u(a^*, \omega_1) > m + \varepsilon_k$, then α quasi-dominates a^*: $u(\alpha, \omega) \geq u(a^*, \omega)$ for all $\omega \in \Omega \setminus \{\omega_1\}$ and by definition of $m = \inf_{\omega\in\Omega} u(a^*, \omega)$, there is an $\omega_2 \in \Omega$, different from ω_1 (since $u(a^*, \omega_1) > m + \varepsilon_k$) with $u(a^*, \omega_2) \leq m + \varepsilon_k$. This implies that $M = u(\alpha, \omega_2) \geq u(a^*, \omega_1) > m + \varepsilon_k \geq u(a^*, \omega_2)$. By Lemma 6, $\alpha \in \varphi(D_{\varepsilon_k,0})$.

  In both subcases, we established that $\alpha \in \varphi(D_{\varepsilon_k,0})$, in contradiction with our assumption. Conclude that $\alpha \in \varphi(D_{\varepsilon_k,0})$.

Step 2: We show that $\beta \in \varphi(D_{0,0})$.

Let $\omega_2 \in \Omega$, $\omega_2 \neq \omega_1$, and consider the decision problems

$$D_1 = \left(\{\alpha, \beta\}, \{\omega_1, \omega_2\}, u_{0,0|\{\alpha,\beta\}\times\{\omega_1,\omega_2\}}\right) \text{ and } D_2 = \left(\{\alpha, \beta\}, \Omega, u_{0,0|\{\alpha,\beta\}\times\Omega}\right).$$

D_2 can be obtained from $D_{0,0}$ by deleting actions. By step 1, $\varphi(D_{0,0}) \cap \{\alpha, \beta\} \neq \emptyset$. So IIA implies that

$$\varphi(D_{0,0}) \cap \{\alpha, \beta\} = \varphi(D_2). \tag{2}$$

Therefore, $\alpha \in \varphi(D_2)$. By DOS, $\varphi(D_1) = \varphi(D_2)$, so $\alpha \in \varphi(D_1)$. Now Lemma 4 implies that $\beta \in \varphi(D_1)$. Since $\varphi(D_1) = \varphi(D_2)$, equation (2) gives that $\beta \in \varphi(D_{0,0})$.

Step 3: Let $\{\delta_k\}_{k\in\mathbb{N}}$ be a sequence of strictly positive real numbers with $\lim_{k\to\infty} \delta_k = 0$. We show that $a' \in \varphi(D_{0,\delta_k})$ for all $k \in \mathbb{N}$. By CONT, we then have $a' \in \varphi(D_{0,0})$.

Consider the decision problem

$$D_3 = \left(A_3, \Omega, u_{0,0_{|A_3\times\Omega}}\right)$$

where $A_3 = (A \cup \{\alpha, \beta\}) \setminus \{a'\}$ for some $\alpha, \beta \notin A$. By steps 1 and 2, $\varphi(D_{0,0}) \cap$

$A_3 \neq \emptyset$, so IIA implies that $\varphi(D_{0,0}) \cap A_3 = \varphi(D_3)$. Hence, from step 2, $\beta \in \varphi(D_3)$.

Let $\delta_k > 0$ and suppose that $a' \notin \varphi(D_{0,\delta_k})$. Since $\varphi(D_{0,\delta_k}) \neq \emptyset$ one obtains that $\varphi(D_{0,\delta_k}) \cap A_3 \neq \emptyset$ and then IIA implies that $\beta \in \varphi(D_{0,\delta_k})$. So, reasoning as in step 1: if $u(a', \omega_1) + \delta_k \geq M$, then a' weakly dominates β and, by WDOM, $a' \in \varphi(D_{\delta_k,0})$; otherwise, a' quasi-dominates β and by Lemma 6: $a' \in \varphi(D_{\delta_k,0})$. In both cases we reach a contradiction. Conclude that $a' \in \varphi(D_{\delta_k,0})$.

Step 4: Finally, we show that $a' \in \varphi(D)$.

By step 3 $a' \in \varphi(D_{0,0}) \cap A$. Hence, IIA implies $\varphi(D_{0,0}) \cap A = \varphi(D)$, and so $a' \in \varphi(D)$. $\square$

These results will help us prove Theorem 3:

Proof of Thm. 3 It is easy to verify that the solution M satisfies all the properties.

Let φ be a solution on $\mathcal{D}$ satisfying all the properties and let $D = (A, \Omega, u) \in \mathcal{D}$. If $\varphi(D) = \emptyset$, then by r-NEM: $M(D) = \emptyset$. So, assume that $\varphi(D) \neq \emptyset$.

Under the assumption that whether or not an action belongs to $\varphi(D)$ depends exclusively on the infimum of its payoffs, it is true that $\varphi(D) = M(D)$. Namely, consider the decision problem $\widehat{D} = (A, \widehat{\Omega}, \widehat{u})$ where $|\widehat{\Omega}| = 1$ and $\widehat{u}(a, \widehat{\omega}) = \inf_{\omega \in \Omega} u(a, \omega)$ for all $(a, \widehat{\omega}) \in A \times \widehat{\Omega}$. We show that

$$\varphi(D) = \varphi(\widehat{D}) \tag{3}$$

Consider the decision problem $\widetilde{D} = (\widetilde{A}, \Omega, \widetilde{u}) \in \mathcal{D}$ obtained from D by adding to the action space a replica $r(a)$ of every action $a \in A$, i.e., $\widetilde{A} = \{a, r(a)\}_{a \in A}$ and with payoffs $\widetilde{u}_{|A \times \Omega} = u$ and $\widetilde{u}(r(a), \omega) = \inf_{\omega \in \Omega} u(a, \omega)$ for all $a \in A$ and $\omega \in \Omega$.

By the assumption: $a \in \varphi(D)$ if and only if $\{a, r(a)\} \subseteq \varphi(\widetilde{D})$. Since $\varphi(D) \neq \emptyset$, deletion of all non-replica actions and IIA imply that

$$a \in \varphi(D) \Leftrightarrow r(a) \in \varphi((\{r(a)\}_{a \in A}, \Omega, \widetilde{u}_{\{r(a)\}_{a \in A} \times \Omega})). \tag{4}$$

ANO and DOS imply that

$$r(a) \in \varphi((\{r(a)\}_{a \in A}, \Omega, \widetilde{u}_{\{r(a)\}_{a \in A} \times \Omega})) \Leftrightarrow a \in \varphi(\widehat{D}). \tag{5}$$

The equality (3) now follows from (4) and (5).

Write $\widehat{\Omega} = \{\widehat{\omega}\}$. By OSR we know that $\varphi(\widehat{D}) = M(\widehat{D}) = \arg\max_{a \in A} \widehat{u}(a, \widehat{\omega})$. Finally, since M satisfies all the properties we also have that $M(\widehat{D}) = M(D)$. Therefore $\varphi(D) = M(D)$.

Now it remains to prove that whether or not an action belongs to $\varphi(D)$ depends exclusively on the infimum of its payoffs.

Let $a^* \in \varphi(D)$ and let $m = \inf_{\omega \in \Omega} u(a^*, \omega)$ and $M = \sup_{\omega \in \Omega} u(a^*, \omega)$. If $m = M$, then $u(a^*, \omega) = m$ for all $\omega \in \Omega$. Let $a \in A$ be such that $\inf_{\omega \in \Omega} u(a, \omega) = m$. If $\sup_{\omega \in \Omega} u(a, \omega) = m$, then $u(a, \omega) = u(a^*, \omega)$ for all $\omega \in \Omega$ and, by ANO, $a \in \varphi(D)$; otherwise, a weakly dominates a^*, so, by WDOM, $a \in \varphi(D)$. Therefore, if $m = M$, then whether or not an action belongs to $\varphi(D)$ depends exclusively on the infimum of its payoffs.

So henceforth assume that $m < M$. This implies in particular that Ω contains at least two elements. Choose $\omega_1, \omega_2 \in \Omega$, $\omega_1 \neq \omega_2$.

Take $D' = (A, \Omega', u') \in \mathcal{D}$ where $\Omega' = \{\omega_1, \omega_2, \omega_3\}$ with $\omega_3 \notin \Omega$ and, for all $a \in A$:

$$u'(a, \omega') = \begin{cases} \sup\limits_{\omega \in \Omega} u(a, \omega) & \text{if } \omega' = \omega_1 \\ \inf\limits_{\omega \in \Omega} u(a, \omega) & \text{otherwise} \end{cases}$$

The table below summarizes the definition of D'.

Actions \ States	ω_1	ω_2	ω_3
$\vdots$		$\vdots$	
a^*	M	m	m
a	$\sup\limits_{\omega \in \Omega} u(a, \omega)$	$\inf\limits_{\omega \in \Omega} u(a, \omega)$	$\inf\limits_{\omega \in \Omega} u(a, \omega)$
$\vdots$		$\vdots$	

Similar to the proof of (3), using Lemma 7 instead of the assumption, it follows that $\varphi(D) = \varphi(D')$.

Define the sequence of decision problems $\{D_k\}_{k \in \mathbb{N}} = \{(A \cup \{\alpha, \beta, \gamma\}, \Omega', u_k)\}_{k \in \mathbb{N}}$ where $\alpha, \beta, \gamma \notin A$, $u_{k|A \times \Omega'} = u'$ and, for all $(a, \omega) \in \{\alpha, \beta, \gamma\} \times \Omega'$,

$$u_k(a, \omega) = \begin{cases} m + \dfrac{1}{2^{k-1}}(M - m) & \text{if } (a, \omega) \in \{(\alpha, \omega_1), (\beta, \omega_2)\} \\ m + \dfrac{1}{2^k}(M - m) & \text{if } (a, \omega) \in \{(\gamma, \omega_1), (\gamma, \omega_2)\} \\ m & \text{otherwise.} \end{cases}$$

The table below summarizes the definition of D_k.

Actions \ States	ω_1	ω_2	ω_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
a^*	M	m	m
$\vdots$	$\vdots$	$\vdots$	$\vdots$
α	$m + \frac{1}{2^{k-1}}(M-m)$	m	m
β	m	$m + \frac{1}{2^{k-1}}(M-m)$	m
γ	$m + \frac{1}{2^{k}}(M-m)$	$m + \frac{1}{2^{k}}(M-m)$	m
$\vdots$	$\vdots$	$\vdots$	$\vdots$

For all $k \in \mathbb{N}$, D_k can be obtained from D' by adding three actions. So, $\varphi(D') \neq \emptyset$ and INH-NEM imply that $\varphi(D_k) \neq \emptyset$. We show by induction that $\gamma \in \varphi(D_k)$ for all $k \in \mathbb{N}$.

Step 1: $\gamma \in \varphi(D_1)$.
D_1 can be obtained from D' by adding actions α, β, and γ in two steps:

First, add α and β to obtain the decision problem $D_1' = (A \cup \{\alpha, \beta\}, \Omega', u_1')$ with $u_1' = u_{1|(A\cup\{\alpha,\beta\})\times\Omega'}$. Lemma 7 implies that $\alpha \in \varphi(D_1')$ if and only if $\beta \in \varphi(D_1')$. Suppose that $\alpha, \beta \notin \varphi(D_1')$. INH-NEM and $\varphi(D') \neq \emptyset$ imply that $\varphi(D_1') \neq \emptyset$, so there is an $a \in \varphi(D_1') \cap A$. Then, by IIA, $\varphi(D_1') \cap A = \varphi(D')$. Hence, $a^* \in \varphi(D_1')$. Lemma 7 then implies that $\alpha, \beta \in \varphi(D_1')$, which is a contradiction. Thus $\alpha, \beta \in \varphi(D_1')$.

Second, add action γ, whose utility is the $(\frac{1}{2}, \frac{1}{2})$-convex combination of the utility of the actions α and β, and by CONV: $\gamma \in \varphi(D_1)$.

Step 2: Let $k \in \mathbb{N}$ and assume that $\gamma \in \varphi(D_n)$ for all $n \in \mathbb{N}, n \leq k$. We show that $\gamma \in \varphi(D_{k+1})$.
The decision problem D_{k+1} can be obtained from D_k in two steps:

First, delete actions α and β from D_k to obtain a new decision problem. By IIA and the assumption that $\gamma \in \varphi(D_k)$, its solution contains γ. Next, introduce actions α and β again, but now with their utility functions equal to those in the problem D_{k+1}. Since α and β have the same infimum and supremum, α belongs to the solution if and only if β belongs to the solution of this new problem. Suppose that α and β do not belong to the solution. By INH-NEM and IIA, γ belongs to the solution. But then Lemma 7 implies that α and β should belong to the solution, which is a contradiction. Thus α and β belong to the solution.

Second, delete γ from this new problem to obtain the decision problem $D_{k+1}' = (A \cup \{\alpha, \beta\}, \Omega', u_{k+1}')$ with $u_{k+1}' = u_{k+1|(A\cup\{\alpha,\beta\})\times\Omega'}$. By IIA $\alpha, \beta \in$

$\varphi(D'_{k+1})$. Next, introduce action γ again, but now with utility function equal to the $(\frac{1}{2}, \frac{1}{2})$-convex combination of the payoffs of actions α and β in D'_{k+1}, so the decision problem D_{k+1} is obtained. By CONV it follows that $\gamma \in \varphi(D_{k+1})$.

Conclude, by induction, that $\gamma \in \varphi(D_k)$ for all $k \in \mathbb{N}$.

Let $D_\infty = (A \cup \{\alpha, \beta, \gamma\}, \Omega', u_\infty)$ be the limiting decision problem of the sequence $\{D_k\}_{k\in\mathbb{N}}$. Notice that $u_{\infty|A\times\Omega'} = u'$ and $u_\infty(\alpha, \omega) = u_\infty(\beta, \omega) = u_\infty(\gamma, \omega) = m$ for all $\omega \in \Omega'$. Since $\gamma \in \varphi(D_k)$ for all $k \in \mathbb{N}$, CONT implies that $\gamma \in \varphi(D_\infty)$.

Take $a \in A$ such that $\inf_{\omega\in\Omega'} u'(a, \omega) = m$. If $\sup_{\omega\in\Omega'} u'(a, \omega) = m$, then $u_\infty(a, \omega) = u'(a, \omega) = m = u_\infty(\gamma, \omega)$ for all $\omega \in \Omega$, so that $a \in \varphi(D_\infty)$ by ANO. Otherwise, a weakly dominates γ and, by WDOM, $a \in \varphi(D_\infty)$. Hence $a \in \varphi(D_\infty) \cap A$, and using IIA it follows that $\varphi(D_\infty) \cap A = \varphi(D') = \varphi(D)$.

Hence, $a \in \varphi(D)$ for all $a \in A$ with $\inf_{\omega\in\Omega} u(a, \omega) = \inf_{\omega\in\Omega} u(a^*, \omega) = m$. □

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