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5-CHROMATIC STRONGLY REGULAR GRAPHS

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Abstract

In this paper, we begin the determination of all primitive strongly regular graphs with chromatic number equal to 5. Using eigenvalue techniques, we show that there are at most 43 possible parameter sets for such a graph. For each parameter set, we must decide which strongly regular graphs, if any, possessing the set are 5-chromatic. In this way, we deal completely with 34 of these parameter sets using eigenvalue techniques and computer enumerations.

1 Introduction

Almost 25 years ago, the second author determined all strongly regular graphs with chromatic number at most 4 [20]. At that time, it was far out of reach to determine the 5-chromatic strongly regular graphs. However, in recent years for several parameter sets, all non-isomorphic strongly regular graphs have been determined, mostly by computer. Many of these sets are candidates for being 5-chromatic. Thus, it became worthwhile to investigate the 5-chromatic strongly regular graphs. This was initiated by the first author in the second chapter of his Ph. D. thesis [14], and the present paper is an updated version of that chapter. We determine all 5-chromatic strongly regular graphs on less than 85 vertices. And for the remaining cases, we show that there are only nine possible parameter sets.

Before we begin, we must establish some terminology and notation. All graphs considered are finite, undirected, and simple (no loops or parallel edges). The vertex set of a graph G will be denoted by $V(G)$. The complement of G will be denoted by $\overline{G}$. Given $x \in V(G)$, a vertex adjacent to x is called a *neighbor* of x. The set of all neighbors of x is called the *neighborhood* of x and will be denoted by $N(x)$. If $|N(x)| = k$ for all $x \in V(G)$, then G will be called *k-regular*. Given distinct $x, y \in V(G)$, the set $N(x) \cap N(y)$ is called the set of *common neighbors* of x and y.

A set of pairwise non-adjacent vertices in a graph G is called a *coclique* of G. A *maximum coclique* in G is a coclique of maximum size in G. The size of a maximum coclique in G is called the *coclique number* of G and will be denoted by $\alpha(G)$. A *proper vertex coloring*, or just a coloring, C of a graph G is a function $C : V(G) \to \{1, \ldots, l\}$ such that $C(x) \neq C(y)$ whenever x and y are adjacent in G. Such a coloring will be called an *l-coloring* of G. Thus, an l-coloring of G is just a partition of $V(G)$ into l cocliques, which we will call *color classes*. If there exists an l-coloring of G, then G is called *l-colorable*. The smallest l such that G is l-colorable is called the *chromatic number* of G and will be denoted by $\chi(G)$. If $\chi(G) = l$, then G will be called *l-chromatic*.

2 Strongly regular graphs

A *strongly regular graph* with parameters v, k ($1 \leq k \leq v-2$), λ, and μ, or an $srg(v,k,\lambda,\mu)$, is a k-regular graph on v vertices such that every pair of adjacent vertices has exactly λ common neighbors and every pair of distinct non-adjacent vertices has exactly μ common neighbors. For a good introduction to the theory of these graphs, see [11], [18], or [29]. For a survey of strongly regular graphs, see [2], [6], [10], or [37]. We shall need some properties of strongly regular graphs. Most material can be found in mentioned references. The first two results are proved by simple counting.

Proposition 2.1 *A graph is an $srg(v,k,\lambda,\mu)$ if and only if its complement is an $srg(v, v-k-1, v-2k+\mu-2, v-2k+\lambda)$.*

Proposition 2.2 *If an $srg(v,k,\lambda,\mu)$ exists, then $k(k-\lambda-1) = (v-k-1)\mu$.*

The next result is a well-known characterization of connected strongly regular graphs as the connected regular graphs with exactly three distinct eigenvalues.

Theorem 2.3 *A connected regular graph G is an $srg(v,k,\lambda,\mu)$ if and only if it has exactly three distinct eigenvalues. One of these eigenvalues is k and the other two r and s ($r > s$) are related to the parameters by*

$$k + rs = \mu \quad \textit{and} \quad r + s = \lambda - \mu \ .$$

The multiplicity of eigenvalue k is equal to 1. The multiplicities f and g of r and s can be computed from

$$f + g + 1 = v \quad \textit{and} \quad fr + gs + k = 0 \ .$$

Additionally, r and s are integers, unless $v \equiv 1 \pmod 4$, $k = (v-1)/2$, $r, s = -\frac{1}{2} \pm \frac{1}{2}\sqrt{v}$, and v is not an integral square.

A strongly regular graph that is connected with connected complement is called *primitive*. A strongly regular graph that is not primitive is called *imprimitive*. The imprimitive strongly regular graphs are precisely the disjoint unions of complete graphs of the same size and the complete multipartite graphs with all color classes of equal size. So their chromatic numbers are trivial. It is straightforward to see that the parameters and eigenvalues of a primitive strongly regular graph satisfy $0 < \mu < k$, $0 < r < k$, and $-k < s < -1$.

Next, we will describe some families of strongly regular graphs that will arise in the forthcoming sections. Given a prime power q with $q \equiv 1 \pmod 4$, the *Paley graph* $P(q)$ is the graph with $V(P(q)) = GF(q)$, with two vertices being adjacent if their difference is a nonzero square in $GF(q)$. It follows that $P(q)$ is an $srg(q, (q-1)/2, (q-5)/4, (q-1)/4)$. It is easily seen that $P(q)$ has an automorphism group, which is transitive on vertices, edges, and non-edges. Moreover, $P(q)$ is isomorphic to its complement.

Given an integer $n \geq 4$, the *triangular graph* $T(n)$ is the line graph of the complete graph K_n. It follows that $T(n)$ is an $srg(n(n-1)/2, 2(n-2), n-2, 4)$. $T(n)$ is known to be characterized by its parameters if $n \neq 8$. When $n = 8$, there are precisely three additional strongly regular graphs with the same parameters as $T(8)$. These are known as the Chang graphs.

Given an integer $n \geq 2$, the *lattice graph* $L_2(n)$ is the line graph of the complete bipartite graph $K_{n,n}$. $L_2(n)$ is an $srg(n^2, 2(n-1), n-2, 2)$, and is characterized by its parameters except when $n = 4$. When $n = 4$, there is precisely one additional strongly regular graph with the same parameters as $L_2(4)$. This graph is known as the Shrikhande graph.

A *system of n linked symmetric (v,k,λ) designs* [9] is a collection $\{X_0, \ldots, X_n\}$ of disjoint finite sets together with incidence relations $\mathcal{I}_{i,j}$, $i,j = 0, \ldots, n$, $i \neq j$, between each pair of distinct sets such that (i) for all $i,j = 0, \ldots, n$, $i \neq j$, the incidence structure $(X_i, X_j, \mathcal{I}_{i,j})$ is a symmetric

(v, k, λ) design, and (ii) for any three distinct sets X_i, X_j, and X_l, and any two points $p \in X_j$ and $q \in X_l$, the number of points $r \in X_i$ such that r is incident with both p and q is x if p and q are incident and is y otherwise. It follows that we must have $(x-y)^2 = k - \lambda$ and $y(k+x-y) = k\lambda$. The *incidence graph* of such a system has the elements of $\cup_{i=0}^{n} X_i$ as its vertices, with two vertices being adjacent when the corresponding points are incident. This incidence graph is strongly regular if and only if $n\lambda = y(n-1)$, in which case it is an $srg(v(n+1), kn, y(n-1), \lambda n)$; see [24].

3 Matrix-theoretic tools

In this section, we will present some matrix-theoretic results that we will need in the forthcoming sections and that are of general use in algebraic graph theory.

Given a real symmetric $v \times v$ matrix A, we will denote the eigenvalues of A (which must be real) by $\lambda_1(A) \geq \ldots \geq \lambda_v(A)$. If A is the $(0,1)$-adjacency matrix of a graph G, then we sometimes write $\lambda_i(G)$ instead of $\lambda_i(A)$, $i = 1, \ldots, v$. If B is a $v_1 \times v_1$ matrix with $v_1 \leq v$, then we say that the eigenvalues of B *interlace* the eigenvalues of A if B has only real eigenvalues and if $\lambda_i(A) \geq \lambda_i(B) \geq \lambda_{v-v_1+i}$ for $i = 1, \ldots, v_1$. We say that the interlacing is *tight* if there exists an integer l, $0 \leq l \leq v_1$, such that $\lambda_i(A) = \lambda_i(B)$ for $i = 1, \ldots, l$ and $\lambda_{v-v_1+i}(A) = \lambda_i(B)$ for $i = l+1, \ldots, v_1$.

The first two matrix results we will need give examples of eigenvalue interlacing. The first of these is sometimes referred to as the *Cauchy inequalities* [30].

Theorem 3.1 *Let A be a real symmetric $v \times v$ matrix partitioned as follows:*

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{1,2}^\top & A_{2,2} \end{pmatrix},$$

where $A_{1,1}$ is square. Then the eigenvalues of $A_{1,1}$ interlace the eigenvalues of A.

Theorem 3.2 [20], [22] *Let A be a real symmetric $v \times v$ matrix partitioned as follows:*

$$A = \begin{pmatrix} A_{1,1} & \ldots & A_{1,v_1} \\ \vdots & \ddots & \vdots \\ A_{1,v_1}^\top & \ldots & A_{v_1,v_1} \end{pmatrix},$$

where $A_{i,i}$ is square for $i = 1, \ldots, v_1$. Let $b_{i,j}$ be the average row sum of $A_{i,j}$ for $i, j = 1, \ldots, v_1$. Let $B = (b_{i,j})$. Then the eigenvalues of B interlace the eigenvalues of A, and if the interlacing is tight, then $A_{i,j}$ has constant row and column sums for $i, j = 1, \ldots, v_1$.

The matrix B is often called the *quotient matrix* of A with respect to the given partition. If all blocks of the partition have constant row and column sums, the partition is called *equitable* (or *regular*). Thus, the above theorem states that if the interlacing is tight, the partition is equitable. If $v_1 = 1$, we have the well-known property that the average row sum of a symmetric matrix lies between the smallest and the largest eigenvalue. For eigenvalue manipulations we shall frequently use the following property [30].

Proposition 3.3 *Let A and B be real symmetric matrices and suppose that A and B commute (that is, $AB = BA$). Then A and B have a common basis of eigenvectors.*

This result implies that the eigenvalues of $A + B$ can be obtained by adding corresponding eigenvalues of A and B. For example, if A has constant row sums and $B = J_v$ (J_v denotes the $v \times v$ all-ones matrix). Since J_v has all but one eigenvalue equal to 0, all but one eigenvalue of $A + J_v$ are also eigenvalues of A. The remaining eigenvalue is the row sum of $A + J_v$. If A and J_v do not commute, we can use the fact that J_v is positive semi-definite, which gives the following inequality.

Proposition 3.4 *Let A be a real symmetric $v \times v$ matrix and let $c \geq 0$. Then*

$$\lambda_i(A + cJ_v) \geq \lambda_i(A) \text{ for } i = 1 \ldots v.$$

The last matrix result we shall need concerns real symmetric matrices with just two distinct eigenvalues.

Theorem 3.5 [20] *Let A be a real symmetric $v \times v$ matrix partitioned as follows:*

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{1,2}^\top & A_{2,2} \end{pmatrix},$$

where $A_{1,1}$ is $v_1 \times v_1$. Suppose that A has eigenvalues r and s, $r > s$, of multiplicities f and $v - f$, respectively. Then

$$\lambda_i(A_{2,2}) = \begin{cases} r & \text{if } 1 \leq i \leq f - v_1, \\ s & \text{if } f + 1 \leq i \leq v - v_1, \\ r + s - \lambda_{f-i+1}(A_{1,1}) & \text{otherwise.} \end{cases}$$

4 Eigenvalues and chromatic number

We start with some eigenvalue inequalities for the chromatic number. The first result is due to Hoffman [27], see also [20].

Theorem 4.1 *If G is a graph on v vertices, then*

$$-\sum_{i=0}^{\chi(G)-2} \lambda_{v-i}(G) \geq \lambda_1(G).$$

As a direct consequence of this theorem we have $-\lambda_v(G)(\chi(G) - 1) \geq \lambda_1(G)$, which gives Hoffman's famous lower bound for the chromatic number:

Corollary 4.2 *If G is a non-empty graph on v vertices, then $\chi(G) \geq 1 - \frac{\lambda_1(G)}{\lambda_v(G)}$.*

The next inequality is less well-known [20], [22].

Theorem 4.3 *Suppose G is a graph on v vertices and let g be the multiplicity of $\lambda_v(G)$, If $\lambda_2(G) \neq 0$, then $\chi(G) \geq \min\{g + 1, 1 - \frac{\lambda_v(G)}{\lambda_2(G)}\}$.*

The latter result is useful for strongly regular graphs, since these graphs have large g. In fact the pentagon is the only primitive strongly regular graph for which $g + 1 < 1 - \frac{\lambda_v(G)}{\lambda_2(G)}$, see [20]. Hence,

Theorem 4.4 [20] *If G is a primitive strongly regular graph with eigenvalues k, r, and s ($k > r > s$), which is not the pentagon, then*

$$\chi(G) \geq 1 - \frac{k}{s} \quad \text{and} \quad \chi(G) \geq 1 - \frac{s}{r}.$$

By use of this theorem it is easy to prove that for a given value of χ there are only finitely many primitive strongly regular graphs with chromatic number χ. In [20], all primitive strongly regular graphs with chromatic number at most 4 are determined. No primitive strongly regular graph has chromatic number less than 3. Three such graphs have chromatic number 3 and 18 such graphs have chromatic number 4. The result is as follows.

Theorem 4.5 *If G is a 3-chromatic primitive strongly regular graph, then G is one of the following graphs: the pentagon $P(5)$ (the unique $srg(5,2,0,1)$), the Petersen graph $\overline{T(5)}$ (the unique $srg(10,3,0,1)$), or $L_2(3)$ (the unique $srg(9,4,1,2)$).*
If G is a 4-chromatic primitive strongly regular graph, then G is one of the following graphs: $\overline{T(6)}$ (the unique $srg(15,6,1,3)$), $L_2(4)$ (one of the two $srg(16,6,2,2)$), $\overline{L_2(4)}$ (one of the two $srg(16,9,4,6)$), the Shrikhande graph (the exceptional $srg(16,6,2,2)$), the Clebsch graph (the unique $srg(16,5,0,2)$ [36]), the Hoffman-Singleton graph (the unique $srg(50,7,0,1)$ [28]), the Gewirtz graph (the unique $srg(56,10,0,2)$ [4], [16]), or one of the eleven incidence graphs of a system of three linked symmetric $(16,6,2)$ designs [31].

The next results will be very useful for obtaining contradictions to the existence of 5-chromatic strongly regular graphs with various parameter sets.

Theorem 4.6 *Let G be a graph on v vertices and let g denote the multiplicity of the smallest eigenvalue $\lambda_v(G)$. Suppose G is colored with $\chi = \lceil 1 - \frac{\lambda_v(G)}{\lambda_2(G)} \rceil$ colors. Let C be the smallest color class. Then $|C| \geq g - \chi + 2$.*

Proof. Consider the graph G' induced by $V(G) \setminus C$. Clearly $\chi(G') = \chi - 1$. Assume that $|C| \leq g - \chi + 1$. Then by Theorem 3.1, G' has eigenvalue $\lambda_v(G)$ with multiplicity at least $g - |C| = \chi(G')$. Now Theorem 4.3 applied to G' gives $\chi(G') \geq 1 - \frac{\lambda_v(G)}{\lambda_2(G)}$, a contradiction. □

The next two results give upper bounds on the size of a coclique. The first is an unpublished result of Hoffman. The second one is due to Cvetković [12].

Theorem 4.7 [20], [22] *Let G be a k-regular graph with v vertices. Let C be a coclique in G. Then $|C| \leq v\lambda_v(G)/(\lambda_v(G) - k)$ with equality if and only if each vertex outside C is adjacent to exactly $-\lambda_v(G)$ vertices in C.*

Theorem 4.8 [12] *Let G be a graph with g non-positive eigenvalues. Then $\alpha(G) \leq g$.*

A coclique that meets Hoffman's bound will be called a *Hoffman coclique* and a coloring of a graph for which Hoffman's chromatic number bound (Corollary 4.2) is tight will be called a *Hoffman coloring*. Note that for a regular graph, the color classes in a Hoffman coloring are Hoffman cocliques. So, by Theorem 4.7, a Hoffman coloring in a regular graph gives an equitable partition of the adjacency matrix. Strongly regular graphs with a Hoffman coloring have been studied by Haemers and Tonchev [24]. They give a characterization in case both bounds of Theorem 4.4 are tight.

Theorem 4.9 [24], [13] *Let G be a primitive strongly regular graph with eigenvalues k, r, and s $(k > r > s)$. Suppose that G has a Hoffman coloring. Then*

$$kr \geq s^2$$

with equality if and only if G is the incidence graph of a system of linked symmetric designs.

5 Parameters of 5-chromatic strongly regular graphs

In this section, we find all possible parameter sets for 5-chromatic primitive strongly regular graphs. The computer program Mathematica [43] was used to generate the list below.

Theorem 5.1 *A 5-colorable primitive strongly regular graph must have one of the following 43 sets of eigenvalues (r,s) and parameters (v,k,λ,μ):*

1.	$(\frac{1}{2} \pm \frac{1}{2}\sqrt{5})$	$(5,2,0,1)$	16.	$(2,-7)$	$(76,21,2,7)$	31.	$(3,-12)$	$(225,48,3,12)$
2.	$(\frac{1}{2} \pm \frac{1}{2}\sqrt{13})$	$(13,6,2,3)$	17.	$(2,-6)$	$(77,16,0,4)$	32.	$(3,-10)$	$(196,39,2,9)$
3.	$(\frac{1}{2} \pm \frac{1}{2}\sqrt{17})$	$(17,8,3,4)$	18.	$(2,-6)$	$(64,18,2,6)$	33.	$(3,-9)$	$(210,33,0,6)$
4.	$(1,-4)$	$(21,10,3,6)$	19.	$(2,-5)$	$(57,14,1,4)$	34.	$(3,-9)$	$(165,36,3,9)$
5.	$(1,-4)$	$(25,16,9,12)$	20.	$(2,-5)$	$(49,16,3,6)$	35.	$(3,-7)$	$(176,25,0,4)$
6.	$(1,-3)$	$(16,5,0,2)$	21.	$(2,-4)$	$(56,10,0,2)$	36.	$(3,-7)$	$(125,28,3,7)$
7.	$(1,-3)$	$(15,6,1,3)$	22.	$(2,-4)$	$(40,12,2,4)$	37.	$(3,-6)$	$(162,21,0,3)$
8.	$(1,-3)$	$(16,9,4,6)$	23.	$(2,-4)$	$(36,14,4,6)$	38.	$(3,-5)$	$(115,18,1,3)$
9.	$(1,-2)$	$(10,3,0,1)$	24.	$(2,-4)$	$(35,16,6,8)$	39.	$(3,-5)$	$(96,19,2,4)$
10.	$(1,-2)$	$(9,4,1,2)$	25.	$(2,-3)$	$(50,7,0,1)$	40.	$(3,-5)$	$(85,20,3,5)$
11.	$(1,-2)$	$(10,6,3,4)$	26.	$(2,-3)$	$(26,10,3,4)$	41.	$(3,-4)$	$(99,14,1,2)$
12.	$(2,-8)$	$(100,22,0,6)$	27.	$(2,-3)$	$(25,12,5,6)$	42.	$(3,-3)$	$(45,12,3,3)$
13.	$(2,-8)$	$(76,30,8,14)$	28.	$(2,-2)$	$(16,6,2,2)$	43.	$(3,-2)$	$(25,8,3,2)$
14.	$(2,-8)$	$(75,32,10,16)$	29.	$(2,-2)$	$(15,8,4,4)$			
15.	$(2,-7)$	$(81,20,1,6)$	30.	$(3,-12)$	$(266,45,0,9)$			

Proof. Let G be a 5-colorable primitive $srg(v,k,\lambda,\mu)$ with eigenvalues k, r, and s $(r > s)$, with multiplicities 1, f, and g, respectively. If G has a non-integral eigenvalue, then $k = (v-1)/2$, $s = (-1-\sqrt{v})/2$, $v \equiv 1 \pmod 4$, and v is not an integral square. Now, by Corollary 4.2, we have $\chi(G) \geq \sqrt{v}$, and so $v \leq 25$. Thus, we must have $v = 5$, 13, or 17. The only strongly regular graphs on 5, 13, or 17 vertices have parameter sets 1, 2, and 3 in the list above.
Next, suppose r and s are both integers. The primitivity of G gives $s \leq -2$, $r \geq 1$, and $\mu > 0$. Also, $k + rs = \mu$ implies $k > -rs$, so $k \geq 3$, and by Corollary 4.2, $5 \geq 1 - k/s > 1 + r$, so $r = 1$, 2, or 3. Theorem 4.4 gives

$$-s \leq 4r \ , \text{ and } \ k \leq 4s \ .$$

This leads to a list of 442 feasible triples (r,s,k). For each triple we computed the other parameters and the multiplicities of the eigenvalues. We checked that $\lambda \geq 0$, $\mu \geq 1$, and that f and g are both integers. Then we obtain the parameter sets 4 through 43 in the list above. □

6 The known cases

In this section, we will deal with the parameter sets for which existence or non-existence of a 5-chromatic strongly regular graph is known.

The graphs with parameter sets 1, 6, 7, 9, 10, 21, and 25 are unique, and 4-colorable by Theorem 4.5. There are two graphs with parameter set 28, $L_2(4)$ and the Shrikhande graph. Both are 4-colorable by Theorem 4.5. Parameter set 8 is the complement of 28. $\overline{L_2(4)}$ is 4-chromatic by Theorem 4.5, but the complement of the Shrikhande graph has coclique number 3, and hence it has chromatic number at least $\lceil 16/3 \rceil = 6$, and hence equal to 6 by Brooks' theorem.

There is only one graph for each of the parameter sets 4, 11, and 29, namely $\overline{T(7)}$, $T(5)$, and $T(6)$, respectively (see Section 2). Both $T(5)$ and $T(6)$ are 5-chromatic since K_5 and K_6 both have edge chromatic number 5. $\overline{T(7)}$ is also 5-chromatic since $\overline{T(6)}$ is 4-chromatic by Theorem 4.5.

There is a unique graph for each of the parameter sets 2 and 3, namely the Paley graphs $P(13)$ and $P(17)$. By Theorem 4.5, $P(13)$ is not 4-colorable. Clearly, $\{0,2,7\} \cup \{1,6,8\} \cup \{3,5,11\} \cup \{4,9\} \cup \{10,12\}$ is a partition of the vertices into five cocliques. So, $P(13)$ is 5-chromatic. In $P(17)$, 0 and 1 are adjacent vertices with three common neighbors, namely 2, 9 and 16. These three vertices are mutually non-adjacent, so the edge $\{0,1\}$ is not contained in a 4-clique. The automorphism group of a Paley graph is edge transitive, so $P(17)$ has no 4-clique. The Paley graph is isomorphic to its complement. Hence, $P(17)$ has no 4-cocliques and therefore $P(17)$ is not 5-chromatic. In fact, the vertices of $P(17)$ can be covered with six translates of $\{0,1,2\}$, which implies that $P(17)$ is 6-chromatic.

There is also a unique graph for each of the parameter sets 5 and 43, namely $\overline{L_2(5)}$ and $L_2(5)$, respectively. Both are 5-chromatic. For $\overline{L_2(5)}$ this is trivial, and a 5-coloring of $L_2(5)$ is given by the symbols of an arbitrary latin square of order 5.

There are exactly 3854 $srg(35,16,6,8)$ [32], [41]. However, none of them are 5-chromatic. Suppose G is such a graph. The eigenvalues of G are $k = 16$, $r = 2$, and $s = -4$. Since $1 - k/s = 5$, a 5-coloring is a Hoffman coloring. In [24] it is proved that G has no Hoffman coloring. Therefore, there does not exist a 5-chromatic $srg(35,16,6,8)$.

Similarly, an $srg(75,32,10,16)$ has eigenvalues $k = 32$, $r = 2$, and $s = -8$. Since $1 - k/s = 5$, a 5-coloring of such a graph would be a Hoffman coloring. By Theorem 4.9, a 5-colorable $srg(75,32,10,16)$ would have to be the incidence graph of a system of four linked symmetric designs. However, this is not possible since it would have a non-integral intersection parameter $y = 10/3$. Therefore, there exists no 5-chromatic $srg(75,32,10,16)$. This fact was observed in [24].

Brouwer [1] proved that there is a unique $srg(77,16,0,4)$. This graph G is the complement of the block graph of the unique quasi-symmetric 3-$(22,6,1)$ design (the extension of the projective plane of order 4; see for example [19]). In [20] it was already observed that this graph is 5-colorable. Indeed, it is well-known (and easy to see) that G has a coclique of size 21 and that the graph induced by the remaining 56 vertices is the Gewirtz graph (the unique $srg(56,10,0,2)$), which is 4-chromatic. So, the $srg(77,16,0,4)$ is 5-chromatic.

There does not exist a strongly regular graph with parameter set 16, 19, or 20 [21], [42], [7].

7 Some theoretic non-existence results

Here we prove that for the parameter sets 13, 15, 30, 32, and 34, no 5-chromatic strongly regular graph exists. We treat them in order of difficulty and start with the easy cases.

Theorem 7.1 *There exists no 5-chromatic* $srg(81,20,1,6)$.

Proof. Let G be such a strongly regular graph with chromatic number 5. The eigenvalues of G are $k = 20$, $r = 2$, and $s = -7$ with multiplicities 1, $f = 60$, and $g = 20$, respectively. We apply Theorem 4.6. Since $5 = \lceil 1 - s/r \rceil$, we find that the smallest color class C has size $|C| \geq g - 5 + 2 = 17$, but the average size is $81/5$, a contradiction. □

In fact, there is a unique $srg(81,20,1,6)$ [3]. By use of GRAPE, it was found that the maximum size of a coclique is 15, which confirms the above result.

Theorem 7.2 *There exists no 5-chromatic* $srg(196,39,2,9)$.

Proof. Let G be such a strongly regular graph with chromatic number 5. The eigenvalues of G are $k = 90$, $r = 3$, and $s = -10$ with multiplicities 1, $f = 147$, and $g = 48$, respectively. Again we apply Theorem 4.6. Since $5 = \lceil 1 - s/r \rceil$, the smallest color class C has size $|C| \geq g - 5 + 2 = 45$, but the average size is $196/5$, a contradiction. □

7.1 The parameter set $(266,45,0,9)$

According to [2], it is not known whether or not an $srg(266,45,0,9)$ exists. However, in this section we will show that no such graph can be 5-chromatic.

Theorem 7.3 *There does not exist a 5-chromatic* $srg(266,45,0,9)$.

Proof. Let G be an $srg(266,45,0,9)$ with adjacency matrix A. Then G has eigenvalues $k = 45$, $r = 3$, and $s = -12$ with multiplicities 1, $f = 209$, and $g = 56$, respectively. Suppose that G is colored with five colors. By Theorem 4.6, the size of the smallest color class is at least $g - 5 + 2 = 53$. This implies that there are four color classes of size 53 and one of size 54. Let C be the color class of size 54. Let G' be the subgraph of G induced by $V(G) \setminus C$ and let A' be the adjacency matrix

of G'. Clearly, G' is 4-chromatic. Consider the matrix $\tilde{A} = A - \frac{3}{19}J_{266}$. Then $\tilde{A}$ has just two distinct eigenvalues, namely 3 with multiplicity 210 and -12 with multiplicity 56. Now we apply Theorem 3.4 to $\tilde{A}$ and find that $A' - \frac{3}{19}J_{212}$ has eigenvalues 3, $-9/19$, -9, and -12 with multiplicities 156, 1, 53, and 2, respectively. Hence, $\lambda_{211}(G') \geq \lambda_{212}(G') \geq -12$ and $\lambda_{210}(G') \geq -9$. Moreover, A' has average row sum $45 - (45 \cdot 54)/212 > 33$, hence $\lambda_1(G') > 33$. Now Theorem 4.1 gives

$$12 + 12 + 9 \geq -\lambda_{212}(G') - \lambda_{211}(G') - \lambda_{210}(G') \geq \lambda_1(G') > 33,$$

a contradiction. □

7.2 The parameter set $(165, 36, 3, 9)$

It is not known whether such a graph exists. But we can say the following.

Theorem 7.4 *There does not exist a 5-chromatic srg*$(165, 36, 3, 9)$.

Proof. Let G be an $srg(165, 36, 3, 9)$. Then G has eigenvalues $k = 36$, $r = 3$, and $s = -9$ with multiplicities 1, $f = 120$, and $g = 44$, respectively. Then a 5-coloring of G must be a Hoffman coloring consisting of five cocliques of size 33. Let G be 5-colored and let C_1 and C_2 be two of the color classes. Partition $V(G)$ into C_1, C_2, and $V(G) \setminus (C_1 \cup C_2)$. Let A be an adjacency matrix for G. Then we can assume that A is in the following form:

$$A = \begin{pmatrix} 0 & A_{1,2} & A_{1,3} \\ A_{1,2}^\top & 0 & A_{2,3} \\ A_{1,3}^\top & A_{2,3}^\top & A_{3,3} \end{pmatrix},$$

where the two 0's on the diagonal are each 33×33 and $A_{1,2}$, $A_{1,2}^\top$, $A_{1,3}^\top$, and $A_{2,3}^\top$ all have constant row sums of $-s = 9$ by Theorem 4.7. Define

$$A_1 = \begin{pmatrix} 0 & A_{1,2} \\ A_{1,2}^\top & 0 \end{pmatrix}$$

and let G_1 be the induced subgraph of G with adjacency matrix A_1. Then G_1 is a 9-regular bipartite graph on 66 vertices and so $\lambda_1(G_1) = 9$ and $\lambda_{66}(G_1) = -9$. By Theorem 3.1, we also have $\lambda_2(A) = 3 \geq \lambda_2(A_1)$, and so G_1 has eigenvalue 9 with multiplicity 1 and so is connected. Now, $A - \frac{1}{5}J_{165}$ has just two distinct eigenvalues, namely $r = 3$ and $s = -9$ with multiplicities $f + 1 = 121$ and $g = 44$, respectively. Therefore, by Theorem 3.5 with $i = 98, 99$, we have

$$\lambda_{98}(A_{3,3} - \tfrac{1}{5}J_{99}) = -6 - \lambda_{24}(A_1 - \tfrac{1}{5}J_{66}) \text{ and } \lambda_{99}(A_{3,3} - \tfrac{1}{5}J_{99}) = -6 - \lambda_{23}(A_1 - \tfrac{1}{5}J_{66}).$$

Now, by Proposition 3.4 with $i = 98, 99$, we have (we often write J instead of J_v)

$$\lambda_{98}(A_{3,3} - \tfrac{1}{5}J) \leq \lambda_{98}(A_{3,3}) \text{ and } \lambda_{99}(A_{3,3} - \tfrac{1}{5}J) \leq \lambda_{99}(A_{3,3}).$$

In fact, since $A_{3,3}$ and J commute, we actually have equality for both cases. Let G_3 be the induced subgraph of G with adjacency matrix $A_{3,3}$. Then G_3 is 3-chromatic and the average row sum of $A_{3,3}$ is 18, and so by Theorems 4.1 and 3.2 we must have

$$\lambda_{23}(A_1 - \tfrac{1}{5}J) + \lambda_{24}(A_1 - \tfrac{1}{5}J) + 12 \geq -\lambda_{99}(G_3) - \lambda_{98}(G_3) \geq \lambda_1(G_3) \geq 18.$$

This implies that

$$\lambda_{23}(A_1 - \tfrac{1}{5}J) + \lambda_{24}(A_1 - \tfrac{1}{5}J) \geq 6.$$

Now, A_1 and $A_1 - \frac{1}{5}J$ have the same eigenvalues except for the eigenvalue 9 of A_1 which becomes $-21/5$ for $A_1 - \frac{1}{5}J$. Therefore, $\lambda_{23}(A_1 - \frac{1}{5}J), \lambda_{24}(A_1 - \frac{1}{5}J) \leq 3$, which implies that $\lambda_{23}(A_1 - \frac{1}{5}J) = \lambda_{24}(A_1 - \frac{1}{5}J) = 3$. Thus, G_1 has eigenvalue 3 with multiplicity at least 24. Since G_1 is bipartite, it also has eigenvalue -3 with multiplicity at least 24. Now, the sum of the squares of the eigenvalues of G_1 is equal to $66 \cdot 9 = 594$. Also, $1 \cdot 9^2 + 24 \cdot 3^2 + 24 \cdot (-3)^2 + 1 \cdot (-9)^2 = 594$. Therefore, the 16 remaining eigenvalues of G_1 must all be 0, and thus we find the following minimal polynomial for A_1.

$$(A_1 - 9I)(A_1 - 3I)A_1(A_1 + 3I)(A_1 + 9I) = 0$$

Since G_1 is 9-regular and connected, the kernel of $(A_1 - 9I)$ is the span of the all-ones vector $\mathbf{j}$. This implies that

$$(A_1 - 3I)A_1(A_1 + 3I)(A_1 + 9I) = cJ$$

for some constant c (in fact $c = 6 \cdot 9 \cdot 12 \cdot 18/66$, but we won't use it). The above polynomial is called the Hoffman polynomial of G_1, see [26]. Next observe that the following matrices have constant diagonal: J, I, A_1, A_1^2 (because G_1 is regular), and A_1^3 (because G_1 is bipartite). Therefore A_1^4 also has a constant diagonal, hence $\mathrm{trace}(A_1^4)$ is divisible by 66. But,

$$\mathrm{trace}(A_1^4) = \sum_{i=1}^{66} \lambda_i(A_1)^4 = 48 \cdot 3^4 + 2 \cdot 9^4 = 2 \cdot 3^5 \cdot 5 \cdot 7.$$

Contradiction. □

7.3 The parameter set $(76, 30, 8, 14)$

Again, it is not known whether such a graph exists, but if it does, it will have chromatic number at least 6.

Theorem 7.5 *There does not exist a 5-chromatic* $srg(76, 30, 8, 14)$.

Proof. Let G be an $srg(76, 30, 8, 14)$ with adjacency matrix A. Then A has eigenvalues $k = 30$, $r = 2$, and $s = -8$ with multiplicities 1, $f = 57$, and $g = 18$, respectively. Suppose that G is 5-chromatic. By Theorem 4.6, each color class has size at least $18 - 5 + 2 = 15$. Therefore, G has one color class C of size 16 and four of size 15. By Theorem 4.7, the subgraph G' induced by $V(G) \setminus C$ is regular of degree 22. Let A' be the adjacency matrix of G'. The matrix $A - \frac{1}{2}J$ has just two eigenvalues: 2 with multiplicity 57 and -8 with multiplicity 19. So, Theorem 3.5 gives the eigenvalues of $A' - \frac{1}{2}J$. They are 2, -6, and -8 with multiplicities 42, 15, and 3, respectively. Since G' is regular, A' and J commute and we find that G' has eigenvalues 22, 2, -6, and -8 with multiplicities 1, 42, 15, and 2, respectively. Clearly, G' is colorable with four classes of size 15. Next, consider the partition of A' corresponding to the color classes of G'. Let B' be the quotient matrix consisting of the average row sums of the blocks of this partition. Clearly, B' has zero entries on the diagonal and row sum 22, hence $\lambda_1(B') = 22$. Eigenvalue interlacing (Theorem 3.2) gives that $\lambda_4(B') \geq -8$, $\lambda_3(B') \geq -8$, and $\lambda_2(B') \geq -6$. But, $\mathrm{trace}(B') = 0 = \sum_{i=1}^{4} \lambda_i(B')$, so the eigenvalues of B' are -8, -8, -6, and 22. The interlacing is tight, hence the color partition is equitable and the entries of B' are integers. Let b be an off-diagonal entry of B'. Then $\left(\begin{smallmatrix} 0 & b \\ b & 0 \end{smallmatrix}\right)$ is a principal submatrix of B' with eigenvalues $\pm b$, hence by Theorem 3.1, $-b \geq -8$, so $b \leq 8$. This leads to two possible rows of B': $(0, 7, 7, 8)$ and $(0, 6, 8, 8)$, and just two possibilities for B':

$$B' = \begin{pmatrix} 0 & 7 & 7 & 8 \\ 7 & 0 & 8 & 7 \\ 7 & 8 & 0 & 7 \\ 8 & 7 & 7 & 0 \end{pmatrix} \quad \text{or} \quad B' = \begin{pmatrix} 0 & 6 & 8 & 8 \\ 6 & 0 & 8 & 8 \\ 8 & 8 & 0 & 6 \\ 8 & 8 & 6 & 0 \end{pmatrix}.$$

However, the second possibility is impossible since it has eigenvalues -10, -6, -6, and 22. Thus, A' admits an equitable partition with quotient matrix B' given on the left above. Next, we partition $V(G)$ into three sets: C, C_1 consisting of the first two color classes of G' (ordered according to the given B'), and C_2 formed by the last two color classes of G'. The quotient matrix of the average row sums of this partition is clearly

$$B = \begin{pmatrix} 0 & 15 & 15 \\ 8 & 7 & 15 \\ 8 & 15 & 7 \end{pmatrix}$$

with eigenvalues -8, -8, and 30. Again the interlacing is tight, so this partition is also equitable. Next, define

$$K = \begin{pmatrix} \frac{1}{16}J_{16} & 0 & 0 \\ 0 & \frac{1}{30}J_{30} & 0 \\ 0 & 0 & \frac{1}{30}J_{30} \end{pmatrix}.$$

Then K has eigenvalues 1 and 0 with multiplicities 3 and 73, respectively. Moreover, K commutes with A since the corresponding partition of A is equitable. Now A, J_{76}, and K have a common basis of eigenvectors. Three eigenvectors for eigenvalue 1 of K are $\mathbf{j}_{76}$ (that is, the all-ones vector of dimension 76), $(-15\mathbf{j}_{16}^\top \ \ 4\mathbf{j}_{60}^\top)^\top$, and $(0 \ \ \mathbf{j}_{30}^\top \ \ -\mathbf{j}_{30}^\top)^\top$. They are also eigenvectors of A for eigenvalues 30, -8, and -8, respectively. Define

$$\tilde{A} = A + 10K - 2I - \tfrac{1}{2}J_{76}.$$

Then $\tilde{A}$ has eigenvalues 0 and -10 with respective multiplicities 60 and 16. Thus, rank $\tilde{A} = 16$. Take a triangle (x, y, z) in G with $x \in C$, $y \in C_1$, and $z \in C_2$ (since adjacent x and y have at most 7 common neighbors in $C \cup C_1$, there is at least one common neighbor z in C_2). Consider the subgraph of G induced by $(C \setminus \{x\}) \cup \{y\} \cup \{z\}$ and let $\tilde{A}_1$ be the corresponding submatrix of $\tilde{A}$. Then

$$\tilde{A}_1 = \begin{pmatrix} \frac{1}{8}J_{15} - 2I & \mathbf{v} & \mathbf{w} \\ \mathbf{v}^\top & -\frac{13}{6} & \frac{1}{2} \\ \mathbf{w}^\top & \frac{1}{2} & -\frac{13}{6} \end{pmatrix},$$

where $\mathbf{v}$ and $\mathbf{w}$ are vectors with 15 entries, of which seven are $\frac{1}{2}$ and eight are $-\frac{1}{2}$. So, without loss of generality, there are just eight possibilities for $\tilde{A}_1$, and it turns out that in all cases rank $\tilde{A}_1 = 17$, which is impossible since rank $\tilde{A} = 16$. □

Remark 7.6 The parameter sets $(76, 30, 8, 4)$ and $(75, 32, 10, 16)$ are related by Seidel switching (see for example [37]). Adding an isolated vertex to a 5-chromatic $srg(75, 32, 10, 16)$ and switching with respect to two color classes gives a 5-chromatic $srg(76, 30, 8, 4)$. Thus, the above theorem confirms the non-existence of a 5-chromatic $srg(75, 32, 10, 16)$.

8 Computer results

In this section, we use the computer algebra system GAP [15] and the share package GRAPE [38] for computing with graphs to determine the 5-chromatic strongly regular graphs with parameter sets 12, 18, 22, 23, 26, 27, and 42.

The philosophy behind GRAPE is that a graph G comes equipped with a user-specified group of automorphisms $\Gamma \leq Aut(G)$ which is used to make GRAPE's graph-theoretic algorithms run more efficiently. GRAPE possesses a function that will return a set of representatives of the set of complete subgraphs of G of size n, where n is specified by the user. This function will return at least one complete subgraph from each orbit of the set of K_n subgraphs under the action of Γ. Thus, if Γ is

the trivial group, this function will return all complete subgraphs of a given size of G. However, we do lose the increased efficiency that a large group of automorphisms would provide. GRAPE also possesses a function to compute the complement of a given graph. Since a complete subgraph of $\overline{G}$ is a coclique of G, we can use these two functions to find all cocliques of a given size in G. This allows us to determine with certainty if a given graph has a coloring with a particular combination of color class sizes, and hence if it has a given chromatic number. This can be time-consuming, but the graphs that we consider in this section are all small enough to make this approach feasible.

8.1 The parameter set $(25, 12, 5, 6)$

There are exactly 15 $srg(25, 12, 5, 6)$. These were found by Paulus [33] and shown to be all such graphs by Rozenfel′d [35]. Adjacency matrices of these graphs are available at [41] and in what follows we will denote these graphs by $G_{25,i}$, $i = 1, \ldots, 15$, where the ordering is the same as on [41]. Now, an $srg(25, 12, 5, 6)$ has smallest eigenvalue $s = -3$, so if such a graph were 5-colorable, it would have a Hoffman coloring consisting of five pairwise disjoint cocliques of size 5. Using GRAPE, it was shown that the only graphs with such a coloring are $G_{25,11}$ and $G_{25,15}$. Hence, there are exactly two 5-chromatic $srg(25, 12, 5, 6)$. Now, there is exactly one latin square of order 5 with an orthogonal mate and so one of the two 5-chromatic $srg(25, 12, 5, 6)$ is the graph arising from this latin square, where the symbols of the orthogonal mate give the color classes; see [24].

Theorem 8.1 *There are exactly two* 5*-chromatic* $srg(25, 12, 5, 6)$.

Remark 8.2 The remaining 13 $srg(25, 12, 5, 6)$ that are not 5-colorable all have six pairwise disjoint cocliques of sizes 5, 4, 4, 4, 4, and 4. Therefore, these 13 graphs are all 6-chromatic.

8.2 The parameter set $(26, 10, 3, 4)$

There are exactly 10 $srg(26, 10, 3, 4)$. These were found by Paulus [33] and shown to be all such graphs by Rozenfel′d [35]. Adjacency matrices of these graphs are available at [41] and in what follows we will denote these graphs by $G_{26,i}$, $i = 1, \ldots, 10$. Now, an $srg(26, 10, 3, 4)$ has smallest eigenvalue $s = -3$, and so it has cocliques of size no greater than $26(-3)/(-3-10) = 6$ by Theorem 4.7. Using GRAPE, it was shown that the only graph with a pair of disjoint cocliques of size 6 is $G_{26,8}$. Thus, a 5-coloring of any of these graphs except $G_{26,8}$ must consist of one color class of size 6 and four color classes of size 5. Using GRAPE, it was shown that of these graphs, only $G_{26,1}$ and $G_{26,2}$ possess such a coloring. It was also shown that $G_{26,8}$ has such a coloring as well. Thus, there are exactly three 5-chromatic $srg(26, 10, 3, 4)$.

Theorem 8.3 *There are exactly three* 5*-chromatic* $srg(26, 10, 3, 4)$.

Remark 8.4 The remaining seven $srg(26, 10, 3, 4)$ that are not 5-colorable all have six pairwise disjoint cocliques of sizes 6, 4, 4, 4, 4, and 4. Therefore, these seven graphs are all 6-chromatic.

8.3 The parameter set $(36, 14, 4, 6)$

McKay and Spence [32] have done a complete enumeration of strongly regular graphs with parameters $(36, 14, 4, 6)$. They found there to be exactly 180 such graphs. Adjacency matrices of these graphs are available at [41] and in what follows we will denote these graphs by $G_{36,i}$, $i = 1, \ldots, 180$. Now, if such a graph is 5-colorable, it must possess a coclique of size $\lceil 36/5 \rceil = 8$. However, using GRAPE, it was shown that only the graphs $G_{36,48}$ and $G_{36,77}$ have a coclique of size 8. What is more, they are both 5-chromatic. Thus, there are exactly two 5-chromatic $srg(36, 14, 4, 6)$. We also obtain the following characterization of the 5-chromatic $srg(36, 14, 4, 6)$.

Theorem 8.5 *There are exactly two* 5*-chromatic* $srg(36, 14, 4, 6)$. *An* $srg(36, 14, 4, 6)$ *has chromatic number* 5 *if and only if it has a coclique of size* 8.

8.4 The parameter set $(40, 12, 2, 4)$

Spence [39] has done a complete enumeration of strongly regular graphs with parameters $(40, 12, 2, 4)$. He found there to be exactly 28 such graphs. Adjacency matrices of these graphs are available at [41] and in what follows we will denote these graphs by $G_{40,i}$, $i = 1, \ldots, 28$. Now, a 5-colorable $srg(40, 12, 2, 4)$ must have a coclique of size $40/5 = 8$. Using GRAPE, it was shown that the only graphs possessing a coclique of size 8 are the 11 graphs $G_{40,i}$, $i = 1, \ldots, 11$. We then used GRAPE and found that each of these 11 graphs possesses a set of five pairwise disjoint cocliques of sizes 9, 9, 8, 8, and 6. Hence, there are exactly 11 $srg(40, 12, 2, 4)$ with chromatic number 5. We also obtain the following characterization of the 5-chromatic $srg(40, 12, 2, 4)$.

Theorem 8.6 *There are exactly* 11 $srg(40, 12, 2, 4)$ *with chromatic number* 5. *An* $srg(40, 12, 2, 4)$ *has chromatic number* 5 *if and only if it has a coclique of size* 8.

8.5 The parameter set $(45, 12, 3, 3)$

Spence [40] has done a complete enumeration of strongly regular graphs with parameters $(45, 12, 3, 3)$. He found there to be exactly 78 such graphs. Adjacency matrices of these graphs are available at [41] and in what follows we will denote these graphs by $G_{45,i}$, $i = 1, \ldots, 78$. Now, an $srg(45, 12, 3, 3)$ has smallest eigenvalue $s = -3$, so if such a graph were 5-colorable, it would have a Hoffman coloring consisting of five pairwise disjoint cocliques of size 9. Using GRAPE, we showed that all of these graphs except $G_{45,24}$, $G_{45,48}$, and $G_{45,71}$ have such a 5-coloring. Hence, there are exactly 75 $srg(45, 12, 3, 3)$ with chromatic number 5. One of these is the collinearity graph of the unique generalized quadrangle of order $(4, 2)$ [34]. Indeed, this generalized quadrangle has a fan (a partition into ovoids), giving a Hoffman coloring in the collinearity graph; see [24].

Theorem 8.7 *There are exactly* 75 $srg(45, 12, 3, 3)$ *with chromatic number* 5.

Remark 8.8 The remaining three $srg(45, 12, 3, 3)$ that are not 5-colorable each have six pairwise disjoint cocliques of sizes 9, 9, 8, 8, 8, and 3. Therefore, these three graphs are each 6-chromatic.

8.6 The parameter set $(64, 18, 2, 6)$

Haemers and Spence [23] have done a complete enumeration of strongly regular graphs with parameters $(64, 18, 2, 6)$. They found there to be exactly 167 such graphs. Adjacency matrices of these graphs are available at [41] and in what follows we will denote these graphs by $G_{64,i}$, $i = 1, \ldots, 167$. Using the adjacency matrices on [41], these 167 graphs were constructed in GRAPE. Now, a 5-colorable $srg(64, 18, 2, 6)$ must have a coclique of size $\lceil 64/5 \rceil = 13$. It was found that the 16 graphs $G_{64,i}$, $i = 5$, 6, 7, 10, 16, 32, 36, 47, 48, 50, 59, 79, 83, 85, 86, 128, do not have a coclique of size 13 and are therefore not 5-colorable. It was also found that only the 11 graphs $G_{64,i}$, $i = 1$, 2, 3, 4, 8, 9, 13, 14, 18, 19, 20, have a coclique of size 16. Since there are exactly 11 $srg(64, 18, 2, 6)$ with chromatic number 4 by Theorem 4.5, these graphs must all be 4-chromatic. Additionally, the two graphs $G_{64,27}$ and $G_{64,54}$ both have 5-colorings with four color classes of size 13 and one color class of size 12, the 45 graphs $G_{64,i}$, $i = 15$, 17, 21, 22, 23, 28, 29, 30, 31, 34, 35, 39, 40, 41, 42, 43, 44, 45, 46, 60, 62, 64, 65, 66, 68, 70, 71, 74, 75, 76, 78, 90, 94, 95, 96, 97, 100, 109, 117, 118, 119, 132, 133, 139, 158, all have 5-colorings with four color classes of size 14 and one color class of size 8, and the 13 graphs $G_{64,i}$, $i = 25$, 38, 56, 57, 61, 63, 69, 84, 91, 108, 110, 130, 137, all have 5-colorings with four color classes of size 15 and one color class of size 4. By trying all possible combinations of color class sizes, it can be shown that the remaining 80 graphs are not 5-colorable. Thus, there are exactly 60 $srg(64, 18, 2, 6)$ with chromatic number 5. We also obtain the following characterization of the 4-chromatic $srg(64, 18, 2, 6)$.

Theorem 8.9 *There are exactly* 60 $srg(64, 18, 2, 6)$ *with chromatic number* 5. *An* $srg(64, 18, 2, 6)$ *has chromatic number* 4 *if and only if it has a coclique of size* 16.

8.7 The parameter set $(100, 22, 0, 6)$

There is a unique strongly regular graph with parameter set 12 [17]. This graph is known as the Higman-Sims graph and can be constructed from the Higman-Sims group, a primitive rank 3 simple group [25].

By Theorem 2.3, the multiplicity of the smallest eigenvalue of this graph is $g = 22$. Therefore, by Theorem 4.8, this graph has cocliques of size no greater than 22. Using the library of primitive groups that is available in GAP, the Higman-Sims graph was constructed within GRAPE as an orbital graph of the Higman-Sims group. It was found that this graph has exactly 100 cocliques of size 22. Clearly, these cocliques are the 100 neighborhoods of a vertex, hence two cocliques of size 22 intersect in exactly 0 or 6 vertices. Therefore, by deleting two vertices in all possible ways from the 100 cocliques of size 22, we see that this graph has at least $22 \cdot 21 \cdot 100/2 = 23100$ cocliques of size 20. However, using GRAPE it was shown that this graph has exactly 23100 cocliques of size 20. Therefore, any coclique of size at least 20 is contained in the neighborhood of a vertex. Assume that the Higman-Sims graph is 5-colorable. Then there is at least one color class C of size $|C| \geq 20$ and the subgraph G' induced by the remaining vertices is 4-colorable. Since C is a subset of the neighborhood of some vertex x, G' contains the graph induced by the non-neighbors of x, which is the unique $srg(77, 16, 0, 4)$ (see [6], [19], or [25]). This graph is 5-chromatic (see Section 6), so G' is not 4-colorable and the Higman-Sims graph is not 5-colorable.

Theorem 8.10 *There does not exist a 5-colorable* $srg(100, 22, 0, 6)$.

Remark 8.11 Since the graph on the non-neighbors of a vertex x is 5-colorable, the Higman-Sims graph can be colored with six colors by giving x one of these five colors, and the neighbors of x the new color.

9 The remaining parameter sets

The parameter sets that remain to be dealt with are numbers 31, 33, 35, 36, 37, 38, 39, 40, and 41. Below we make some observations about these sets.

According to [2], it is not even known if a single strongly regular graph with parameter set 31, 33, 35, 37, 38, or 41 exists.

By Theorems 4.7 and 4.9, a 5-chromatic strongly regular graph with parameter set 31 must be the incidence graph of a system of four linked symmetric $(45, 12, 3)$ designs. Unfortunately, these are large and have not been enumerated like the systems of three linked symmetric $(16, 6, 2)$ designs were in [31].

Parameter sets 33, 35, and 37 are the three smallest triangle-free sets for which existence is in question, the fourth smallest set being $(266, 45, 0, 9)$, which we have seen is not 5-colorable. There are only seven triangle-free primitive strongly regular graphs known to exist, and there seems to be nothing known about parameter sets 33, 35, and 37. There also seems to be nothing known about sets 38, and 41.

There are at least three strongly regular graph with parameter set 39, see [5]. Sven Reichard and Misha Klin (private communication) have checked by computer that none of the three has a 20-coclique, and hence none is 5-chromatic.

A 5-coloring of a strongly regular graph with parameter set 36 or 40 would have to be a Hoffman coloring. Set 40 is on Haemers's and Tonchev's list [24] of small strongly regular graphs that are feasible for a Hoffman coloring. It is currently the fourth smallest set for which existence of such a coloring is unknown (the set $(36, 15, 6, 6)$ was settled in the negative in [8]). Perhaps the techniques of [8] could be used to enumerate all strongly regular graphs with parameter sets 36 and 40 that possess a Hoffman coloring. However, these graphs are much larger and so this might not be feasible. Parameter set 40 also belongs to the collinearity graph of the unique generalized quadrangle of order 4; see [34]. This graph is called *geometric*. However, there probably exist non-geometric graphs with the same parameter set. A Hoffman coloring of the geometric $srg(85, 20, 3, 5)$ would

correspond to a partition of the points of the generalized quadrangle into ovoids (a *fan*). However, the generalized quadrangle does not have a fan [34]. Similarly, set 36 belongs to the collinearity graph of a generalized quadrangle of order $(4, 6)$, of which there is one example known [34]. This known example has no fan, so the collinearity graph is not 5-colorable. The generalized quadrangle may be unique. However, there may exist many such graphs besides the geometric one(s).

10 Conclusion

In this final section, we summarize our results.

Theorem 10.1 *Let G be a 5-chromatic primitive strongly regular graph. Then G is isomorphic to one of the graphs on the following list:*

(1) $T(5)$ *(the unique* $srg(10, 6, 3, 4)$*),*

(2) $P(13)$ *(the unique* $srg(13, 6, 2, 3)$*),*

(3) $T(6)$ *(the unique* $srg(15, 8, 4, 4)$*),*

(4) $\overline{T(7)}$ *(the unique* $srg(21, 10, 3, 6)$*),*

(5) $L_2(5)$ *(the unique* $srg(25, 8, 3, 2)$*),*

(6) *one of the two* $srg(25, 12, 5, 6)$ $G_{25,i}$ $(i = 11, 15)$*,*

(7) $\overline{L_2(5)}$ *(the unique* $srg(25, 16, 9, 12)$*),*

(8) *one of the three* $srg(26, 10, 3, 4)$ $G_{26,i}$ $(i = 1, 2, 8)$*,*

(9) *one of the two* $srg(36, 14, 4, 6)$ $G_{36,i}$ $(i = 48, 77)$ *with coclique number* 8*,*

(10) *one of the* 11 $srg(40, 12, 2, 4)$ $G_{40,i}$ $(i = 1, \ldots, 11)$ *with a coclique of size* 8*,*

(11) *one of the* 75 $srg(45, 12, 3, 3)$ $G_{45,i}$ $(i \neq 24, 48, 71)$*,*

(12) *one of the* 60 $srg(64, 18, 2, 6)$ $G_{64,i}$ $(i = 15, 17, 21, 22, 23, 25, 27, 28, 29, 30, 31, 34, 35, 38, 39, 40, 41, 42, 43, 44, 45, 46, 54, 56, 57, 60, 61, 62, 63, 64, 65, 66, 68, 69, 70, 71, 74, 75, 76, 78, 84, 90, 91, 94, 95, 96, 97, 100, 108, 109, 110, 117, 118, 119, 130, 132, 133, 137, 139, 158)$ *with coclique number less than* 16*,*

(13) *the unique* $srg(77, 16, 0, 4)$*,*

or G is possibly the incidence graph of a system of four linked symmetric $(45, 12, 3)$ *designs (an* $srg(225, 48, 3, 12)$*), or G has one of the parameter sets* $(85, 20, 3, 5)$ *(non-geometric),* $(96, 19, 2, 4)$*,* $(99, 14, 1, 2)$*,* $(115, 18, 1, 3)$*,* $(125, 28, 3, 7)$*,* $(162, 21, 0, 3)$*,* $(176, 25, 0, 4)$*, or* $(210, 33, 0, 6)$*.*

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