Cooperative Games and Disjunctive Permission Structures*

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Abstract

In many (internal) organization structures economic decisions are made through a chain of decision makers. In this paper we give a game theoretic analysis of such hierarchical organization structures: Every participant has to get permission for his actions from at least one chain consisting of superiors. This assumption forms the foundation of the *disjunctive approach* to cooperative games with a permission structure. A computational method for the study of these disjunctive games with a permission structure is provided.

We show that the disjunctive approach implies that there is competition among superiors over the leadership of a subordinate, which may lead to a higher payoff of the subordinate in the presence of more superiors. This feature is used to show that even in the presence of small transaction costs the formation of a hierarchical production organization may be Pareto superior to the situation without such a hierarchical firm.

JEL codes: C71, C72.

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1 Introduction

Our goal is to analyze the consequences of the adoption of a hierarchical authority structure on the set of players in the context of a cooperative game with transferable utilities. In this analysis we suppose that the authority structure is exogenously given and puts certain constraints on the behavior of the players in the game.

The motivation to analyze the consequences of a hierarchical authority structure on a cooperative game is that in many economic organizations one adopts a hierarchical authority structure. Here we consider an abstract authority structure in which certain players dominate certain other players in the sense that the superiors have well specified veto power over the activities undertaken by their subordinates. The consequences of these specified forms of constraints on any cooperative game are then studied using the general analytical and computational framework outlined in Owen (1986).[1]

For example consider a production situation with three players, described by the set $N = \{1, 2, 3\}$. Suppose that only player 1 is productive and creates an output of one unit. This can be described by a *cooperative game with transferable utilities* given by $v : 2^N \to \mathbb{R}$ with $v(E) = 1$ if $1 \in E \subset N$, and $v(E) = 0$ otherwise. This is an example of a *unanimity game*. Next suppose that players 2 and 3 are both superiors of player 1. What are the consequences of this hierarchical authority structure for the productivity as well as the rewards of the three decision makers involved? There are several possibilities to specify the power of the superiors in the authority situation over their subordinates. In the *conjunctive approach* developed in Gilles, Owen and van den Brink (1992), Derks and Gilles (1995), and van den Brink and Gilles (1996) it is assumed that every player has complete veto power over the actions undertaken by his subordinates. Here and in van den Brink (1997) this assumption is modified to the other standard case in which a subordinate only has to get permission from *at least one* superior within the hierarchical authority structure. Thus an action of a certain player has to be authorized by a chain of subsequent superiors within the hierarchy. To distinguish this approach from the conjunctive approach we denote this behavioral model as the *disjunctive approach*.[2]

Let us consider the three-player example of a hierarchical production situation. In a conjunctive hierarchy player 1 has to obtain permission to produce from both his superiors, who are players 2 and 3. Hence, this conjunctive production situation

[1] Owen (1986) provides an analysis of cooperative games with limited communication. His method can however be transfered easily to the analysis of games with authority structures as is the case in Gilles et al. (1992) and this paper.

[2] For an alternative approach we refer to Winter (1989).

can be represented as a game $v_c : 2^N \to \mathbb{R}$ with $v_c(N) = 1$ and $v_c(E) = 0$ for any $E \subsetneq N$. (The game v_c is usually called the "conjunctive restriction" of the original game v.) In contrast, within a disjunctive hierarchy player 1 has to obtain permission to produce from either player 2 or player 3 or both. Such a production situation can be represented by a game $v_d : 2^N \to \mathbb{R}$ with $v_d(E) = 1$ if $E \in \{12, 13, 123\}$[3] and $v_d(E) = 0$ otherwise. The game v_d is referred to as the *disjunctive restriction* of the original production game v.

In general we represent a situation with a hierarchical authority structure as a *cooperative game with a permission structure,* consisting of a set of players, a cooperative game with transferable utilities describing the potential outputs of the various coalitions, and a mapping that assigns to every player a subset of (direct) subordinates describing the authority structure. Now a coalition can only form if every member is authorized to participate and thus for each member there is at least one direct superior within that coalition. This leads to a reduced collection of formable coalitions. A coalition can only generate its potential output if it is *autonomous* within the hierarchical authority structure. By adopting that only autonomous coalitions generate their potential output, we get a restriction of the original cooperative game, which is called the *disjunctive restriction.* This exactly what is pursued in the three player example.

With the use of the disjunctive approach we are able to evaluate the consequences of the adoption of a hierarchical production organization. In the case that this hierarchy has a single topman, there may arise competition regarding the authority over a subordinate among his direct superiors. In this analysis we use the *Shapley value* of the disjunctive restriction as a utility function that assigns to each player in the situation an expected payoff. In general the subordinate as well as the topman in the hierarchy achieve higher payoffs than the competing direct superiors of the productive subordinate.

As a natural consequence we derive that in the presence of small transaction costs a disjunctively hierarchical organization of production may be Pareto superior to a market organization. Here the owner of the production facility separates ownership from control by allowing a middle tier of managers within the production organization. The competitiveness among those managers increases the expected payoff of the owner of the production facility. On the other hand, the hierarchical production organization provides shelter for the market participants in the form of a reduction of their transaction costs. This analysis goes beyond the Coasian approach to the

[3] Here we use for convenience the notation $12 = \{1, 2\}$, $123 = \{1, 2, 3\}$, etc..

nature of the firm (Coase (1937)) in the sense that a separation of ownership and control as well as the hierarchical rules depend on the size of the transaction costs in the market.

We emphasize that the analysis and model presented in this paper should be distinguished from the literature on link and network formation as developed in, e.g., Myerson (1977), Owen (1986), Aumann and Myerson (1988), and Qin (1996). In those contributions the structures studied are based on equivalent communication relationships in which the participants have equal status. Here we deal instead with non-equivalent, hierarchical authority relations in which the participants have unequal status.

2 Disjunctive permission structures

The notion of a game with a permission structure has been introduced by Gilles, Owen and van den Brink (1992). Let $N = \{1, \ldots, n\}$ be a finite set of players. A cooperative game with transferable utilities — or a *TU-game* — on N is a function $v: 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. The collection of all TU-games on N is denoted by $\mathcal{G}^N$. We remark that $\mathcal{G}^N$ is a $(2^n - 1)$-dimensional real vector space.

A *permission structure* on N is a mapping $S: N \to 2^N$. In our setting $j \in S(i)$ means that player i is a *superior* of player j in the permission structure S on N. We call a player $j \in S(i)$ a *successor* of i in S. Finally, the mapping $\widehat{S}: N \to 2^N$ is the transitive closure of S, i.e., for every player $i \in N$ $j \in \widehat{S}(i)$ if there exists a finite sequence $j_1, \ldots, j_K$ in N ($K \in \mathbb{N}$) such that $i = j_1$, $j = j_K$, and for every $1 \leqq k \leqq K - 1$ it holds that $j_{k+1} \in S(j_k)$. The players $j \in \widehat{S}(i)$ are called the *subordinates* of player i in S. A permission structure S *acyclic* if for every player $i \in N$ it holds that $i \notin \widehat{S}(i)$. In this paper we assume that each permission structure is acyclic.

Furthermore, we define $B_S := \{i \in N \mid S^{-1}(i) = \emptyset\}$, where for every player $i \in N$ we have $S^{-1}(i) = \{j \in N \mid i \in S(j)\}$. It is clear that for every acyclic permission structure S on N it holds that $B_S \neq \emptyset$. A permission structure S on N is *strictly hierarchical* if it is acyclic and $|B_S| = 1$, where for every finite set P we denote by $|P| \in \mathbb{N}$ the cardinality of that set. Without loss of generality we may assume that for any strictly hierarchical permission structure S we have $B_S = \{1\}$.

A triple (N, v, S) is a **game with an acyclic permission structure** when $v \in \mathcal{G}^N$ and S is an acyclic permission structure on N. A game with a permission structure describes the potential output of the players as well as a (hierarchical) authority structure on that collection of players. However, it does not yet indicate how the authority structure S affects the potential outcomes of cooperative behavior described by the game v.

In this paper we assume that the authorization of at least one direct superior is necessary and sufficient for a player to enter in cooperation with other players. In this case a coalition $E \subset N$ is "formable" if for every member $i \in E$ there is at least one direct superior present in that coalition.

Definition 2.1 *A coalition $E \subset N$ is* ***(disjunctively) autonomous*** *in the acyclic permission structure S if for every $i \in E \setminus B_S$: $S^{-1}(i) \cap E \neq \emptyset$.*

Definition 2.1 can be restated by noting that a coalition $E \subset N$ is disjunctively autonomous if and only if for every player $i \in E$ there exists a collection of players $\{j_1, \ldots, j_m\} \subset E$ with $j_1 \in B_S$, $j_m = i$, and for every $1 \leqq k \leqq m-1$ we have $j_{k+1} \in S(j_k)$.

We denote by $\Psi_S \subset 2^N$ the collection of all disjunctively autonomous coalitions in the acyclic permission structure S. With reference to the introduction the collection Ψ_S exactly describes the class of all formable coalitions based on the *disjunctive approach*.

Lemma 2.2 *Let S be an acyclic permission structure. Then $\emptyset, N \in \Psi_S$ and for every $E, F \in \Psi_S$: $E \cup F \in \Psi_S$.*

Proof. Evidently $\emptyset \in \Psi_S$ as well as $N \in \Psi_S$. Take $E, F \in \Psi_S$. Let $i \in E \setminus B_S$. Then by Definition 2.1 $\emptyset \neq S^{-1}(i) \cap E \subset S^{-1}(i) \cap [E \cup F]$. Similarly for $i \in F \setminus B_S$. This shows that $E \cup F \in \Psi_S$. ■

In Example 2.5 we will show that Ψ_S does not have to be closed for taking intersections. The knowledge that finite unions of autonomous coalitions are again autonomous leads us to the introduction of the maximal autonomous subcoalition of any given coalition in the setting of an acyclic permission structure.

Definition 2.3 *Let S be an acyclic permission structure on N and let $E \subset N$. Then the subcoalition given by*

$$\psi(E) := \cup\{F \in \Psi_S \mid F \subset E\}$$

is the ***(disjunctively) autonomous part*** *of E in S.*

From Lemma 2.2 it follows that for every coalition $E \subset N$ $\psi(E) \in \Psi_S$ and so $\psi(E)$ is the largest autonomous subcoalition of E in S. Moreover, $\psi(\psi(E)) = \psi(E)$ and $\Psi_S \equiv \{E \subset N \mid E = \psi(E)\}$. If a coalition is not autonomous, it can only "form" indirectly through the addition of other players to create an autonomous superset. These related coalitions are introduced in the following definition.

Definition 2.4 *Let $E \subset N$ be a coalition. A coalition $F \subset N$ is an* ***(disjunctive) authorizing set*** *for E in the acyclic permission structure S if*

1. *$F \in \Psi_S$ and $E \subset F$, and*

2. *there does not exist a $G \in \Psi_S$ such that $E \subset G \subset F$ and $G \neq F$.*

The collection of all authorizing sets for E in S is denoted by $\mathfrak{A}_S(E) \subset \Psi_S$.

Clearly, $E \in \Psi_S$ if and only if for every member $i \in E$ there is an authorizing set $F_i \in \mathfrak{A}_S(\{i\})$ with $F_i \subset E$. Another characterization of autonomous coalitions with the use of authorizing sets is given by

$$\Psi_S = \{E \subset N \mid \mathfrak{A}_S(E) = \{E\}\} \text{ and } \Psi_S = \bigcup_{E \subset N} \mathfrak{A}_S(E).$$

Furthermore we mention that for every non-empty coalition $\emptyset \neq E \subset N$ it holds that $\mathfrak{A}_S(E) \neq \emptyset$.

As a special case we have the "empty" permission structure S_0 defined by $S_0(i) = \emptyset$ for every $i \in N$. Obviously, $\Psi_{S_0} = 2^N$ and for every $E \subset N$ we have $\mathfrak{A}_{S_0}(E) = \{E\}$ and $\psi_{S_0}(E) = E$. Later we will use the empty permission structure to show that the disjunctive approach indeed generalizes the standard approach to cooperative games.

We conclude this section with an example to illustrate these concepts.

Example 2.5 Take $N = \{1, 2, 3, 4, 5\}$. We introduce the permission structure S by $S(1) = \{2, 3\}$, $S(2) = \{4\}$, $S(3) = \{4, 5\}$, and $S(4) = S(5) = \emptyset$. We can represent this strictly hierarchical permission structure by a directed graph on the set of players N. This graph is given in Figure 1.

From the graph it is clear that authorizing sets for individual players are just collections of players on paths in the graph from that particular player to the leader in the hierarchy, player 1. We can deduce that

$$\mathfrak{A}_S(\{4\}) = \{ \{1, 2, 4\}, \{1, 3, 4\} \} \subset \Psi_S.$$

This immediately shows that the intersection of two (disjunctively) autonomous coalitions does not have to be autonomous. Namely, $\{1, 4\} = \{1, 2, 4\} \cap \{1, 3, 4\} \notin \Psi_S$.

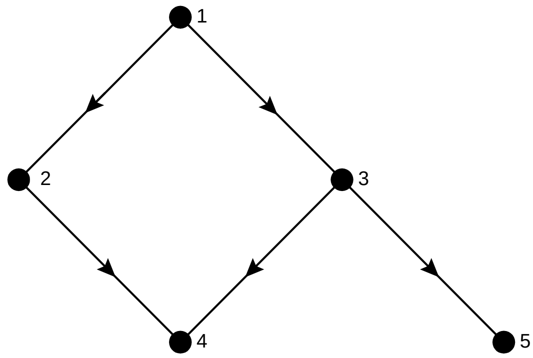

Figure 1: Hierarchy used in Example 2.5.

To illustrate autonomous parts of a particular coalition we take the coalition $E = \{1, 2, 5\} \notin \Psi_S$. It is evident that its autonomous part is given by $\psi(E) = \{1, 2\} \in \Psi_S$. This illustrates the general property that for every non-autonomous coalition $E \notin \Psi_S$ we have that $\psi(E) \subsetneq E$.

3 Disjunctive restrictions

Let $v \in \mathcal{G}^N$ be a TU-game on the set of players N. We assume that $v(E)$ represents the potential output of the coalition $E \subset N$ in case this coalition forms. However, if E is not disjunctively autonomous in the permission structure S, the authority exercised prevents the coalition E from forming. In fact only its autonomous part $\psi(E) \subset E$ is able to form within S.

We therefore introduce a mapping $\mathcal{P}_S\colon \mathcal{G}^N \to \mathcal{G}^N$, which assigns to every game $v \in \mathcal{G}^N$ its *disjunctive restriction*, given as the game $\mathcal{P}_S(v) \in \mathcal{G}^N$ with

$$\mathcal{P}_S(v)(E) := v(\psi(E)), \quad E \subset N.$$

Obviously, for the empty permission structure $\mathcal{P}_{S_0}(v) = v$ for every $v \in \mathcal{G}^N$.

In order to analyze the properties of the mapping $\mathcal{P}_S$ on $\mathcal{G}^N$ we recall two well known bases for the $(2^n - 1)$-dimensional real vector space $\mathcal{G}^N$, namely the standard basis and the unanimity basis. The *standard basis* of $\mathcal{G}^N$ is given by the games $\{z_E \mid E \subset N,\ E \neq \emptyset\}$ defined by

$$z_E(F) = \begin{cases} 1 & \text{if} \quad E = F \\ 0 & \text{if} \quad E \neq F \end{cases}.$$

It is easy to see that in terms of the standard basis the game $v \in \mathcal{G}^N$ can be expressed as

$$v = \sum_{\substack{E \subset N \\ E \neq \emptyset}} v(E) \cdot z_E. \tag{1}$$

The *unanimity basis* of $\mathcal{G}^N$ consists of the collection of unanimity games $\{u_E \mid E \subset N,\ E \neq \emptyset\}$ given by

$$u_E(F) = \begin{cases} 1 & \text{if } E \subset F \\ 0 & \text{otherwise} \end{cases}.$$

In terms of the unanimity basis the game $v \in \mathcal{G}^N$ can be expressed as

$$v = \sum_{\substack{E \subset N \\ E \neq \emptyset}} \Delta_v(E) \cdot u_E, \tag{2}$$

where the quantity $\Delta_v(E)$ is referred to as the *dividend* of coalition E in game v given by $\Delta_v(E) = \sum_{F \subset E} (-1)^{|E \setminus F|}\ v(F)$.

The next lemma provides a deeper insight into the properties of the mapping $\mathcal{P}_S$ for any given acyclic permission structure S using the two bases of the space $\mathcal{G}^N$. The proof is very similar to the proof of the characterization of the Conjunctive restriction as given in Gilles, Owen and van den Brink (1992) and is therefore omitted.

Lemma 3.1 *Let S be an acyclic permission structure on N. The mapping $\mathcal{P}_S$ is a linear projection of the rank $|\Psi_S|$ on $\mathcal{G}^N$. Its kernel — or null space — is spanned by the standard basis games $\{z_E \mid E \notin \Psi_S\}$ and its image is spanned by the unanimity games $\{u_E \mid E \in \Psi_S\}$.*

Next we investigate the dividends of the coalitions within the disjunctive restriction of a certain game in a given acyclic permission structure. Authorizing sets turn out to be of crucial importance in this analysis.

Let $E \subset N$ be a coalition. We define $\mathfrak{A}^*_S(E) \subset 2^N$ as the collection of all finite unions of authorizing sets for E, i.e., $F \in \mathfrak{A}^*_S(E)$ if and only if there exist $F_q \in \mathfrak{A}_S(E)$ $(1 \leqq q \leqq Q)$ such that $F = \bigcup_{q=1}^{Q} F_q$. It is clear that

$$\mathfrak{A}_S(E) \subset \mathfrak{A}^*_S(E) \subset \Psi_S.$$

Theorem 3.2 *Let $v \in \mathcal{G}^N$. Then its disjunctive restriction on S is given by*

$$\mathcal{P}_S(v) = \sum_{E \in \Psi_S} \left\{ \sum_{F \in \mathfrak{A}_S^{-1}(E)} \Delta_v(F) + \sum_{F \in \widehat{\mathfrak{A}}_S(E)} \delta_E(F) \cdot \Delta_v(F) \right\} \cdot u_E,$$

where

1. $\mathfrak{A}_S^{-1}(E) := \{F \subset N \mid E \in \mathfrak{A}_S(F)\}$,

2. $\widehat{\mathfrak{A}}_S(E) := \{F \subset N \mid E \in \mathfrak{A}_S^*(F) \setminus \mathfrak{A}_S(F)\}$, *and*

3. for every $E \in \Psi_S$ *and* $F \in \mathfrak{A}_S^*(E)$: $\delta_E(F) = \Delta_{w_F}(E) \in \mathbb{Z}$ *with* $w_F = \mathcal{P}_S(u_F)$ *and* $\mathbb{Z}$ *the collection of all whole numbers.*

A proof of Theorem 3.2 is included in Appendix A of this paper.

The numbers $\delta_E(F) \in \mathbb{Z}$ for coalitions $E, F \subset N$ as introduced in Theorem 3.2 are clearly independent of the game v, and therefore determined completely by the structure as described by S. We propose as an unproven conjecture that $\delta_E(F) \in \{-1, 0, 1\}$ for all $E, F \in 2^N$. In certain cases this can be confirmed and a formula for these numbers can indeed be derived. To illustrate this, an analysis is given in Appendix B of this paper. However, a general proof of the conjecture is not available yet.

We complete our discussion of the disjunctive restriction of a cooperative game by focussing on the properties of the restricted game. We recall that a game v is called

monotone if for all $E, F \subset N$ with $E \subset F$: $v(E) \leqq v(F)$,

superadditive if for all $E, F \subset N$ with $E \cap F = \emptyset$ we have $v(E) + v(F) \leqq v(E \cup F)$,

convex if for all $E, F \subset N$ it holds that $v(E) + v(F) \leqq v(E \cup F) + v(E \cap F)$, and

balanced if the core of the game v is not empty, i.e., there exists $x \in \mathbb{R}^N$ with $\sum_{i \in N} x_i = v(N)$ and $\sum_{i \in E} x_i \geqq v(E)$ for every $E \subset N$.

The next result states that all properties as described above, except the convexity of a game, are natural properties of disjunctive restrictions of arbitrary monotone games on strictly hierarchical permission structures.

Theorem 3.3 *Let* (v, S) *be a monotonic game with an acyclic permission structure. Then the following properties hold:*

1. $\mathcal{P}_S(v)$ *is a monotone game.*

2. If v *is superadditive, then* $\mathcal{P}_S(v)$ *is superadditive as well.*

3. If v *is balanced, then* $\mathcal{P}_S(v)$ *is balanced as well.*

4. If S *is strictly hierarchical, then* $\mathcal{P}_S(v)$ *is monotone, superadditive, and balanced.*

Proof. Let (v, S) be such that v is monotone and S is acyclic.

1. Take $E, F \subset N$ with $E \subset F$. Then $\psi(E) \subset \psi(F)$ and by monotonicity of v
$$\mathcal{P}_S(v)(E) = v(\psi(E)) \leqq v(\psi(F)) = \mathcal{P}_S(v)(F).$$

2. Let v be monotone and superadditive, and let $E, F \subset N$ with $E \cap F = \emptyset$. Then, $\psi(E) \cap \psi(F) = \emptyset$. Also, $\psi(E) \cup \psi(F) \in \Psi_S$ and, thus, $\psi(E) \cup \psi(F) \subset \psi(E \cup F)$. Now by monotonicity and superadditivity of v this implies that
$$\begin{aligned}\mathcal{P}_S(v)(E) + \mathcal{P}_S(v)(F) = v(\psi(E)) + v(\psi(F)) &\leqq v(\psi(E) \cup \psi(F)) \\ &\leqq v(\psi(E \cup F)) = \mathcal{P}_S(v)(E \cup F).\end{aligned}$$

3. Let v be monotone and balanced. Then there exists $x \in \mathbb{R}^N$ with $\sum_{i \in N} x_i = v(N)$ and $\sum_{i \in E} x_i \geqq v(E)$ for every $E \subset N$. We show that x is a core allocation of $\mathcal{P}_S(v)$ as well. Namely,
$$\begin{aligned}\sum_{i \in N} x_i = v(N) = v(\psi(N)) = \mathcal{P}_S(v)(N) \text{ and} \\ \sum_{i \in E} x_i \geqq v(E) \geqq v(\psi(E)) = \mathcal{P}_S(v)(E).\end{aligned}$$

4. Without loss of generality we may assume that S is a strictly hierarchical permission structure with $1 \in N$ at the top-level, i.e., $B_S = \{1\}$. This implies that for every coalition $E \subset N$ either $1 \in \psi(E)$ or $\psi(E) = \emptyset$.
First, that $\mathcal{P}_S(v)$ is monotone follows immediately from the proved assertion 1.
Second, $\mathcal{P}_S(v)$ has a non-empty core. Namely, take $x \colon N \to \mathbb{R}$ with
$$x_i = \begin{cases} v(N) & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}.$$
x is a core imputation of $\mathcal{P}_S(v)$. Namely by definition we have that $\mathcal{P}_S(v)(E) > 0$ only if $\psi(E) \neq \emptyset$. Hence, $\mathcal{P}_S(v)(E) > 0$ implies that $1 \in E$. Thus if $\mathcal{P}_S(v)(E) > 0$:
$$\mathcal{P}_S(v)(E) = v(\psi(E)) \leqq v(E) \leqq v(N) = x_1 = \sum_{i \in E} x_i.$$
Third, $\mathcal{P}_S(v)$ is superadditive.
Take $E, F \subset N$ with $E \cap F = \emptyset$. From the fact that S is strictly hierarchical it clearly follows that either $1 \in E$ or $1 \in F$ or $1 \notin E \cup F$. Thus, either $\psi(E) = \emptyset$ or $\psi(F) = \emptyset$ or both. Without loss of generality we may assume that $\psi(F) = \emptyset$. Then
$$\mathcal{P}_S(v)(E) + \mathcal{P}_S(v)(F) = v(\psi(E)) \leqq v(\psi(E \cup F)) = \mathcal{P}_S(v)(E \cup F).$$

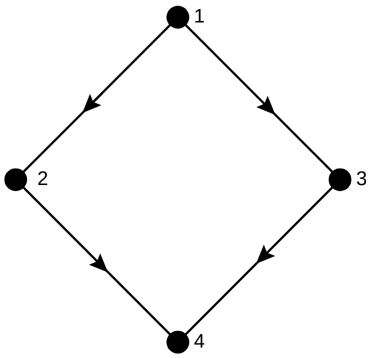

Figure 2: Hierarchy used in Example 3.4.

This completes the proof of the assertion. ■

Convexity is a particularly desirable property of a game, since it implies balancedness and superadditivity in combination with some geometric properties of the core. The next example however shows that although a game may be monotone and convex and the permission structure is strictly hierarchical, the disjunctive restriction does not have to be convex.

Example 3.4 Take $N = \{1, 2, 3, 4\}$. We introduce the permission structure S by $S(1) = \{2, 3\}$, $S(2) = \{4\}$, $S(3) = \{4\}$, and $S(4) = \emptyset$. We can represent this strictly hierarchical permission structure by a graph on the set of players N. This graph is given in Figure 2.
Let v be a game given by $v(N) = 4$, $v(134) = v(124) = 3$, $v(14) = 2$, and $v(E) = 0$ otherwise. It may be clear that v is monotone and convex, and therefore also superadditive and balanced. Take $w = \mathcal{P}_S(v)$. Then w is given by $w(N) = 4$, $w(124) = w(134) = 3$, and $w(E) = 0$ otherwise. This shows that w is indeed monotone, superadditive, and balanced, but also that w is not convex. Namely,

$$w(124) + w(134) = 6 > 4 = w(N) + w(14).$$

4 A computational method

Next we develop a method to compute the disjunctive restriction of an arbitrary game using multilinear extensions introduced by Owen (1972). Let $v \in \mathcal{G}^N$. Then its *multilinear extension* (MLE) is the function $f_v\colon [0,1]^N \to \mathbb{R}$ given by

$$f_v(x_1,\ldots,x_n) = \sum_{E\subset N} \Delta_v(E)\cdot \left\{\prod_{i\in E} x_i\right\}.$$

Evidently f_v is a multilinear function, which coincides with the worth $v(E)$ at the extreme points of the unit cube $[0,1]^N \subset \mathbb{R}^N_+$. We interpret the MLE of a game as a probabilistic expectation. Namely, we can rewrite $f_v(x_1,\ldots,x_n) = \mathbb{E}[v(\mathcal{E})]$, where $\mathcal{E}$ is a random variable whose values are subsets of N given the probabilities $\Pr\{i\in\mathcal{E}\} = x_i$ and under the assumptions that the n events $\{i\in\mathcal{E}\}$ are stochastically independent.

It is our goal to derive the MLE of the disjunctive restriction $\mathcal{P}_S(u_E)$ of the unanimity game u_E for $E\subset N$. With this in mind we introduce two operators. The operator $\otimes$ is denoted as *independent multiplication* and is completely characterized by the following properties:

- For every $(x_1,\ldots,x_n)\in[0,1]^N$ and all $E,F\subset N$:

$$\left(\prod_{i\in E} x_i\right)\otimes\left(\prod_{i\in F} x_i\right) = \prod_{i\in E\cup F} x_i.$$

- Let a,b,c be three multilinear functions on $[0,1]^N$. Then

$$(a+b)\otimes c = (a\otimes c)+(b\otimes c) \text{ and}$$

$$a\otimes(b+c) = (a\otimes b)+(a\otimes c).$$

The second operator is denoted as *disjunctive addition* $\oplus$ and for every two multilinear functions a and b on $[0,1]^N$ it is defined by

$$a\oplus b := 1-(1-a)\otimes(1-b).$$

Let $\mathcal{E}$ be the random coalition variable with independent probabilities $(x_1,\ldots,x_n)\in[0,1]^N$. Now for all $F,G\subset N$ define the events $\mathbf{A} := \{F\subset\mathcal{E}\}$ and $\mathbf{B} := \{G\subset\mathcal{E}\}$. Then $\Pr\{\mathbf{A}\} = \prod_{i\in F} x_i$ and $\Pr\{\mathbf{B}\} = \prod_{j\in G} x_j$. We conclude from the above that

$$\Pr(\mathbf{A}\wedge\mathbf{B}) = \Pr\{F\cup G\subset\mathcal{E}\} = \prod_{i\in F\cup G} x_i = \Pr(\mathbf{A})\otimes\Pr(\mathbf{B}).$$

Similarly we derive that

$$\Pr(\mathbf{A}\vee\mathbf{B}) = \Pr(\mathbf{A})\oplus\Pr(\mathbf{B}).$$

Next let S be an acyclic permission structure. Now take $i \in N$ and take probabilities $x \in [0,1]^N$. We define $h_i(x)$ as the probability that there exists an authorizing set for $\{i\}$ in the random coalition $\mathcal{E}$. Since such an authorizing set is in fact a permission path from some $j_1 \in B_S$ to i, we know that $\mathcal{E}$ contains such a set if and only if

1. $i \in \mathcal{E}$ and
2. there is at least one superior $j \in S^{-1}(i)$ which has an authorizing set in $\mathcal{E}$.

This leads to the conclusion that

$$h_i(x) = x_i \otimes \bigoplus_{j \in S^{-1}(i)} h_j(x).$$

Since for $j \in B_S$ it simply holds that $h_j(x) = x_j$ we now have derived a recursive method for computing the multilinear function $h_i(x)$ $(i \in N)$, which expresses the probability that i has an authorizing set in $\mathcal{E}$.

Finally from the computational rules it follows that for any coalition $E \subset N$ the probability that E has an authorizing set in $\mathcal{E}$ is given by

$$h_E(x) = \bigotimes_{i \in E} h_i(x).$$

We thus conclude that the MLE of $\mathcal{P}_S(u_E)$, where u_E is the unanimity game belonging to E, is exactly given by the multilinear function h_E. Since we now have the MLE of the disjunctive restriction of any unanimity game on S we therefore have the MLE of the disjunctive restriction of any game v to the acyclic permission structure S.

In the following example and the next subsections we use this method to compute the MLE of certain games.

Example 4.1 By applying the computational method developed in this subsection we analyze a game with a strictly hierarchical permission structure, which does not satisfy the requirements in Proposition B1 as given in Appendix B. We show that the formula given there indeed does not hold and thus this result cannot be extended. Take $N = \{1, \ldots, 6\}$ as the set of players and let S be a strictly hierarchical permission structure described in Figure 3.

To compute $\mathcal{P}_S\left(u_{\{6\}}\right)$ we follow the outlined recursive procedure for computing all multilinear functions h_i, $i \in N$. We get the following expressions:

$$h_1(x) = x_1$$

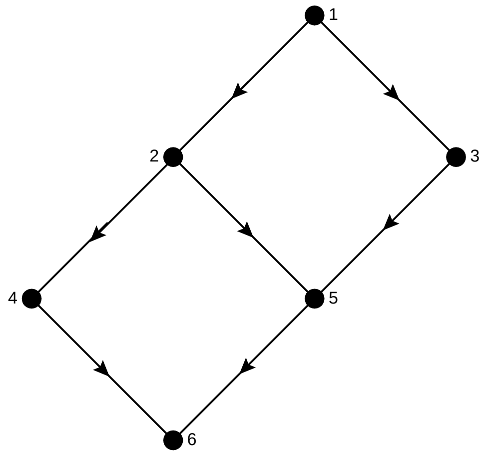

Figure 3: Hierarchy used in Example 4.1.

$$\begin{aligned} h_2(x) &= x_1x_2 \\ h_3(x) &= x_1x_3 \\ h_4(x) &= x_1x_2x_4 \\ h_5(x) &= x_5[h_2(x) \oplus h_3(x)] = \\ &= x_1x_3x_5 + x_1x_2x_5 - x_1x_2x_3x_5 \\ h_6(x) &= x_6[h_4(x) \oplus h_5(x)] = \\ &= x_1x_3x_5x_6 + x_1x_2x_5x_6 + x_1x_2x_4x_6 - x_1x_2x_3x_5x_6 - x_1x_2x_4x_5x_6 \end{aligned}$$

As argued before h_6 is the MLE of $\mathcal{P}_S\left(u_{\{6\}}\right)$ and thus

$$\mathcal{P}_S\left(u_{\{6\}}\right) = u_{\{1,3,5,6\}} + u_{\{1,2,5,6\}} + u_{\{1,2,4,6\}} - u_{\{1,2,3,5,6\}} - u_{\{1,2,4,5,6\}}.$$

Now in $\mathcal{P}_S\left(u_{\{6\}}\right)$ we have that $N \in \mathfrak{A}^*_S(\{6\})$. Also, $N \notin \mathfrak{A}_S(\{6\})$. From the formula given in Proposition B1 it now would follow that $\delta_N(\{6\}) \in \{-1, 1\}$. However, from the expression above and the fact that $\Delta_{u_{\{6\}}}(E) = 0$ if $E \neq \{6\}$. Thus, it has to be concluded that $\delta_N(\{6\}) = 0$. This shows that even in relatively simple cases, that do not satisfy the requirement as formulated in Proposition B1, we have a refutation of the conjecture that in general $\delta_H(E) \in \{-1, 1\}$.

5 Some applications

In this section we present several applications of the results derived in the previous sections. First, we use the Shapley value to analyze the competitive features within an acyclic permission structures. The Shapley value, as introduced by Shapley (1953), can be regarded as a normative allocation rule, satisfying certain fairness properties, and thus can be used to express the value of a certain player in a certain situation. We show that the Shapley values of superiors of the same subordinate decreases as more players enter that hierarchical level or tier, while the value of the subordinate increases. Second, this competitive feature is used to show that in a certain situation in the presence of small transaction costs the formation of a disjunctive permission structure is Pareto superior to a market organization.

5.1 The exercise of authority in hierarchies

Consider a hierarchical production organization in which there is only a single productive participant, or "worker." We adhere to the literature on the theory of the firm by assuming that this worker is at the bottom of the hierarchy.[4] Within such a situation we show that in a disjunctive hierarchy this worker can exploit the multiplicity of second tier managers and can claim a higher payoff.

Formally, let $N = \{1, 2\} \cup P$ with $p = |P| \geqq 1$. Player $1 \in N$ is assumed to be the unique chief officer in the hierarchical production organization while $2 \in N$ is the unique worker. The set P describes the tier of direct superiors of the worker. Thus we arrive at a strictly hierarchical permission structure S given by

$$S(i) = \begin{cases} P & \text{if } i = 1 \\ \{2\} & \text{if } i \in P \\ \emptyset & \text{if } i = 2 \end{cases} .$$

The description is completed by taking $v = u_{\{2\}}$ and remarking that the triple (N, v, S) now represents a situation as discussed above. In the disjunctive approach all direct superiors $i \in P$ can independently authorize any productive activity of the worker $2 \in N$. We show that they actually "compete" over the right to exercise that authority.

Formally we can express this type of "competition" by regarding the disjunctive restriction $\mathcal{P}_S(v)$. We apply the computational method introduced before. The MLE's for the different singletons are given by:

$$\begin{aligned} h_1(x) &= x_1 \\ h_i(x) &= x_1 x_i \quad \text{where } i \in P \end{aligned}$$

[4]Here we refer to van den Brink (1996) for a discussion of this assumption and its consequences for the description of hierarchical production organizations.

$$h_2(x) \quad = \quad \sum_{k=1}^{p}(-1)^{k+1} \sum_{\substack{E \subset P \\ |E|=k}} x_1 x_2 \prod_{i \in E} x_i$$

Here h_2 is the MLE of $\mathcal{P}_S(v)$. We can easily derive that

$$\mathcal{P}_S(v) = \sum_{k=1}^{p}(-1)^{k+1} \sum_{\substack{E \subset P \\ |E|=k}} \widehat{u}_E,$$

where for every $E \subset P$ we have

$$\widehat{u}_E(F) = \begin{cases} 1 & \text{if } E \cup \{1,2\} \subset F \\ 0 & \text{otherwise} \end{cases} .$$

The consequences of competition between the superiors in P can be shown by the Shapley value $\varphi_j(\mathcal{P}_S(v))$ of the players $j \in N$ in the game $\mathcal{P}_S(v)$. We derive

$$\varphi_1(\mathcal{P}_S(v)) = \varphi_2(\mathcal{P}_S(v)) = \sum_{k=1}^{p} \frac{(-1)^{k+1}}{k+2} \frac{p!}{k!\,(p-k)!} = \frac{p(p+3)}{2(p+1)(p+2)},$$

and for every $i \in P$

$$\varphi_i(\mathcal{P}_S(v)) = \sum_{k=1}^{p} \frac{(-1)^{k+1}}{k+2} \frac{(p-1)!}{(k-1)!\,(p-k-1)!} = \frac{2}{p(p+1)(p+2)}.$$

This shows that for larger p the Shapley values of players 1 and 2 are increasing, while the values of the competing superiors $i \in P$ are diminishing. A further analysis of these properties is given in the axiomatization of the Shapley value for games with a disjunctive permission structure in van den Brink (1997).

5.2 Markets and hierarchical production organizations

In this subsection we consider the production of an output directly through the market mechanism versus the formation of a disjunctive production organization. We show that in the presence of relatively small transaction costs related to the use of the price mechanism such a hierarchical production organization is Pareto superior.

We consider a very simple production situation. Given is a production process which converts one (composite) input into one output. All incomes are generated by the sale of this single output on an unspecified commodity market. We assume that the owner of the production process can achieve a certain level of output by herself denoted by $\rho > 0$. Additional output is achieved by obtaining additional units of the input. This situation can be represented by the specifically designed TU-game

$w \in \mathcal{G}^N$ in which the player set is given by $N := \{0\} \cup P$, where $0 \in N$ is the production process and $i \in P$ are owners of additional units of input, and for which we let

$$w = \varrho \cdot u_0 + \sum_{i \in P} u_{0i}, \tag{3}$$

where u_0, respectively u_{0i}, is the unanimity game with respect to $\{0\}$, respectively $\{0, i\}$. Obviously, the input of player $i \in P$ generates one additional unit of the output.

We modify the game w in different fashions to describe two organization structures, one through the market mechanism and one through a hierarchy.

A market organization.

First we consider a market organization with one-sided transaction costs. The owner of the production facility 0 is assumed to purchase additional units of the input market with one-sided transaction costs. Here the owners of the input $i \in P$ are subjected to a cost of $c > 0$ for each unit sold. Hence, the owner 0 of the production process has no explicit market transaction costs. The output of the production organization in this situation can be described by the game

$$v_1 := w - c \sum_{i \in P} u_i, \tag{4}$$

where u_i is the unanimity game for $\{i\}$ with $i \in P$. We assume that for all input providers $i \in P$ the transaction costs are the same.

The expected payoff in the situation that all transactions take place through the market mechanism is now given by the Shapley value of the game v_1, i.e.,

$$\varphi_0 (v_1) = \varrho + \frac{p}{2}, \text{ and } \varphi_i (v_1) = \frac{1}{2} - c, \quad i \in P,$$

where we let $p = |P| \geqq 1$.

A disjunctive hierarchy.

An alternative organization would be to separate ownership and control of the production process and allow the input market participants $i \in P$ to enter into an organization structure involving partial control of the production process. Such a hierarchical production organization is described by an acyclic permission structure $S \colon N' \to 2^{N'}$, where $N' := \{a, b\} \cup P$. Here player a is the owner of the production process 0, while b is the controlled production process itself. Thus, we consider that the original production process 0 is separated into an "owner" and the production process itself, described by $\{a, b\}$. If all $i \in P$ enter the hierarchical production organization we describe the resulting hierarchy S by

$$S(i) = \begin{cases} P & \text{if } i = a \\ \{b\} & \text{if } i \in P \\ \emptyset & \text{if } i = b \end{cases}.$$

The permission structure S gives exactly the same structure as discussed in Subsection 5.1. Thus, we expect that there will be competition between the second tier managers $i \in P$ over the leadership of the production technology b.

From the description it is clear that owners of the input take partial control of the production facility itself. Now the output of this hierarchical production organization is described by $v_2 := \mathcal{P}_S(w')$, where

$$w' = \varrho \cdot u_b + \sum_{i \in P} u_{ib} \tag{5}$$

is a modification of the original game w given in equation (3) by replacing 0 by b. This replacement indicates the actual separation of ownership and control with regard to the production process. We assume that the separated owner is not productive, but has the top position in the hierarchy. Now

$$\varphi_a(v_2) = \varphi_b(v_2) = \varrho \cdot \frac{p(p+3)}{2(p+1)(p+2)} + \frac{p}{3}, \text{ and}$$

$$\varphi_i(v_2) = \frac{2\varrho}{p(p+1)(p+2)} + \frac{1}{3}, \quad i \in P.$$

A comparison of both organizations.

With the use of the Shapley values as evaluations of the different production organizations we now are able to give a comparison. In particular we identify when the disjunctively hierarchical production situation with separation of ownership and control is Pareto superior to the market mechanism with one-sided transaction costs.

1. The owner of the production process has to gain by allowing the disjunctive organization of the production. This is the case when

$$\varphi_0(v_1) < \varphi_a(v_2) + \varphi_b(v_2).$$

This is equivalent to

$$\varrho < \frac{1}{12} p(p+1)(p+2). \tag{6}$$

2. An individual input supplier $i \in P$ has to gain from entering the hierarchical production organization. This is the case when

$$\varphi_i(v_1) < \varphi_i(v_2).$$

This results into the requirement that

$$c > \frac{1}{6} - \frac{2\varrho}{p(p+1)(p+2)}. \tag{7}$$

From the analysis above we deduce that for any transaction cost $c > 0$ the organization of production by a disjunctive hierarchy is weakly Pareto superior to a direct organization of production through the market mechanism if condition (6) is satisfied. If one allows a lower bound on the transaction costs c, one can deduce that for many combinations of productivity (ϱ) and size (p), a disjunctive hierarchy is strongly Pareto superior to the use of the market mechanism.

6 A comparison with the Conjunctive approach

In the conjunctive approach to games with a permission structure analyzed in Gilles, Owen, and van den Brink (1992), Derks and Gilles (1995), and van den Brink and Gilles (1996) it is assumed that *each* superior has veto power over the actions undertaken by his subordinates. Hence, a player has to get permission of *all* his superiors before he can engage in productive cooperative behavior.

Let S be an acyclic permission structure. A coalition $E \subset N$ is formable within the conjunctive approach if each superior of each member is also a member of E, i.e., if for every $i \in E$: $S^{-1}(i) \subset E$. Thus, the collection of all *conjunctively autonomous* coalitions is given by

$$\Phi_S := \{E \subset N \mid E \cap S(N \setminus E) = \emptyset\}.$$

As a preliminary result we compare this collection with the collection of all disjunctively autonomous sets Ψ_S.

Lemma 6.1 *For every acyclic permission structure S it holds that $\Phi_S \subset \Psi_S$, i.e., every conjunctively autonomous coalition is also disjunctively autonomous.*

Proof. Take a conjunctively autonomous coalition $E \in \Phi_S$. From $E \cap S(N \setminus E) = \emptyset$ it follows that for every player $i \in E$ there is no $j \in N \setminus E$ such that $i \in S(j)$. ∎

A similar comparison is made in Proposition 3.1 in van den Brink (1997), who investigates the consequences of the removal of certain authority relations from the permission structure on the class of autonomous sets. He found that for strictly hierarchical permission structures the class of disjunctively autonomous coalitions is increasing in the number of authority relations, while the class of conjunctively autonomous coalitions is decreasing in the number of authority relations. Thus, the conjunctive exercise of authority discourages coalition formation, while the disjunctive exercise of authority facilitates coalition formation.

Next define for every coalition $E \subset N$ its *conjunctively autonomous part* by $\sigma(E) := \cup\{F \in \Phi_S \mid F \subset E\}$. Following Gilles et al. (1992) we now introduce a mapping $\mathcal{R}_S\colon \mathcal{G}^N \to \mathcal{G}^N$, which for every $v \in \mathcal{G}^N$ is defined by $\mathcal{R}_S(v)(E) := v(\sigma(E)),\ E \subset N$. We call $\mathcal{R}_S(v)$ the *conjunctive restriction* of $v \in \mathcal{G}^N$ on S. A second comparison of the disjunctive and conjunctive approaches to games with a permission structure is to compare the ranges of the mappings $\mathcal{P}_S$ and $\mathcal{R}_S$.

Lemma 6.2 *For every acyclic permission structure S we have that* $\operatorname{Im}\mathcal{R}_S \subset \operatorname{Im}\mathcal{P}_S$.

The proof of this lemma results directly from application of Lemma 6.1, Lemma 3.1, and the results shown in Gilles et al. (1992).

We remark that there exist games of which the conjunctive and the disjunctive restrictions are different, i.e., $\operatorname{Im}\mathcal{P}_S \setminus \operatorname{Im}\mathcal{R}_S \neq \emptyset$. (These examples can easily be constructed with the use of acyclic permission structures such as described in, e.g., Example 2.5.)

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Appendix A: Proof of Theorem 3.2.

The proof of Theorem 3.2 is conducted through a series of lemmas. In the sequel we take a fixed coalition $E \subset N$. Furthermore, we define $w_E := \mathcal{P}_S(u_E)$ as the disjunctive restriction of the unanimity game u_E on the acyclic permission structure S on N.

Lemma A1. For every non autonomous coalition $F \notin \Psi_S$ with the property that $E \subset \psi(F)$: $\Delta_{w_E}(F) = 0$.

PROOF. By definition it holds that

$$\Delta_{w_E}(F) = \sum_{H \subset F} (-1)^{|F|-|H|} w_E(H).$$

Let $G := F \setminus \psi(F)$. Clearly $G \cap E = \emptyset$. Since F is not autonomous it is evident that $G \neq \emptyset$. Next take $j \in G$ and let $H \subset F \setminus \{j\}$.
If $E \subset \psi(H)$, then clearly $w_E(H) = w_E(H \cup \{j\}) = 1$.
If $E \setminus \psi(H) \neq \emptyset$, then obviously $w_E(H) = w_E(H \cup \{j\}) = 0$.
This shows that for every $H \subset F \setminus \{j\}$

$$w_E(H) - w_E(H \cup \{j\}) = 0.$$

Now rewrite

$$\Delta_{w_E}(F) = \sum_{H \subset F} (-1)^{|F|-|H|} w_E(H)$$

$$= \sum_{H \subset F \setminus \{j\}} \left((-1)^{|F|-|H|} w_E(H) + (-1)^{|F|-|H|-1} w_E(H \cup \{j\}) \right)$$

$$= \sum_{H \subset F \setminus \{j\}} (-1)^{|F|-|H|} \left(w_E(H) - w_E(H \cup \{j\}) \right) = 0.$$

This shows the assertion. ∎

Lemma A2. For every coalition $F \in \mathfrak{A}_S(E)$: $\Delta_{w_E}(F) = 1$.

PROOF. For every strict subset G of F it holds that either $G \notin \Psi_S$ or $E \setminus G \neq \emptyset$. (This is a consequence of the definition of the collection $\mathfrak{A}_S(E)$.)
In both cases it follows that $E \setminus \psi(G) \neq \emptyset$. This implies that for every strict subcoalition G of F it holds that $w_E(G) = 0$ and so $\Delta_{w_E}(G) = 0$. Furthermore, $w_E(F) = u_E(\psi(F)) = u_E(F) = 1$, which implies that

$$\Delta_{w_E}(F) = w_E(F) - \sum_{G \subset F,\ G \neq F} \Delta_{w_E}(G) = 1.$$

This shows the assertion. ∎

Lemma A3. For every coalition $F \notin \mathfrak{A}^*_S(E)$: $\Delta_{w_E}(F) = 0$.

Proof. Define $\widehat{F} := \cup\{G \in \mathfrak{A}_S(E) \mid G \subset F\}$. By definition it holds that $\widehat{F} \subset F$. For the case that $\widehat{F} = \emptyset$ it is evident that $w_E(F) = u_E(\psi(F)) = 0$ since then from the definition of authorizing sets $E \setminus \psi(F) \neq \emptyset$. This immediately shows that $\Delta_{w_E}(F) = 0$.
So we may restrict ourselves to the case that $\widehat{F} \neq \emptyset$. Clearly $\widehat{F} \in \mathfrak{A}_S^*(E) \subset \Psi_S$. Hence, by the fact that $F \notin \mathfrak{A}_S^*(E)$ it follows that $F \setminus \widehat{F} \neq \emptyset$. But it also holds that $E \subset \widehat{F} \subset \psi(F)$.
If $F \notin \Psi_S$, then by Lemma A1 it holds that $\Delta_{w_E}(F) = 0$. So, we suppose that $F \in \Psi_S$, i.e., $F = \psi(F)$. Take $G := F \setminus \widehat{F} \neq \emptyset$ and let $j \in G$. Take $H \subset F \setminus \{j\}$. Then it is obvious that $E \subset \widehat{F} \subset \psi(H)$. So, $w_E(H) = u_E(\psi(H)) = 1$. It is also clear that $w_E(H \cup \{j\}) = 1$. We therefore may conclude that for every coalition $H \subset F \setminus \{j\}$ it holds that $w_E(H) - w_E(H \cup \{j\}) = 0$. Hence,

$$\Delta_{w_E}(F) = \sum_{H \subset F} (-1)^{|F|-|H|} w_E(H)$$

$$= \sum_{H \subset F \setminus \{j\}} (-1)^{|F|-|H|} \left(w_E(H) - w_E(H \cup \{j\})\right) = 0.$$

This shows the assertion. ■

With the use of the lemmas as stated and proved above we are able to prove Theorem 3.2.

Proof of Theorem 3.2.
Let $v \in \mathcal{G}^N$ and define $w = \mathcal{P}_S(v)$ as its disjunctive restriction on the acyclic permission structure S. Now we write the game v as follows:

$$v = \sum_{E \subset N} \Delta_v(E) \cdot u_E.$$

Take any coalition $E \subset N$ then by combination of Lemma A1, Lemma A2 and Lemma A3 we derive that for $w_E := \mathcal{P}_S(u_E)$ it holds that

$$w_E = \sum_{H \in \mathfrak{A}_S(E)} u_H + \sum_{\substack{H \in \mathfrak{A}_S^*(E) \\ H \notin \mathfrak{A}_S(E)}} \Delta_{w_E}(H) \cdot u_H.$$

Next define for every $H \in \mathfrak{A}_S^*(E)$

$$\delta_H(E) := \Delta_{w_E}(H) = \sum_{G \subset H} (-1)^{|H|-|G|} \cdot w_E(G),$$

then it is evident that these numbers are independent of the original game v. Moreover, since for every $G \subset N$ it holds that $w_E(G) \in \{0, 1\}$ we may conclude that these numbers are whole. This implies by linearity of the mapping $\mathcal{P}_S$ that

$$w = \sum_{E \subset N} \Delta_v(E) \cdot \mathcal{P}_S(u_E)$$

$$= \sum_{E \subset N} \Delta_v(E) \cdot \left\{ \sum_{H \in \mathfrak{A}_S(E)} u_H + \sum_{\substack{H \in \mathfrak{A}^\star_S(E) \\ H \notin \mathfrak{A}_S(E)}} \delta_H(E) \cdot u_H \right\}.$$

Rewriting this formula leads to

$$w = \sum_{E \in \Psi_S} \left\{ \sum_{F \in \mathfrak{A}_S^{-1}(E)} \Delta_v(F) + \sum_{F \in \widehat{\mathfrak{A}}_S(E)} \delta_E(F) \cdot \Delta_v(F) \right\} u_E.$$

This completes the proof of Theorem 3.2. ■

Appendix B: Determination of δ_E values

In this appendix we show that for certain situations the δ_H can be determined precisely. This is stated as a proposition below.

Definition *Let $E \subset N$ be a coalition. Take any $F \subset N$ such that $E \subset F$. The **essentiality** of the coalition F for E in the acyclic permission structure S is the natural number*

$$\eta_E(F) := \#\{G \in \mathfrak{A}_S(E) \mid G \subset F\}.$$

It immediately follows that a coalition $E \subset N$ is disjunctively autonomous if and only if for every $F \subset N$ with $E \subset F$ it holds that $\eta_E(F) = 1$.

Proposition B1 Let S be an acyclic permission structure on N and let $E \subset N$ be a coalition such that for every coalition $F \in \mathfrak{A}_S(E)$ there exists a player $i \in F$ such that $i \notin G$ for every $G \in \mathfrak{A}_S(E) \setminus \{F\}$. Then for every coalition $H \in \mathfrak{A}^*_S(E)$ it holds that

$$\delta_H(E) = (-1)^{\eta_E(H)+1} \in \{-1, 1\}.$$

Proof. By the definition of the number $\delta_F(E)$ and the proof of Theorem 3.2, we only have to show that under the conditions as put on the coalition $E \subset N$ it holds that for every $F \in \mathfrak{A}^*_S(E)$

$$\Delta_{w_E}(F) = (-1)^{\eta_E(F)+1}.$$

In order to prove this we use induction on the natural number $\eta_E(F)$. First suppose that $\eta_E(F) = 1$, then by Lemma A2 in the appendix

$$F \in \mathfrak{A}_S(E) \text{ and } \Delta_{w_E}(F) = 1 = (-1)^2.$$

Let $K := \eta_E(F) \geqq 2$ and assume by the induction hypothesis that the formula as given above is true for all coalitions $G \in \mathfrak{A}^*_S(E)$ with $\eta_E(G) \leqq K - 1$. Obviously it holds that $w_E(F) = 1$ and so

$$\Delta_{w_E}(F) = 1 - \sum_{\substack{G \subset F \\ G \neq F}} \Delta_{w_E}(G).$$

Since by Lemma A3 $\Delta_{w_E}(G) = 0$ for every coalition $G \notin \mathfrak{A}^*_S(E)$ we may restrict ourselves to the coalitions $G \subset F$ with $G \in \mathfrak{A}^*_S(E)$ and $G \neq F$.
By the induction hypothesis

$$\Delta_{w_E}(F) = 1 - \sum_{\substack{G \subset F,\ G \neq F \\ G \in \mathfrak{A}^*_S(E)}} (-1)^{\eta_E(G)+1}.$$

By the conditions as put on the coalition E it follows that F contains precisely $\frac{K!}{k!(K-k)!}$ subcoalitions $G \in \mathfrak{A}^*_S(E)$ with the property that $\eta_E(G) = k$, where $1 \leqq k \leqq n$. This implies that

$$\Delta_{w_E}(F) = 1 - \sum_{k=1}^{K-1} \frac{K!}{k!\,(K-k)!} (-1)^{k+1} = \begin{cases} 2 & \text{for } K \text{ even} \\ 0 & \text{for } K \text{ odd} \end{cases}.$$

This completes the proof of Proposition B1. ■