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Monotonic Allocation Schemes in Clan Games

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Abstract: Total clan games are characterized using monotonicity, veto power of the clan members, and a concavity condition reflecting the decreasing marginal contribution of non-clan members to growing coalitions. This decreasing marginal contribution is incorporated in the notion of a bi-monotonic allocation scheme, where the value of each coalition is divided over its members in such a way that the clan members receive a higher, and the non-clan members a lower share as the coalitions grow larger. Each core element of a total clan game can be extended to both a population monotonic and a bi-monotonic allocation scheme. In total clan games where the clan consists of a single member (the so-called big boss) the use of the nucleolus as an allocation mechanism gives rise to a bi-monotonic allocation scheme.

Keywords: cooperative games, population monotonic allocation scheme, bi-monotonic allocation scheme, clan games, big boss games.

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1 Introduction

Potters *et al.* (1989) introduce clan games to model social conflicts between 'powerful' players (clan members) and 'powerless' players (non-clan members), in which the powerful players have veto power, and the powerless players operate more profitably in unions than on their own. Economic applications of such clan games to bankruptcy problems, production economies, and information acquisition are provided in Potters *et al.* (1989), Muto *et al.* (1988), and Brânzei *et al.* (2000).

In this paper we consider total clan games: cooperative situations in which the game itself and each of its subgames can be modelled as a clan game. Classes of games giving rise to total clan games are considered in Tijs *et al.* (2000), who study a class of inventory games, and Brânzei *et al.* (2000), who study a class of information collecting games. In both these classes, the clan consists of a single player. Total clan games are characterized by monotonicity of the characteristic function, the veto power of the clan members, and a concavity condition similar to the standard definition of concave games (Shapley, 1971): the marginal contribution of a non-clan member to a coalition containing all clan members decreases as the coalition that he joins grows larger.

Taking the decreasing marginal influence of non-clan members into account, the notion of population monotonic allocation schemes as introduced in Sprumont (1990) is adapted in such a way that clan members are still assigned increasing shares, but non-clan members are actually allocated a smaller amount in larger coalitions. This monotonicity in two directions — increasing payoffs to clan members, decreasing payoffs to non-clan members as the coalitions grow larger — leads us to call such allocations *bi-monotonic.* Finally, a stability condition requiring each allocation to give rise to a core allocation in the subgames is imposed to make sure that coalitions cannot profit from rejecting the allocation scheme and operating on their own.

Each core element of a total clan game can be extended to both a population monotonic and a bi-monotonic allocation scheme. Moreover, additional appeal for bi-monotonic allocation schemes is provided by showing that in total clan games where the clan consists of a single member (the so-called big boss, see Muto *et al.*, 1988), the use of the nucleolus as an allocation mechanism gives rise to a bi-mas.

The set-up of this paper is as follows. After some matters of notation, total clan games are defined and characterized in Section 2. In Section 3, bi-monotonic allocation schemes are introduced and it is shown that each core element of a total clan game can be extended to both a population monotonic and to a bi-monotonic allocation scheme. In Section 4, the nucleolus as an allocation mechanism in total big boss games is shown to yield a bi-monotonic allocation scheme. A final example indicates that this result does not extend to total clan games in which the clan consists of more than one player.

Notation: For two sets A and B, we write $A \subseteq B$ is A is a subset of B, and $A \subset B$ if A is a proper subset of B. For a finite set N, $2^N = \{S \mid S \subseteq N\}$ denotes the collection of subsets of N. A cooperative game with transferable utility is a tuple (N, v) consisting of a finite set N of players and a characteristic function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. Let $S \in 2^N \setminus \{\emptyset\}$. The subgame $(S, v_{|2^S})$ obtained from (N, v) by restricting attention to the coalitions contained in S is denoted, with a slight abuse of notation, by (S, v). The imputation set of (N, v) is denoted

$$I(N, v) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for every } i \in N\},$$

the core of (N, v) is denoted

$$C(N, v) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for each } S \in 2^N\}.$$

For each coalition $S \in 2^N \setminus \{\emptyset\}$ and each player $i \in S$, define

$$M_i(S, v) := v(S) - v(S \setminus \{i\})$$

to be the marginal contribution of player i to coalition S. A player $i \in N$ is a *veto player* if $v(S) = 0$ whenever $i \notin S$. The game (N, v) is *monotonic* if for each $S, T \in 2^N$ with $S \subset T$: $v(S) \leqslant v(T)$, or, equivalently, if for each $S \in 2^N \setminus \{N\}$ and each player $i \in N \setminus S$: $v(S \cup \{i\}) \geq v(S)$.

2 Total Clan Games

In this section, total clan games are defined and two characterizations of such games are provided. Clan games were introduced in Potters *et al.* (1989) to model conflicts between powerful players (clan members) and less influential players (non-clan members). Each clan member is a veto player. In addition, it is more profitable for any coalition of non-clan members to enter into negotiations with the clan as a group than to act as an individual; this is referred to as the union property.

Formally (see Potters *et al.*, 1989, p. 276), a game (N, v) is a *clan game* with clan $C \in 2^N \setminus \{\emptyset, N\}$, if it satisfies the following four conditions:

(a) Nonnegativity: $v(S) \geq 0$ for each coalition $S \subseteq N$.

(b) Nonnegative marginal contributions to the grand coalition: $M_i(N, v) \geq 0$ for each player $i \in N$.

(c) Clan property: every player $i \in C$ is a veto player, i.e., $v(S) = 0$ for each coalition S with $C \nsubseteq S$.

(d) Union property: $v(N) - v(S) \geq \sum_{i \in N \setminus S} M_i(N, v)$ if $C \subseteq S$.

For notational convenience, define $\mathcal{P}(C) := \{S \subseteq N \mid C \subseteq S\}$ as the collection of coalitions including all clan members. According to the clan property, these are the only coalitions that can possibly attain a positive value.

In many games (N, v) arising from practical situations, the subgames (S, v) inherit the structure of the original game (N, v):

- every subgame of a flow game (cf. Kalai and Zemel, 1982) is a flow game;
- every subgame of a linear production game (cf. Owen, 1975) is a linear production game;
- every subgame of a bankruptcy game (cf. Aumann and Maschler, 1985) is a bankruptcy game,

to name but a few. Similarly, it is easy to imagine that in practical situations giving rise to clan games, the same distinction between powerful and powerless players will exist in its subgames. A concrete example of a class of total clan games in which the clan consists of a single player can be found in Brânzei *et al.* (2000); see also Section 4.

We refer to clan games in which every subgame that contains the clan is again a clan game, as total clan games. Formally, a game (N, v) is a *total clan game* with clan $C \in 2^N \setminus \{\emptyset, N\}$, if (S, v) is a clan game (with clan C) for every coalition $S \in \mathcal{P}(C)$. Attention is restricted to coalitions in $\mathcal{P}(C)$, since the clan property of (N, v) implies that in the other subgames the characteristic function is simply the zero function.

Theorem 2.1 *Let (N, v) be a game and $C \in 2^N \setminus \{\emptyset, N\}$. The following claims are equivalent:*

(α): *(N, v) is a total clan game with clan C;*

(β): *(N, v) is monotonic, every player $i \in C$ is a veto player, and*

$$\forall S, T \in \mathcal{P}(C): \text{ if } S \subset T, \text{ then } v(T) - v(S) \geq \sum_{i \in T \setminus S} M_i(T, v). \tag{1}$$

(γ): *(N, v) is monotonic, every player $i \in C$ is a veto player, and*

$$\forall S, T \in \mathcal{P}(C): \text{ if } S \subset T \text{ and } i \in S \setminus C, \text{ then } M_i(S, v) \geq M_i(T, v). \tag{2}$$

Proof. $(\alpha) \Rightarrow (\beta)$: Assume that (N, v) is a total clan game with clan C. To prove that (N, v) is monotonic, let $S \subset N$ and $i \in N \setminus S$.

- If $C \not\subseteq S$, then $v(S) = 0 \leqslant v(S \cup \{i\})$ by properties (c) and (a) of the clan game (N, v);
- If $C \subseteq S$, then $v(S \cup \{i\}) - v(S) = M_i(S \cup \{i\}, v) \geq 0$ by property (b) applied to the clan game $(S \cup \{i\}, v)$.

This proves that (N, v) is monotonic. Every player $i \in C$ is a veto player by the clan property (c) of (N, v). Property (1) follows immediately from the union property (d) applied to the clan game (T, v).

$(\beta) \Rightarrow (\gamma)$: It suffices to prove (1) $\Rightarrow$ (2). We first prove that for all $U \in 2^N$ and $j, k \in N$:

$$\text{if } C \subset U, j \in U \setminus C, \text{ and } k \in N \setminus U, \text{ then } M_j(U,v) \geq M_j(U \cup \{k\}, v). \tag{3}$$

Let U, j, k be as in (3). Then

$$\begin{aligned} M_j(U,v) + M_k(U \cup \{k\}, v) &= v(U) - v(U \setminus \{j\}) + v(U \cup \{k\}) - v(U) \\ &= v(U \cup \{k\}) - v(U \setminus \{j\}) \\ &\geq M_j(U \cup \{k\}, v) + M_k(U \cup \{k\}, v), \end{aligned}$$

where the inequality follows from (1) with $S = U \setminus \{j\}$ and $T = U \cup \{k\}$. This proves (3). Write $T \setminus S = \{i_1, \ldots, i_k\}$. Repeated application of (3) yields

$$M_i(S,v) \geq M_i(S \cup \{i_1\}) \geq \ldots \geq M_i(S \cup \{i_1, \ldots, i_k\}) = M_i(T,v).$$

$(\gamma) \Rightarrow (\alpha)$: Assume that (γ) holds. To show: every subgame (T,v) with $T \in \mathcal{P}(C)$ is a clan game with clan C. Monotonicity of (N,v) and the fact that $v(\emptyset) = 0$ imply that v is nonnegative and all marginal contributions are indeed nonnegative. The clan property is trivial. To show that (T,v) also satisfies the union property, let $S \in \mathcal{P}(C), S \subset T$. Write $T \setminus S = \{i(1), \ldots, i(k)\}$. Then

$$\begin{aligned} v(T) - v(S) &= \sum_{m=1}^{k} M_{i(m)}(S \cup \{i_1, \ldots, i_{i(m)}\}, v) \\ &\geq \sum_{m=1}^{k} M_{i(m)}(T,v) \\ &= \sum_{i \in T \setminus S} M_i(T,v), \end{aligned}$$

where the inequality follows from (2). □

While (1) simply writes out the union property of subgames, the characterization of total clan games using inequality (2) provides an interesting link with concave games (Shapley, 1971). Recall that a game (N,v) is *concave* if for every pair of coalitions $S, T \in 2^N$ and every $i \in N$:

$$\text{if } i \in S \subseteq T, \text{ then } M_i(S,v) \geq M_i(T,v).$$

Comparing this with (2) indicates that total clan games require much fewer concavity conditions.

3 Monotonic Allocation Schemes

Sprumont (1990) introduces the notion of a *population monotonic allocation scheme* (pmas) for a cooperative game (N,v). A pmas specifies for each coalition $S \subseteq N$ an allocation of $v(S)$ over its members. Moreover, it reflects the intuition that there is 'strength in numbers': the share allocated to every player increases as the coalition to which he belongs grows larger. When we consider population monotonic allocation schemes in total clan games, we restrict attention to the allocation of $v(S)$ for coalitions $S \in \mathcal{P}(C)$, since other coalitions have value zero by the clan property. Every pmas constructed in this paper, however, can be extended to a pmas where also coalitions in $2^N \setminus \mathcal{P}(C)$ are taken into account, by simply allocating zero to each of the players in such a coalition.

Formally, consider a total clan game (N, v) with clan $C \in 2^N \setminus \{\emptyset, N\}$. A *pmas* for the game (N, v) is a vector $(x_{S,i})_{S \in \mathcal{P}(C), i \in S}$ of real numbers such that

$$\forall S \in \mathcal{P}(C): \quad \sum_{i \in S} x_{S,i} = v(S)$$

and

$$\forall S, T \in \mathcal{P}(C), \forall i \in S: \quad \text{if } S \subset T, \text{ then } x_{S,i} \leqslant x_{T,i}.$$

An imputation $y \in I(N, v)$ is *pmas extendable* if there exists a pmas $(x_{S,i})_{S \in \mathcal{P}(C), i \in S}$ such that $x_{N,i} = y_i$ for each player $i \in N$. Sprumont (1990, p. 382) proves that in a convex game, every core element is pmas extendable. Total clan games have a nonempty core (Potters *et al.*, 1989, p. 279) and every core element is pmas extendable.

Theorem 3.1 *Let (N, v) be a total clan game with clan $C \in 2^N \setminus \{\emptyset, N\}$ and let $y \in C(N, v)$. Then y is pmas extendable.*

Proof. Potters *et al.* (1989, p. 279) prove that

$$C(N, v) = \{z \in I(N, v) : z_i \leqslant M_i(N, v) \text{ for each } i \in N \setminus C\}.$$

Hence there exists, for each player $i \in N$, a number $\alpha_i \in [0, 1]$ such that

$$\begin{aligned} y_i &= \alpha_i M_i(N, v) && \text{if } i \in N \setminus C, \\ y_i &= \alpha_i \left[v(N) - \textstyle\sum_{j \in N \setminus C} \alpha_j M_j(N, v) \right] && \text{if } i \in C, \\ \textstyle\sum_{i \in C} \alpha_i &= 1. \end{aligned}$$

Intuitively: the non-clan members each receive a fraction of their marginal contribution to the grand coalition, whereas the clan members divide the remainder. Define for each $S \in \mathcal{P}(C)$ and $i \in S$:

$$x_{S,i} = \begin{cases} \alpha_i M_i(N, v) & \text{if } i \in S \setminus C, \\ \alpha_i \left[v(S) - \sum_{j \in S \setminus C} \alpha_j M_j(N, v) \right] & \text{if } i \in C. \end{cases}$$

Clearly $x_{N,i} = y_i$ for each player $i \in N$. We proceed to prove that $(x_{S,i})_{S \in \mathcal{P}(C), i \in S}$ is a pmas. Since $\sum_{i \in C} \alpha_i = 1$, it follows that $\sum_{i \in S} x_{S,i} = v(S)$. Now let $S, T \in \mathcal{P}(C)$ and $i \in S \subset T$.

- If $i \notin C$, then $x_{S,i} = x_{T,i} = \alpha_i M_i(N, v)$.

- If $i \in C$, then

$$\begin{aligned} x_{T,i} - x_{S,i} &= \alpha_i \left[v(T) - \sum_{j \in T \setminus C} \alpha_j M_j(N,v) \right] - \alpha_i \left[v(S) - \sum_{j \in S \setminus C} \alpha_j M_j(N,v) \right] \\ &= \alpha_i \left[v(T) - v(S) - \sum_{j \in T \setminus S} \alpha_j M_j(N,v) \right] \\ &\geq \alpha_i \left[v(T) - v(S) - \sum_{j \in T \setminus S} M_j(N,v) \right] \\ &\geq \alpha_i \left[v(T) - v(S) - \sum_{j \in T \setminus S} M_j(T,v) \right] \\ &\geq 0, \end{aligned}$$

  where the first inequality follows from nonnegativity of the marginal contributions, the second inequality follows from (2), and the final inequality from (1).

Consequently, $(x_{S,i})_{S \in \mathcal{P}(C), i \in S}$ is a pmas. □

In a pmas, every allocation $(x_{S,i})_{i \in S}$ is a core element of the subgame (S, v); cf. Sprumont, p. 380. This yields the desirable additional property that in none of the subgames a coalition has an incentive to split off and oppose its allocation.

Whereas a pmas allocates a larger payoff to each player as the coalitions grow larger, property (2) suggests a slightly different approach in total clan games: the marginal contribution of non-clan members actually *decreases* in a larger coalition. Think for instance of the clan members as owners of production facilities and the non-clan members as laborers: one can easily imagine their marginal product of labor to be decreasing in the face of more co-workers. Taking the decreasing influence of non-clan members into account, one might actually allocate a smaller amount to the non-clan members in larger coalitions. Moreover, to still maintain some stability, such allocations should still give rise to core allocations in the subgames. An allocation scheme satisfying these properties is called a *bi-monotonic allocation scheme* (bi-mas). Formally, consider a total clan game (N, v) with clan $C \in 2^N \setminus \{\emptyset, N\}$. A bi-mas for the game (N, v) is a vector $(x_{S,i})_{S \in \mathcal{P}(C), i \in S}$ of real numbers such that

- $\sum_{i \in S} x_{S,i} = v(S)$ for each $S \in \mathcal{P}(C)$;
- $x_{S,i} \leqslant x_{T,i}$ if $S, T \in \mathcal{P}(C)$, $S \subset T$, and $i \in S \cap C$;
- $x_{S,i} \geq x_{T,i}$ if $S, T \in \mathcal{P}(C)$, $S \subset T$, and $i \in S \setminus C$;
- $(x_{S,i})_{i \in S}$ is a core element of the subgame (S, v) for each coalition $S \in \mathcal{P}(C)$.

An imputation $y \in I(N,v)$ is *bi-mas extendable* if there exists a bi-mas $(x_{S,i})_{S\in\mathcal{P}(C),i\in S}$ such that $x_{N,i} = y_i$ for each player $i \in N$. Every core element in a total clan game is bi-mas extendable.

Theorem 3.2 *Let (N,v) be a total clan game with clan C and let $y \in C(N,v)$. Then y is bi-mas extendable.*

Proof. Take $(\alpha_i)_{i\in N} \in [0,1]^N$ as in the proof of Theorem 3.1. Define for each $S \in \mathcal{P}(C)$ and $i \in S$:

$$x_{S,i} = \begin{cases} \alpha_i M_i(S,v) & \text{if } i \in S \setminus C, \\ \alpha_i \left[v(S) - \sum_{j\in S\setminus C} \alpha_j M_j(S,v)\right] & \text{if } i \in C. \end{cases}$$

We proceed to prove that $(x_{S,i})_{S\in\mathcal{P}(C),i\in S}$ is a bi-mas. Since $\sum_{i\in C} \alpha_i = 1$, it follows that $\sum_{i\in S} x_{S,i} = v(S)$. Now let $S,T \in \mathcal{P}(C)$ and $i \in S \subset T$.

- If $i \in N \setminus C$, then $x_{S,i} = \alpha_i M_i(S,v) \geq \alpha_i M_i(T,v) = x_{T,i}$ by (2).
- If $i \in C$, then

$$\begin{aligned} x_{T,i} - x_{S,i} &= \alpha_i \left[v(T) - \sum_{j\in T\setminus C} \alpha_j M_j(T,v)\right] - \alpha_i \left[v(S) - \sum_{j\in S\setminus C} \alpha_j M_j(S,v)\right] \\ &= \alpha_i \left[v(T) - v(S) - \sum_{j\in T\setminus S} \alpha_j M_j(T,v) + \sum_{j\in S\setminus C} \alpha_j \left(M_j(S,v) - M_j(T,v)\right)\right] \\ &\geq \alpha_i \left[v(T) - v(S) - \sum_{j\in T\setminus S} \alpha_j M_j(T,v)\right] \\ &\geq 0, \end{aligned}$$

  where the first inequality follows from (1) and nonnegativity of the $(\alpha_j)_{j\in T\setminus S}$, and the second inequality follows from (2).

Finally, for each coalition $S \in \mathcal{P}(C)$, the vector $(x_{S,i})_{i\in S}$ is shown to be a core allocation of the clan game (S,v). Let $S \in \mathcal{P}(C)$. According to Potters *et al.* (1989, p. 279):

$$C(S,v) = \{z \in I(S,v) \mid z_i \leqslant M_i(S,v) \text{ for each } i \in S \setminus C\}.$$

Let $i \in S \setminus C$. Then $x_{S,i} = \alpha_i M_i(S,v) \leqslant M_i(S,v)$. Also, $\sum_{i\in S} x_{S,i} = v(S)$, so $(x_{S,i})_{i\in S}$ satisfies efficiency. To prove individual rationality, discern three cases:

- Let $i \in S \setminus C$. Then $x_{S,i} = \alpha_i M_i(S,v) \geq 0 = v(\{i\})$;
- Let $i \in S \cap C$ and $|C| = 1$. Then $C = \{i\}$ and by construction $\alpha_i = \sum_{j\in C} \alpha_j = 1$. Hence $x_{S,i} \geq x_{C,i} = \alpha_i v(C) = v(\{i\})$;

- Let $i \in S \cap C$ and $|C| > 1$. Then $x_{S,i} \geq x_{C,i} = \alpha_i v(C) \geq 0 = v(\{i\})$, since every player in C is a veto player.

Consequently, $(x_{S,i})_{S \in \mathcal{P}(C), i \in S}$ is a bi-mas. □

In the proof of Theorem 3.2, an explicit construction was used to extend arbitrary core elements in total clan games to a bi-monotonic allocation scheme. The next section indicates that in total clan games in which the clan consists of a single player, the nucleolus, applied to the original game and its subgames, yields a bi-mas.

4 The nucleolus in total big boss games

A practical example of total clan games can be found in Brânzei *et al.* (2000), who consider information collecting situations and their corresponding cooperative games. They model situations where an action taker in an uncertain situation can improve his action choices by gathering information from players more informed about the situation. Only the action taker can achieve a reward and is thus a veto player in the information collecting game. In fact, under an additional concavity condition Brânzei *et al.* (2000) show that these games are total clan games with a clan consisting of a single player, the action taker. Following Muto *et al.* (1988), they refer to such games as (total) *big boss* games. The inventory games of Tijs *et al.* (2000) provide an additional class of total big boss games. In total big boss games, the allocation scheme that assigns to each player in a subgame his payoff in the nucleolus, is a bi-monotonic allocation scheme. A final example indicates that this is not necessarily the case in total clan games where the clan contains more than one player.

Theorem 4.1 *Let (N, v) be a total clan game with clan $C \in 2^N \setminus \{\emptyset, N\}$ consisting of a single player: $|C| = 1$. Define $x_{S,i}$ for each $S \in \mathcal{P}(C)$ and $i \in S$ as follows:*

$$x_{S,i} := Nu_i(S, v),$$

where $Nu_i(S, v)$ is the payoff to player $i \in S$ in the nucleolus of the clan game (S, v). Then $(x_{S,i})_{S \in \mathcal{P}(C), i \in S}$ is a bi-mas.

Proof. Without loss of generality, $C = \{1\}$. Since the nucleolus is an element of the core whenever this set is nonempty, it is clear that for each coalition $S \in \mathcal{P}(C)$ indeed $\sum_{i \in S} x_{S,i} = \sum_{i \in S} Nu_i(S, v) = v(S)$ and $(x_{S,i})_{i \in S} \in C(S, v)$. Remains to prove the two monotonicity properties.

Let $S \in \mathcal{P}(C)$. An explicit formula for the nucleolus of the big boss game (S, v) is provided by Muto *et al.* (1988, Thm. 4.2):

$$x_{S,i} = Nu_i(S, v) = \begin{cases} v(N) - \sum_{j \in S \setminus \{1\}} \frac{1}{2} M_j(S, v) & \text{if } i = 1, \\ \frac{1}{2} M_i(S, v) & \text{if } i \in S \setminus \{1\}. \end{cases}$$

Now let $S, T \in \mathcal{P}(C)$, $S \subset T$, and $i \in S$.

- If $i \in S \setminus \{1\}$, then $x_{T,i} = \frac{1}{2}M_i(T,v) \leqslant \frac{1}{2}M_i(S,v) = x_{S,i}$ by (2);
- If $i = 1$, then

$$\begin{aligned} x_{T,i} - x_{S,i} &= \left[v(T) - \sum_{j \in T \setminus \{1\}} \frac{1}{2} M_j(T,v) \right] - \left[v(S) - \sum_{j \in S \setminus \{1\}} \frac{1}{2} M_j(S,v) \right] \\ &= v(T) - v(S) - \sum_{j \in T \setminus S} \frac{1}{2} M_j(T,v) \\ &\geq v(T) - v(S) - \sum_{j \in T \setminus S} M_j(T,v) \\ &\geq 0, \end{aligned}$$

  where the first inequality follows from nonnegativity of the marginal contributions and the second inequality from (1) applied to the clan/big boss game (T, v).

Hence, $(x_{S,i})_{S \in \mathcal{P}(C), i \in S}$ is a bi-mas. □

The fact that the nucleolus in total big boss games satisfies the properties of bi-monotonic allocation schemes provides additional support for this concept. Unfortunately, the nucleolus does not necessarily extend to a bi-mas in total clan games with more than one clan member.

Example 4.2 Consider the four-player game (N, v) with

$$v(S) = \begin{cases} 6 & \text{if } S = \{1,2,3\}, \\ 99 & \text{if } S = \{1,2,4\}, \\ 105 & \text{if } S = N, \\ 0 & \text{otherwise.} \end{cases}$$

The clan is $C = \{1, 2\}$ and $\mathcal{P}(C) = \{\{1,2\}, \{1,2,3\}, \{1,2,4\}, N\}$. It is easy to see that (N, v) is monotonic and that every player in C is a veto player. Moreover, the game satisfies the conditions in (2):

- for $S = \{1,2,3\}, T = N, i = 3 : M_i(S,v) = 6 - 0 = 6 = 105 - 99 = M_i(T,v)$, and
- for $S = \{1,2,4\}, T = N, i = 4 : M_i(S,v) = 99 - 0 = 99 = 105 - 6 = M_i(T,v)$.

Hence (N, v) is a total clan game with clan $C = \{1, 2\}$. Potters *et al.* (1989, p. 283) prove that the nucleolus of a game (N, v) with clan C is given by

$$Nu_i(N,v) = \begin{cases} t & \text{if } i \in C, \\ \min\{t, \frac{1}{2}M_i(N,v)\} & \text{if } i \in N \setminus C, \end{cases}$$

where $t \geq 0$ is the unique real number such that $\sum_{i \in N} Nu_i(N,v) = v(N)$. The game (N,v) has nucleolus $(34,34,3,34) \in \mathbb{R}^N$ with $t = 3$, the game $(\{1,2,4\},v)$ has nucleolus $(33,33,33) \in \mathbb{R}^{\{1,2,4\}}$. Hence player 4 receives a higher share in the coalition N than in coalition $\{1,2,4\}$, even though he is a non-clan member. Consequently, the nucleolus applied to (N,v) and its subgames does not yield a bi-mas. $\triangleleft$

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