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# Discussion Paper

No. 2006–125

## **QUANTITY CONSTRAINED GENERAL EQUILIBRIUM**

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December 2006

ISSN 0924-7815

# Quantity Constrained General Equilibrium

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December 12, 2006

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## Abstract

In a standard general equilibrium model it is assumed that there are no price restrictions and that prices adjust infinitely fast to their equilibrium values. In case of price restrictions a general equilibrium may not exist and rationing on net demands or supplies is needed to clear the markets. In the mid 1970s it was shown that in case of upper and lower bound restrictions on the prices there exists a quantity constrained equilibrium at which not both demand and supply of a good are rationed simultaneously and there is rationing on the net supply or net demand of a good only if the price of that good is on its lower or upper bound, respectively.

For an arbitrary set of admissible prices it was recently proposed to let the rationing schemes be determined by the components of a vector being a direction in which the prices are restricted to move. When the set of restricted prices is convex and compact, it was shown that there exists a connected set of such quantity constrained equilibria, containing two trivial no-trade equilibria without trade opportunities. In this paper we refine the concept of quantity constrained equilibrium and propose a specific quantity constrained equilibrium which may serve as a general equilibrium in case of price restrictions. At this equilibrium demand rationing and supply rationing are in balance with each other, so that trade opportunities are maximal and therefore trivial no-trade and other equilibria with less trade opportunities are excluded. Moreover, in equilibrium only relative prices matter. Any homogenous transformation or normalization of the set of admissible prices yields the same set of quantity constrained general equilibria up to scaling of the price vectors.

*Key words: exchange economy, price restrictions, general equilibrium, rationing.*

JEL-code: C62, C63, C68, D51.

# 1 Introduction

Perfect competition is a basic assumption in economic theory. Agents are assumed to be price takers and therefore express their demand and supply at the prevailing prices on the market. Trade takes place only at a price system for which for every commodity total demand equals total supply. It is assumed that there is no restriction on the prices and that prices adjust infinitely fast. Walras (1874) considered the problem of the existence of a general competitive equilibrium, corresponding to a price system at which all markets clear simultaneously. Under very general conditions the existence of such a Walrasian equilibrium was shown in the 1950s by Debreu (1959) and others.

Unemployment and excess supply on commodity and factor markets are apparently serious problems in many countries. Price restrictions and regulations, such as price controls to reduce inflation, see Cox (1980), Nguyen and Whalley (1986, 1990), and Ginsburgh and Van der Heyden (1988), price systems resulting from models with imperfect competition or incomplete markets, see Bénassy (1993), Drèze (1997) and Herings and Polemarchakis (2000), or for instance minimum wages and price indexation, often prevent prices from adjusting in the right direction, so that prices can not reach their Walrasian equilibrium values. Nevertheless also under price regulations trade takes place on the markets. When trade occurs against disequilibrium prices, markets may clear through rationing instead of prices, e.g. by imposing quantity rationing, queueing systems or production quota. In the mid 1970s Drèze (1975) and Benassy (1975) independently developed market clearing mechanisms for economies with price regulations by using quantity rationing. In that approach an agent chooses a most preferred consumption bundle, subject to both a budget constraint and quantity constraints on net demands and supplies. In order to let the market function frictionless, the quantity rationing may effect excess supply or excess demand, but not both simultaneously. When price rigidities are given by lower and upper bounds on the prices of the commodities, Drèze and Benassy proved the existence of an equilibrium, at which at least one a priori chosen commodity is not being rationed at all, for example the numeraire good or money. Further supply (demand) rationing on a nonnumeraire commodity can only be binding when its price is on its lower (upper) bound.

In the 1980s both van der Laan and Kurz argued that in practice rationing on net demand is rarely observed and difficult to implement. This motivated these authors to consider Drèze equilibria with only rationing on the net supplies. In van der Laan (1980, 1982) and Kurz (1982) the existence of a Drèze equilibrium, satisfying that only rationing on the net supplies occurs and at least one commodity is not rationed at all, has been proven. Such an equilibrium is called a supply constrained or unemployment equilibrium. In case the set of admissible prices is a cube determined by lower and upper bounds, it has been shown in Herings, Talman and Yang (1996) by using a simplicial technique, that there

exists a connected set of constrained equilibria, containing two trivial no-trade equilibria. At one of these equilibria all commodities are fully rationed in their supply and all prices are on their lower bound, whereas at the other trivial equilibrium all commodities are fully rationed in their demand and all prices are on their upper bound. The set also contains for every commodity a Drèze equilibrium at which that commodity is not being rationed at all. One of these Drèze equilibria is a supply constrained equilibrium with no rationing on the demands, another one is a demand constrained equilibrium with no rationing on the supplies.

In Herings, van der Laan and Talman (2004) general sets of admissible prices, allowing for an arbitrary compact and convex set of positive prices, are considered. For the general case it can not be assured that there is only rationing for a commodity if its price is equal to its (global) minimum or maximum value. In Herings et al. (2004) the rationing scheme is determined by a direction in which the prices are restricted to move, being a vector pointing outwards from the set of admissible prices. They show that there exists a connected set of so-called quantity constrained equilibria (QCEs) containing two trivial no-trade equilibria, one equilibrium with full rationing on the supplies and the sum of the prices being minimized, whereas at the other trivial no-trade equilibrium all consumers are fully rationed on their demands and the sum of the prices is maximized. These two equilibria do not allow for any trade opportunities for the consumers.

In this paper we propose to refine the quantity constrained equilibrium concept in case of restricted prices by maximizing the trade opportunities for the agents. We require that at a quantity constrained equilibrium the potential rationing levels on demands and supplies should be in balance with each other. For example, there should be no demand rationing on all markets and also no supply rationing on all markets. This automatically excludes the trivial no-trade equilibria at which there is one-sided full rationing on all markets but it also excludes other QCEs with too much demand rationing or too much supply rationing. We call such an equilibrium a quantity constrained general equilibrium (QCGE) and show that such an equilibrium always exists if the set of admissible prices is a nonempty convex and compact set in the interior of the price space.

It is well known that at a Walrasian equilibrium in an exchange economy without price restrictions only relative prices matter. When all prices in a Walrasian equilibrium are multiplied with a positive constant, the new price vector is also a Walrasian equilibrium price vector, yielding the same equilibrium allocation. We will show that this property also holds for a quantity constrained general equilibrium. When all prices in a QCGE are multiplied with a positive constant and the new price vector is still an element of the set of admissible prices, the new price vector is also a QCGE price vector for the same rationing scheme and yielding the same equilibrium allocation. This property does not

hold for quantity constrained equilibria that are not general equilibria. It also implies that when prices are normalized the set of quantity constrained general equilibrium allocations is the same. More general, if the set of admissible prices is transformed in a homogenous way, the set of quantity constrained general equilibrium allocations does not change. This also implies that if the set of admissible prices is unbounded or not convex but can be homogeneously transformed to a compact and convex set a quantity constrained general equilibrium will exist. Also because of this homogeneity property the concept of a quantity constrained general equilibrium seems to be in case of price restrictions the equilibrium concept which is closest to a Walrasian equilibrium. In case a Walrasian equilibrium price vector happens to be an element of the set of admissible prices, then this equilibrium is also a quantity constrained (general) equilibrium, one at which no (binding) rationing takes place and therefore having maximal trade opportunities.

The paper is organized as follows. Section 2 describes the model of Drèze and introduces the concept of quantity constrained general equilibrium. For the compact, convex case with positive prices the existence results are given in Section 3. Homogeneity properties cases are discussed in Section 4, including the unbounded case.

## 2 The model

We consider an exchange economy  $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P)$ . In this economy there are  $m$  consumers, indexed  $i = 1, \dots, m$ , and  $n$  commodities, indexed  $j = 1, \dots, n$ . For  $k$  a positive integer, we denote  $I_k = \{1, \dots, k\}$ . Each consumer  $i \in I_m$  is characterized by a consumption set  $X^i$ , a preference preordering  $\succeq^i$  on  $X^i$ , and a vector of initial endowments  $w^i$ . The vector  $w$  is defined by  $w = \sum_{i \in I_m} w^i$ . We assume that the economy  $\mathcal{E}$  is faced with a set  $P \subset \mathbb{R}_+^n$  of admissible prices.

The following standard assumptions X, U and W with respect to the economy  $\mathcal{E}$  are made.

### Assumption X

For every consumer  $i \in I_m$ , the consumption set  $X^i$  is a closed and convex subset of  $\mathbb{R}_+^n$  and  $X^i + \mathbb{R}_+^n \subset X^i$ .

### Assumption U

For every consumer  $i \in I_m$ , the preference preordering  $\succeq^i$  on  $X^i$  is complete, continuous, strongly monotonic, and strictly convex.<sup>1</sup>

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<sup>1</sup>A preference preordering  $\succeq^i$  is said to be strongly monotonic if  $\bar{x}^i, \hat{x}^i \in X^i$ ,  $\bar{x}^i \leq \hat{x}^i$ , and  $\bar{x}^i \neq \hat{x}^i$  implies  $\hat{x}^i \succ^i \bar{x}^i$ . A preference preordering  $\succeq^i$  is said to be strictly convex when for any pair  $\bar{x}^i, \hat{x}^i \in X^i$ ,

### Assumption W

For every consumer  $i \in I_m$ , the vector of initial endowments  $w^i$  belongs to the interior of  $X^i$ .

The assumption of strict convexity allows us to work with demand functions instead of demand correspondences. Concerning the set of admissible prices we make the following assumption.

### Assumption P

The set  $P$  of admissible prices is a nonempty, convex and compact subset in the interior of  $\mathbb{R}_+^n$ .

The assumption that the set of admissible prices lies in the interior of  $\mathbb{R}_+^n$  allows us to stay away from prices equal to zero. The other assumptions are needed to prove existence, although boundedness is not essential, as we will see later on.

The set  $P$  of admissible prices may or may not contain a Walrasian price system for the economy  $\mathcal{E}$ , being a nonzero price system  $p^* \in \mathbb{R}_+^n$  such that  $\sum_{i \in I_m} x^{*i} = w$  with  $x^{*i}$  a best element for  $\succeq^i$  in the budget set  $\{x^i \in X^i \mid p^* \cdot x^i \leq p^* \cdot w^i\}$  of consumer  $i$ ,  $i \in I_m$ . When not, one may introduce an equilibrium concept involving vectors of quantity constraints on the net demands and the net supplies of the commodities. Given a price system  $p \in P$ , a rationing scheme on (net) supply  $\ell \in -\mathbb{R}_+^n$ , and a rationing scheme on (net) demand  $u \in \mathbb{R}_+^n$ , the constrained budget set of consumer  $i \in I_m$  is given by

$$B^i(p, \ell, u) = \{x^i \in X^i \mid p \cdot x^i \leq p \cdot w^i \text{ and } \ell_k w_k \leq x_k^i - w_k^i \leq u_k w_k, \forall k \in I_n\}.$$

The number  $\ell_k(u_k)$  is the fraction of the total endowment of commodity  $k$  each consumer is allowed to supply (demand) maximally. The corresponding constrained demand  $d^i(p, \ell, u)$  of consumer  $i$  is defined as the best element for  $\succeq^i$  in  $B^i(p, \ell, u)$ . Because of the strict convexity and strong monotonicity assumptions and since  $B^i(p, \ell, u)$  is not empty and compact, this element is unique and lies on the budget hyperplane, i.e.  $p \cdot d^i(p, \ell, u) = p \cdot w^i$ . If  $x_k^i - w_k^i = \ell_k w_k$  for some  $i \in I_m$  and  $k \in I_n$ , we say that consumer  $i$  is constrained or rationed on his supply, and if  $x_k^i - w_k^i = u_k w_k$  for some  $i \in I_m$  and  $k \in I_n$ , we say that consumer  $i$  is constrained or rationed on his demand.

In equilibrium we require that prices and rationing schemes are such that aggregate constrained demand equals aggregate supply, i.e., all markets clear. Moreover, the rationing schemes should allow for maximal trade opportunities. This means that markets have to be frictionless in the sense that if some consumer is constrained on his demand (supply)

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such that  $\bar{x}^i \neq \hat{x}^i$ ,  $\bar{x}^i \sim^i \hat{x}^i$ , holds  $\lambda \bar{x}^i + (1 - \lambda) \hat{x}^i \succ^i \bar{x}^i$  for  $\lambda \in (0, 1)$ .

of a commodity, no consumer is constrained on his supply (demand) of that commodity. Moreover, the rationing on demand and the rationing on supply over all markets should be balanced in the sense that in equilibrium there should be neither too much demand rationing nor too much supply rationing. For example, if there is full supply rationing on all markets, there are no trade opportunities at all, because no consumer is allowed to supply anything. Similarly, in case of full demand rationing on all markets, there are no trade opportunities either, since no agent is able to demand more than his endowment.

Further, there should be a direct link between the rationing scheme and the price restriction. If there are no restriction on the price, i.e., the price vector lies in the interior of the set of admissible prices, there should be no rationing at all, neither on the demand sides nor on the supply sides, because prices are flexible enough to move towards an equilibrium price. In case the prices are restricted, i.e., the price vector lies on the boundary of the set of admissible prices, there is some direction in which the prices can not move any further. In equilibrium, this direction should completely determine the rationing levels in case of rationing. If a component of this direction is positive only demand rationing might occur on the corresponding market, and if a component of it is negative only supply rationing might occur. Moreover, the (absolute) values of the components of the direction in which the price vector is not able to move should determine in equilibrium how severe the rationing has to be, the higher the value, the more severe the rationing should be. A direction into which a restricted price vector  $p$  can not move is a vector that points outward at  $p$  to the set of admissible prices. The set of vectors that at a given price vector  $p$  point outwards to the set  $P$  of admissible prices is called the subgradient at  $p$  to  $P$ . Formally, for  $p \in P$  the subgradient  $G(p)$  at  $p$  to the set  $P$  is given by

$$G(p) = \{r \in \mathbb{R}^n \mid \hat{p} \cdot r \leq p \cdot r \text{ for any } \hat{p} \in P\}.$$

**Definition 2.1 (Quantity Constrained General Equilibrium)**

A Quantity Constrained General Equilibrium QCGE for the economy  $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P)$  is a price system  $p^* \in P$ , a rationing scheme  $(\ell^*, u^*) \in -\mathbb{R}_+^n \times \mathbb{R}_+^n$ , and, for every consumer  $i \in I_m$ , a consumption bundle  $x^{*i} \in X^i$  such that

- (i) for all  $i \in I_m$ ,  $x^{*i} = d^i(p^*, \ell^*, u^*)$ ;
- (ii)  $\sum_{i=1}^m x^{*i} = w$ ;
- (iii) for all  $k \in I_n$ :  $x_k^{*h} - w_k^h = \ell_k^* w_k$  for some  $h \in I_m$  implies  $x_k^{*i} - w_k^i < u_k^* w_k, \forall i \in I_m$ , and, analogously,  $x_k^{*h} - w_k^h = u_k^* w_k$  for some  $h \in I_m$  implies  $x_k^{*i} - w_k^i > \ell_k^* w_k, \forall i \in I_m$ ;



(iv) there exists  $r^* \in G(p^*)$  such that  $p^* \cdot r^* = 0$  and for all  $k \in I_n$  it holds that if there exists  $i \in I_m$  such that  $\ell_k^* w_k = x_k^{*i} - w_k^i$ , then  $\ell_k^* = -1 - r_k^*$ , and if there exists  $i \in I_m$  such that  $u_k^* w_k = x_k^{*i} - w_k^i$ , then  $u_k^* = 1 - r_k^*$ .

In this definition, the rationing schemes on supply and demand are assumed to be uniform, i.e. the same for each consumer. This assumption can be easily relaxed. Condition (i) requires that the consumption of each consumer equals his constrained demand, i.e., each consumer is maximizing his utility given the equilibrium prices and rationing scheme. Condition (ii) is the market clearing condition. Condition (iii) implies that there can be no simultaneous rationing on both sides of any market, so that all markets are frictionless. Condition (iv) links the rationing scheme to the price restriction. In case of rationing, the level of rationing is completely determined by a vector  $r^*$  which is orthogonal to the price vector and a direction in which the prices can not further move. If  $r_k^* = 0$  for some good  $k$ , no consumer can be rationed on his demand or supply of good  $k$ , since in equilibrium for every consumer  $i$  it holds that  $-w_k < -w_k^i \leq x_k^{*i} < w_k$ .

In case there is no price restriction at all at the equilibrium price  $p^*$ , i.e.,  $p^*$  lies in the (full-dimensional) interior of  $P$ , then the subgradient of  $P$  at  $p^*$  is just the origin and prices can move freely in any direction. In this case  $r^* = 0^n$  and no rationing will take place. A QCGE at an unrestricted price vector is therefore a Walrasian equilibrium.

In case there is some price restriction at the equilibrium price  $p^*$ , i.e., the price vector  $p^*$  is on the boundary of the set  $P$ , the subgradient of  $P$  at  $p^*$  contains also nonzero vectors and therefore  $r^*$  might be a nonzero vector. If  $r^* = 0^n$  the equilibrium is again a Walrasian equilibrium and the equilibrium price vector happens to be on the boundary of the set of admissible prices. If  $r_k^* \geq 0$  no supply rationing can occur on market  $k$  and if  $r_k^* \leq 0$  no demand rationing can occur on market  $k$ . Since  $r^*$  is a direction in which the prices can not further move at  $p^*$ , demand rationing for a good can only occur if prices can not be moved in a direction at which the price of that good is increased and supply rationing for a good can only occur if in that direction the price of that good cannot be decreased. Moreover, in case of rationing the values of the components of this direction  $r^*$  completely determine the levels of rationing. A higher positive  $r_k^*$  yields more severe demand rationing since in case of demand rationing on market  $k$  it holds that  $u_k^* = 1 - r_k^*$ , whereas a more negative  $r_k^*$  yields more severe supply rationing since in case of supply rationing it holds that  $\ell_k^* = -1 - r_k^*$ . In case of demand rationing on market  $k$  the component  $r_k^*$  lies between 0 and 1 and in case of supply rationing on market  $k$  the component  $r_k^*$  lies between 0 and  $-1$ . When  $r_k^* = 1$  there is complete demand rationing on market  $k$  and no consumer is allowed to demand more of commodity  $k$  than his initial endowment, and since markets clear in equilibrium no consumer will supply something of good  $k$  either. Similarly, when  $r_k^* = -1$  there is complete supply rationing on market  $k$  and no consumer is able to supply

anything of good  $k$  and therefore in equilibrium no consumer can demand something of it either.

The condition  $p^* \cdot r^* = 0$  implies that in equilibrium demand and supply rationing are in balance with each other. Too much demand rationing or too much supply rationing restricts the trade opportunities of the consumers. For example, if  $r_k^* > 0$  for all  $k$ , there could be demand rationing on all markets, which limits the trade opportunities for all consumers. Similarly, if  $r_k^* < 0$  for all  $k$ , there might be supply rationing on all markets, so that the trade opportunities are also limited. In general, these outcomes lead to not very satisfactory equilibria or even to trivial no-trade equilibria. The condition  $p^* \cdot r^* = 0$  means that either  $r^* = 0^n$ , in which case there is no rationing at all and we have a Walrasian equilibrium without any rationing, or some of the components of  $r^*$  are positive and some other components of  $r^*$  are negative. Only for the positive components there can be demand rationing and only for the negative components there can be supply rationing. In this way, in equilibrium there can be not too much demand rationing or too much supply rationing on all markets, so that equilibria with too few or no trade opportunities are excluded. Remark that although a nonzero vector  $r^*$  has both positive and negative components this does not mean that there will be always both demand and supply rationing. Only demand rationing or only supply rationing may occur but not simultaneously on all markets.

### 3 Existence results

To show the existence of a QCGE, we follow Herings et al. (2004) and introduce a full-dimensional compact and convex set  $Q \subset \mathbb{R}^n$  with smooth boundary and containing the set of admissible prices  $P$  in its interior and define for every  $q \in Q$  a price vector  $p(q) \in P$ , a direction  $r(q)$  in the subgradient  $G(p(q))$ , and a rationing scheme  $(\ell(q), u(q)) \in -\mathbb{R}_+^n \times \mathbb{R}_+^n$ . The set  $Q$  is taken to be set

$$Q = \{q \in \mathbb{R}^n \mid \|q - p\|_2 \leq 1 \text{ for some } p \in P\}.$$

For  $q \in Q$ , the admissible price vector  $p(q) \in P$  is defined by the orthogonal projection of  $q$  on  $P$ , i.e.,

$$p(q) = \arg \min_{p \in P} \|p - q\|_2.$$

Since by Assumption P, the set  $P$  is convex and compact, for every  $q \in Q$  it holds that  $p(q)$  is uniquely defined and continuous in  $q$  and that  $q - p(q) \in G(p(q))$ . For  $q \in Q$ , the vector  $r(q) \in \mathbb{R}^n$  is defined by

$$r(q) = 0^n \text{ when } q \in P$$

and, for  $k \in I_n$ ,

$$r_k(q) = \frac{q_k - p_k(q)}{\max_{h \in I_n} |q_h - p_h(q)|} \|q - p(q)\|_2, \text{ when } q \in Q \setminus P.$$

The function  $r(\cdot)$  is a continuous function on  $Q$  and for all  $q \in Q$  it holds that  $r(q) \in G(p(q))$  and  $-1 \leq r_k(q) \leq 1$  for all  $k \in I_n$ . Finally, for  $q \in Q$  the rationing scheme  $(\ell(q), u(q)) \in -\mathbb{R}_+^n \times \mathbb{R}_+^n$  is defined by

$$\ell_k(q) = -1 - r_k(q) \text{ and } u_k(q) = 1 - r_k(q), \quad k \in I_n.$$

We now define for any consumer  $i \in I_m$  his reduced budget set  $B^i(q)$  by

$$B^i(q) = B^i(p(q), \ell(q), u(q)), \quad q \in Q$$

and his reduced excess demand  $d^i: Q \rightarrow \mathbb{R}^n$  by

$$d^i(q) = \{x^i \in B^i(q) | x^i \succeq^i y^i, \text{ for all } y^i \in B^i(q)\}.$$

From Herings et al. (2004) it follows that  $d^i$  is a continuous function on  $Q$  and so is the reduced excess demand function  $z: Q \rightarrow \mathbb{R}^n$  defined by

$$z(q) = \sum_{i \in I_m} d^i(q) - w.$$

The next result shows that if  $q^*$  is a zero point of  $z$  satisfying  $p(q^*) \cdot r(q^*) = 0$ , then  $q^*$  induces a QQCGE.

### Lemma 3.1

Let  $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P)$  be an economy satisfying Assumptions  $X$ ,  $U$ ,  $W$  and  $P$  and let  $q^*$  be a zero point of the corresponding reduced excess demand function  $z$ , i.e.  $z(q^*) = 0^n$ , satisfying  $p(q^*) \cdot r(q^*) = 0$ . Then the price system  $p^* = p(q^*)$ , the rationing scheme  $(\ell^*, u^*) = (\ell(q^*), u(q^*)) \in -\mathbb{R}_+^n \times \mathbb{R}_+^n$  and, for each  $i \in I_m$ , the bundle  $x^{*i}$  given by  $x^{*i} = d^i(q^*)$ , constitute a quantity constrained general equilibrium.

### Proof.

We have to show that the conditions (i), (ii), (iii) and (iv) of Definition 2.1 hold. By construction it holds for all  $i \in I_m$  that  $d^i(q^*) = d^i(p(q^*), \ell(q^*), u(q^*))$ , so that  $x^{*i} = d^i(p^*, \ell^*, u^*)$ , and hence condition (i) holds. Since  $z(q^*) = 0^n$  and  $z(q^*) = \sum_{i \in I_m} d^i(q^*) - w = \sum_{i \in I_m} x^{*i} - w$ , also condition (ii) holds. From  $\sum_{i \in I_m} x^{*i} = w$  it follows for every  $i \in I_m$  that  $0^n \leq x^{*i} \leq w$  and therefore from Assumption  $W$  that  $-w_k < x_k^{*i} - w_k^i < w_k$ ,  $k \in I_n$ . Moreover, for any  $k \in I_n$ , it follows that  $\ell_k^* \leq -1$  when  $r_k(q^*) \geq 0$  and  $u_k^* \geq 1$  when  $r_k(q^*) \leq 0$ . So, if for some  $h \in I_m$  it holds that  $x_k^{*h} - w_k^h = \ell_k^* w_k$ , we must have  $\ell_k^* > -1$  and therefore  $r_k(q^*) < 0$  and thus  $u_k^* \geq 1$ , and so  $x_k^{*i} - w_k^i < u_k^* w_k$  for all  $i \in I_m$ . Analogously,

if for some  $h \in I_m$  it holds that  $x_k^{*h} - w_k^h = u_k^* w_k$ , we must have  $u_k^* < 1$  and therefore  $r_k(q^*) > 0$  and thus  $\ell_k^* \leq -1$ , and so  $x_k^{*i} - w_k^i > \ell_k^* w_k$  for all  $i \in I_m$ . This proves condition (iii). To prove condition (iv), consider the vector  $r^* = r(q^*)$ . Since  $r(q^*) \in G(p(q^*))$ , it holds that  $r^* \in G(p^*)$ . Moreover, it holds for all  $k \in I_n$  that  $\ell_k^* = -1 - r_k^*$  and  $u_k^* = 1 - r_k^*$ . Therefore, condition (iv) is also satisfied if  $p^* \cdot r^* = p(q^*) \cdot r(q^*) = 0$ . Q.E.D.

From Lemma 3.1 it follows that the question of existence of a QCGE reduces to the existence of a zero point  $q^*$  of  $z$  in  $Q$  satisfying  $p(q^*) \cdot r(q^*) = 0$ . Let  $Q_0$  and  $Q_1$  be given by

$$Q_0 = \{q \in Q \mid e \cdot q \leq e \cdot \tilde{q} \text{ for all } \tilde{q} \in Q\}$$

and

$$Q_1 = \{q \in Q \mid e \cdot q \geq e \cdot \tilde{q} \text{ for all } \tilde{q} \in Q\},$$

where  $e \in \mathbb{R}^n$  is the vector with all components equal to one. Clearly, since  $Q$  is compact, the sets  $Q_0$  and  $Q_1$  are both non-empty. Moreover, the intersection of  $Q_0$  and  $Q_1$  is empty, since  $Q$  is full-dimensional. The existence of a quantity constrained general equilibrium follows from the next result, shown in Herings et al. (2004), saying that  $Q$  contains a connected set  $C$  of zero points of  $z$  having a nonempty intersection with both  $Q_0$  and  $Q_1$ .

**Lemma 3.2** *Let  $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P)$  be an economy satisfying Assumptions X, U, W and P. Then there exists a connected set  $C$  of zero points of the corresponding reduced excess demand function  $z$  in  $Q$  satisfying  $C \cap Q_0 \neq \emptyset$  and  $C \cap Q_1 \neq \emptyset$ .*

From this lemma the next main result follows.

**Theorem 3.3**

*Let  $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P)$  be an economy satisfying Assumptions X, U, W and P, then a quantity constrained general equilibrium of  $\mathcal{E}$  exists.*

**Proof.**

From Lemma 3.2 it follows that there exists a connected set  $C \subset Q$  of zero points of the corresponding reduced excess demand function  $z$  such that  $C \cap Q_0 \neq \emptyset$  and  $C \cap Q_1 \neq \emptyset$ . For  $q^0 \in C \cap Q_0$  it holds that  $r(q^0) = -e$  and therefore  $p(q^0) \cdot r(q^0) < 0$ , whereas for  $q^1 \in C \cap Q_1$  it holds that  $r(q^1) = e$  and therefore  $p(q^1) \cdot r(q^1) > 0$ . Since  $C$  is a connected set having a nonempty intersection with both  $Q_0$  and  $Q_1$  and since both functions  $p(\cdot)$  and  $r(\cdot)$  are continuous on  $Q$ , there exists  $q^* \in C$  satisfying  $p(q^*) \cdot r(q^*) = 0$ . From Lemma 3.1 it follows that  $q^*$  induces a quantity constrained general equilibrium of  $\mathcal{E}$  Q.E.D.

## 4 Properties

In this section we show that a quantity constrained general equilibrium is not sensitive for a homogenous transformation of the set of admissible prices. We call the set  $P'$  a homogenous transformation of the set  $P$  if for every  $p' \in P'$  there exists a  $p \in P$  satisfying  $p = \lambda p'$  for some  $\lambda > 0$  and for every  $p \in P$  there exists a  $p' \in P'$  satisfying  $p' = \mu p$  for some  $\mu > 0$ . Notice that for a particular point  $p' \in P'$  there can be more than one  $p \in P$  satisfying  $p = \lambda p'$  for some  $\lambda > 0$ . If  $P'$  is a homogenous transformation of a set of admissible prices  $P$  and the economy further consists of the same consumers, then in general the sets of quantity constrained equilibria differ for both economies. This is caused by the fact that the subgradients of  $P$  at  $p$  and of  $P'$  at  $p'$  may differ from each other when  $p = \lambda p'$  for some  $\lambda > 0$ . However, for quantity constrained general equilibria this is not the case, since as will be shown in the next theorem any QCGE for the economy with  $P$  as admissible set of prices induces a QCGE with the same rationing schemes and allocation for the economy with  $P'$  as the set of admissible prices.

### Theorem 4.1

*Suppose  $P'$  is a homogenous transformation of  $P$  and let  $(p^*, (\ell^*, u^*), (x^{*i})_{i=1}^m)$  be a QCGE of the economy  $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P)$  satisfying Assumptions  $X$ ,  $U$ ,  $W$  and  $P$ . Then for any  $p' \in P'$  satisfying  $p' = \mu p^*$  for some  $\mu > 0$ , and such a  $p'$  exists, it holds that  $(p', (\ell^*, u^*), (x^{*i})_{i=1}^m)$  is a QCGE of the economy  $\mathcal{E}' = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P')$ .*

### Proof.

Since  $(p^*, (\ell^*, u^*), (x^{*i})_{i=1}^m)$  is a QCGE of the economy  $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P)$ , there exists  $r^* \in G(p^*)$  satisfying condition (iv) with respect to this equilibrium. Since  $r^* \in G(p^*)$  and  $p^* \cdot r^* = 0$ , it holds that  $p \cdot r^* \leq 0$  for all  $p \in P$ . Now take any  $p' \in P'$  satisfying  $p' = \mu p^*$  for some  $\mu > 0$ . Such a  $p'$  exists because  $P$  and  $P'$  are homogenous transformations from each other. We now show that  $(p', (\ell^*, u^*), (x^{*i})_{i=1}^m)$  is a QCGE in the economy  $\mathcal{E}'$ . Clearly, for every consumer  $i \in I_m$ , it holds that  $B'_i(p', \ell^*, u^*) = B_i(p^*, \ell^*, u^*)$ , where  $B'_i(p', \ell^*, u^*)$  is the constrained budget set of consumer  $i$  in the economy  $\mathcal{E}'$  at price vector  $p'$  and rationing scheme  $(\ell^*, u^*)$ . Therefore, for all  $i \in I_m$  it holds that  $x^{*i} = d'_i(p', \ell^*, u^*)$ , where  $d'_i(p', \ell^*, u^*)$  is the constrained demand of consumer  $i$  in the economy  $\mathcal{E}'$  at price vector  $p'$  and rationing scheme  $(\ell^*, u^*)$ . Hence,  $(p', (\ell^*, u^*), (x^{*i})_{i=1}^m)$  satisfies condition (i) of a QCGE in the economy  $\mathcal{E}'$ . Since the rationing schemes and allocation are the same, conditions (ii) and (iii) of a QCGE in the economy  $\mathcal{E}'$  are automatically satisfied at  $(p', (\ell^*, u^*), (x^{*i})_{i=1}^m)$ . To show that also condition (iv) of a QCGE in the economy  $\mathcal{E}'$  is satisfied at  $(p', (\ell^*, u^*), (x^{*i})_{i=1}^m)$ , we still have to show that  $r^* \in G'(p')$  and  $p' \cdot r^* = 0$ , where  $G'(p')$  is the subgradient of  $P'$  at  $p'$ . Take any  $p \in P'$  and let  $p''$  be any point in  $P$  satisfying  $p = \mu p''$  for some  $\mu > 0$ . From  $p'' \in P$  it follows that  $p'' \cdot r^* \leq 0$  and from

$p^* \cdot r^* = 0$  it follows that  $p' \cdot r^* = 0$ . Therefore,

$$p \cdot r^* = \mu p'' \cdot r^* \leq 0 = p' \cdot r^*.$$

Since  $p$  is an arbitrary point in  $P'$ , we obtain that  $r^* \in G'(p')$ , which completes the proof. Q.E.D.

Notice that it is not assumed that the set  $P'$  is bounded or convex. From the theorem it also follows that if  $p^*$  is a QCGE price vector and  $p$  in  $P$  is a multiple vector of  $p^*$ , i.e.,  $p = \lambda p^*$  for some  $\lambda > 0$ , then  $p$  is also a QCGE price vector with the same rationing scheme and allocation as for  $p^*$ . This is caused by the fact that for the vector  $r^*$  that exists in condition (iv) with respect to  $p^*$  it holds that  $p \cdot r^* = 0$  and  $r^* \in G(p)$ , so that condition (iv) also holds with respect  $p$  for the same  $r^*$ , rationing scheme and allocation.

### Corollary 4.2

*Let  $(p^*, (\ell^*, u^*), (x^{*i})_{i=1}^m)$  be a QCGE of the economy  $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P)$  satisfying Assumptions X, U, W and P, and let  $p' \in P'$  be such that  $p' = \lambda p^*$  for some  $\lambda > 0$ . Then  $(p', (\ell^*, u^*), (x^{*i})_{i=1}^m)$  is also a QCGE of the economy  $\mathcal{E}$ .*

This result is very well known for a Walrasian equilibrium in an exchange economy without restricted prices. When in such an economy all components of a Walrasian equilibrium price vector are multiplied with the same (positive) constant, the new price vector is also a Walrasian equilibrium price vector, yielding the same allocation. In this way the set of prices can be normalized, for example by summing up all prices to some constant or taking one of the prices equal to one. Theorem 4.1 tells us that such a normalization is also allowed in case there are price restrictions and we take as equilibrium concept the quantity constrained general equilibrium. Any normalization that leads to a homogenous transformation of the set of admissible prices yields the same set of quantity constrained general equilibria. For example, dividing every component of any admissible price vector by the sum of the components of that vector leads to a homogenous transformation with sum of the prices equal to one and yields therefore the same set of equilibrium allocations. Another example is to divide every component of any admissible price vector by a specific component of that vector. Then a homogenous transformation of the set of admissible prices is obtained at which the price of one of the goods is always equal to one. The set of QCGEs is the same, but the prices are normalized by taking one of the goods as the numeraire having price equal to one.

Theorem 4.1 implies that for quantity constrained general equilibria only relative prices matter. When in equilibrium relative prices do not change and prices are still admissible, then again an equilibrium is obtained for the same rationing scheme and allocation. In this

way also the existence of a quantity constrained general equilibrium can be analyzed in case the set of admissible prices is unbounded. For example, if the set of admissible prices is a convex closed cone in (the interior of)  $\mathbb{R}_+^n$ , so that  $p' \in P$  whenever  $p' = \lambda p$  for some  $\lambda > 0$  and  $p \in P$ , then there exists a ray of quantity constrained equilibrium price vectors. Each price vector on this ray corresponds to the same equilibrium rationing scheme and allocation. The existence of such a ray of equilibrium prices is guaranteed by the fact that the set  $P$  of admissible prices can be homogeneously transformed to a convex and compact set  $P'$  of prices, e.g., by taking the sum of the prices equal to 1. Every QCGE with respect to the latter set is also a QCGE with respect to the set  $P$  and corresponds to a ray of QCGEs with respect to the latter set. If the set  $P$  of admissible prices is unbounded but not a convex and closed cone, a quantity constrained general equilibrium is guaranteed to exist if there exist a homogenous transformation of  $P$  which is a compact and convex subset in  $\mathbb{R}_+^n$ .

### Corollary 4.3

*Suppose that the Assumptions X, U and W hold for the economy  $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, P)$  and suppose also that there exists a homogenous transformation of  $P$  which is a nonempty compact and convex subset in the interior of  $\mathbb{R}_+^n$ . Then there exists a QCGE for the economy  $\mathcal{E}$ .*

We remark that a homogenous transformation of  $P$ , which is a nonempty compact and convex subset in the interior of  $\mathbb{R}_+^n$ , may not exist, even if  $P$  is a nonempty closed and convex set in the interior of  $\mathbb{R}_+^n$ . In that case a QCGE may not exist.

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