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LANDFILL CONSTRUCTION AND CAPACITY EXPANSION

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Landfill Construction and Capacity Expansion*

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Abstract

We study the optimal capacity and lifetime of landfills taking into account their sequential nature. Such an optimal capacity is characterized by the so-called Optimal Capacity Condition. Particular versions of this condition are obtained for two alternative settings: first, if all the landfills are to have the same capacity, and second, if each of them is allowed to have a different capacity. In the second case we obtain an Optimal Control problem, with mixed elements of both continuous and discrete time. The resulting optimization problems involve dividing a time horizon of planning into several subintervals of endogenously decided length. The results obtained may be useful to address other economic problems such as private and public investments, consumption decisions on durable goods, etc.

Keywords: Landfills, Non-renewable resources, Optimal Capacity, Optimal Control, Set-up costs.

JEL codes: Q30, C61.

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1 Introduction

Waste management has become an important problem from a social, technical and economic point of view (see, for example, Quadrio-Curzio et al 1994, Beede and Bloom 1995, Porter 2002 or Fullerton and Kinnaman 2002). Among all the available techniques for solid waste management, landfilling has traditionally been, and still is, the most widely used. As noted by Williams 1994 "landfilling is the waste treatment option that nobody wants, but everybody needs. Simply, there is no combination of waste treatment techniques which do not need, to some extent, the use of landfills". The rest of treatment techniques generate some by-products that can not be fully eliminated and have to be landfilled.

The management of landfills has changed dramatically during the last 20 years for both economic and environmental reasons, including the growing price of land in densely populated urban areas and the increased concern for the effects of dumps on our health and the environment. At the start of the 1970's, there were 20,000 landfills in the United States, but by the end of the 1980s only 6,000 and by 1998 barely 2,000 (U.S. EPA 1988; Repa 2000). Small landfills closed and big landfills grew in number and size. By the end of the 1980s, a few hundred landfills handled half of all the municipal solid waste generated in the United States. These data can help to understand the increasing importance of carefully designing the landfills to be used. In this paper we focus on two key feature of such a design: the optimal capacity and lifetime of landfills, which will be studied from a dynamic point of view.

Jacobs and Everett 1992, Ready and Ready 1995, Huhtala 1997, Gaudet, Moreaux and Salant 2001 and André and Cerdá 2001 consider the sequential nature of landfills: once a landfill is full it can be replaced at some cost, by constructing a new one. The new landfill will also be depleted and so on. As a consequence, the capacity of a landfill should not be decided just by considering its own associated costs, but also the costs linked to the following ones. Therefore, instead of optimally designing a landfill, the appropriate approach is that of designing an optimal sequence of landfills. In all the papers cited above, except André and Cerdá 2001, landfill capacity is a given and therefore the problem of obtaining the optimal capacity (which is the main focus of the present article) is not explicitly considered.

The building of landfills is characterized by high setup costs (given by the tasks of building and preparing the new landfills to be used, as well as closing the full ones), as compared to the operating costs (basically given by the transportation and processing of residuals). Deciding the capacity of the landfills has some relevance for the setup costs and also for the switching time of a sequence of landfills. On the one hand, the smaller the capacity of the landfill to be constructed, the smaller the construction cost but, on the other hand, the shorter the lifetime of such a landfill, so that the construction of a new landfill will have to be undertaken sooner. This conflict between present and future costs gives rise to a dynamic decision problem implying that a planning time horizon has to be divided into several subintervals, the length of which is endogenously determined.

This problem has a strong resemblance with so-called capacity expansion problems such as those proposed, for example, by Sinden (1960), Manne (1961), Srinivasan (1967), Nickell (1977) or Bean (1992). In these problems, a decision maker has to add new facilities of similar types over time to meet a rising demand for their services. As a consequence, the installed capacity displays a stair-shaped pattern and, given the increasing demand, there is typically an excess capacity which displays a sawtooth pattern. Note, nevertheless, an important difference: in a typical capacity expansion problem the necessity to expand capacity comes from the rising demand. If demand were constant, there would not be any reason to expand capacity. In the problem presented in this paper, the necessity for capacity expansions comes from the fact that landfill space is an exhaustible resource. Even if demand (in this case, the flow of waste) happens to be constant, the available capacity will diminish progressively and will eventually reach a zero value when the amount of landfilled waste equals the installed capacity. Then, a new landfill (with optimally decided capacity) will be set up, so that we also have this typical sawtooth pattern for the available capacity of landfills.

In André and Cerdá 2001 the optimal capacity of a sequence of landfills is analyzed and the concept of Optimal Capacity Condition is introduced. The present paper offers some further results concerning the properties of the optimal capacity of a sequence of landfills. Specifically, section 2 studies the problem of determining a single optimal capacity for all the landfills of a sequence, both with an infinite and a finite horizon of planning. In the second case, the effect of the horizon length is analyzed and we obtain a counter-intuitive result on the possibility of (optimally) installing an excess of capacity. Section 3 presents a different version of the optimal capacity decision considering the possibility of a different capacity for every landfill, both with an infinite and a finite planning horizon. In the first case an instability result is proved. In the second case, the solution techniques are discussed in depth and a specific algorithm is suggested to solve a particular case with linear construction costs. Some sensitivity analysis results are given concerning the effects of the parameters on the solution and all the results are interpreted from an economic point of view. The main findings of the paper are summarized in section 4.

2 Basic Problem: constant capacity

Assume that a given amount $Q(t)$ of waste[1] is generated at each instant t of a continuous-time, infinite planning horizon $[0, \infty)$. The whole amount of waste is to be landfilled in a sequence of landfills of (endogenously decided) capacity Y which, in this section, is assumed to be the same for all the landfills. If a landfill of capacity Y is built at time $t = 0$, it will last until time $t = T$, implicitly determined by the equation $\int_0^T Q(t)\,dt = Y$. For simplicity, we assume that $Q(t)$ is constant, so that $Q(t) = Q \ \forall\ t$ and, as a consequence, T is determined by $T = \dfrac{Y}{Q}$. The second landfill (open at T) will last until $t = 2T$. In general, the i^{th} landfill will last from $(i-1)T$ until $t = iT$. The cost of building a landfill of capacity Y is given by the $C^{(2)}$ cost function $C(Y)$, which satisfies $C'(Y) > 0$ and $C''(Y) \geq 0$ (so that, it is an increasing and convex function). There is a positive time discount rate δ. A planner solves the problem of finding that value of the capacity Y that minimizes the discounted aggregation of building costs

$$J(Y) = \sum_{i=0}^{\infty} e^{-i\delta T} C(Y)$$

which, using $T = \dfrac{Y}{Q}$, can be written as

$$J(Y) = \sum_{i=0}^{\infty} e^{-i\delta \frac{Y}{Q}} C(Y) = \frac{C(Y)}{1 - e^{-\delta \frac{Y}{Q}}} \tag{1}$$

subject to the minimum and maximum capacity constraints $\underline{Y} \leq Y \leq \overline{Y}$. The last expression in (1) was computed using the formula to sum up the infinite terms of a geometric convergent progression (given that $e^{-\delta \frac{Y}{Q}} < 1, \forall\ \delta, Q, > 0$).

Two remarks regarding the cost function $C(Y)$ are useful. First, note that $C(Y)$ can be thought to measure the (discounted) aggregation of building and closure costs of a landfill. To formalize this issue, let $G_1(Y)$ denote the building costs and $G_2(Y)$ the closure costs of a landfill built at time $t = 0$ with capacity

Y. The present value (evaluated at $t = 0$) of the aggregation of both costs is given by

$$G(Y,T) \equiv G_1(Y) + e^{-\delta T} G_2(Y)$$

but, once Y is decided and Q being exogenous, T is given by $T = \frac{Y}{Q}$, so that $G(Y,T)$ collapses to a function depending only on Y and the parameters of the model:

$$G(Y,T) \equiv G_1(Y) + e^{-\delta T} G_2(Y) = G_1(Y) + e^{-\delta \frac{Y}{Q}} G_2(Y) \equiv C(Y)$$

Second, in order to keep the discussion as straightforward as possible, in this paper we will refer to $C(Y)$ as measuring purely economic costs. Nevertheless, this function could also be constructed to measure an aggregation of economic, social and environmental costs. For operational purposes, of course, the second and the third components would need some non-market valuation method which is beyond the scope of this paper.

The problem in (1) can be summarized as choosing between building many small, cheap landfills or a few large, expensive landfills (or any intermediate possibility). Note that deciding the capacity Y is equivalent to deciding the lifetime T, in such a way that the whole planning horizon is divided into a sequence of intervals of length T. This is precisely the most outstanding feature of this problem: the decision variable Y affects, not only the building costs, but also the length of the temporal intervals and, as a consequence, the discounted value of such costs through the term $i\delta T$ in (1).

The first order Kuhn-Tucker conditions are

$$J'(Y) + \lambda + \mu = 0,$$

$$\lambda\left(\overline{Y} - Y\right) = 0, \quad \mu\left(Y - \underline{Y}\right) = 0, \quad \lambda \geq 0, \quad \mu \leq 0,$$

where

$$J'(Y) = \frac{C'(Y)\left(1 - e^{-\delta \frac{Y}{Q}}\right) - \dfrac{\delta}{Q} e^{-\delta \frac{Y}{Q}} C(Y)}{\left(1 - e^{-\delta \frac{Y}{Q}}\right)^2}, \tag{2}$$

λ and μ being the multipliers associated with the maximum and minimum capacity constraints. We will focus on interior solutions (i.e., with $\underline{Y} < Y < \overline{Y}$ holding) because of their more interesting economic interpretation. Given that in an interior solution $\lambda = \mu = 0$, $J'(Y) = 0$, and rearranging (2), we obtain

$$C'(Y) = \frac{\delta}{\left(e^{\delta\frac{Y}{Q}} - 1\right) Q} C(Y). \tag{3}$$

We call (3) the *Optimal Capacity Condition* (OCC), which has the following economic interpretation: for every landfill, the marginal cost and the marginal gain of increasing capacity Y have to coincide. The former is measured by the first derivative of C (left hand side of (3)). The marginal gain of increasing Y (measured by the right hand side of (3)) comes from the fact that, if Y increases, the setup costs of all the future landfills to be constructed are delayed and, as a consequence, the present value of such costs becomes smaller. When both sides are equal, it is not possible to reduce the value of the objective function by increasing or reducing Y. The sufficient second order condition is $J'' \geq 0$ that, by derivating (2), using the fact $\left(1 - e^{-\delta\frac{Y}{Q}}\right) > 0$ (to eliminate the denominator of the derivative), and rearranging, can be expressed as

$$\left(1 - e^{-\delta\frac{Y}{Q}}\right)^2 C''(Y) + \frac{\delta^2}{Q^2}\left(e^{-\delta\frac{Y}{Q}} + e^{-2\delta\frac{Y}{Q}}\right) C(Y) \geq 2\frac{\delta}{Q} e^{-\delta\frac{Y}{Q}}\left(1 - e^{-\delta\frac{Y}{Q}}\right) C'(Y). \tag{4}$$

Observe that the convexity of the cost function (or equivalently the assumption $C''(Y) \geq 0$) is not -as usually in a cost minimization problem- a sufficient optimality condition (it is not a necessary condition either) because of the double effect that a marginal increment of Y has on the objective function: on the one hand, it increases the building cost of all the landfills and, on the other hand, it increases the duration of all of them, reducing the discounted value of such costs.

2.1 Example: linear cost function

Assume that the setup costs are given by the linear cost function $C(Y) = a + bY$, where a and b are two positive parameters representing fixed and marginal costs respectively. Substituting $C(Y) = a + bY$ and $C'(Y) = b$ in (3) and rearranging, we obtain the following Optimal Capacity Condition for this case:

$$a + bY = \frac{bQ}{\delta}\left(e^{\delta\frac{Y}{Q}} - 1\right). \tag{5}$$

and the second order conditions holds for any $Y \geq 0$[2]. We can not solve (5) for Y as an explicit function of the parameters. In order to obtain an insight into the effect of each parameter, we present some sensitivity analysis results.

As shown in the appendix (subsection 5.1), the following results hold in the solution:

$$\frac{\partial Y}{\partial a} > 0, \qquad \frac{\partial Y}{\partial Q} \lesseqgtr 0, \qquad \frac{\partial^2 Y}{\partial Q \partial a} \equiv \frac{\partial}{\partial a}\left(\frac{\partial Y}{\partial Q}\right) > 0$$
$$\frac{\partial Y}{\partial b} < 0, \qquad \frac{\partial Y}{\partial \delta} \lesseqgtr 0. \qquad \frac{\partial^2 Y}{\partial Q \partial b} \equiv \frac{\partial}{\partial b}\left(\frac{\partial Y}{\partial Q}\right) < 0. \tag{6}$$

Note the economic meaning of this results: the larger the fixed cost of construction a, the larger the optimal capacity of all the landfills in order to enlarge their lifetime and avoid having to construct many (small) landfills and, as a consequence, we obtain $\partial Y/\partial a > 0$. Conversely, the larger the marginal cost of construction b, the smaller the optimal capacity, as it becomes more expensive to construct large landfills. When the instantaneous waste generation Q increases, in order to keep the solution feasible, we have to construct either more landfills or larger landfills. According to (6), when Q increases, the larger the fixed construction costs, the more convenient it is to construct larger landfills $\left(\partial^2 Y/\partial Q\partial a > 0\right)$ and the larger the marginal costs, the more convenient is to keep feasibility, not by increasing the capacity of landfills, but building more of them $\left(\partial^2 Y/\partial Q\partial b < 0\right)$. The sensitivity analysis results concerning the discount rate δ, are ambiguous and depend on the specific case under study. Note that increasing δ leads to an increase in the weight given in the objective function to short-term versus long-term costs. As a consequence, on the one hand, it becomes worthwhile to decrease present costs (which implies reducing landfill capacity Y) but, on the other hand, it also becomes worthwhile to delay future costs as much as possible (which implies increasing landfill capacity Y).

2.2 Finite Horizon

If we transform (1) into a finite-horizon problem, some qualitative differences arise, deserving some discussion. Assume that the decision problem consists of constructing an arbitrary number of landfills to manage the waste generated in the finite period $[0, \tau]$. Let K denote the total number of landfills indexed by $i = 0, 1, \ldots, K-1$. The solution consists of two positive values for Y and K, denoted by $\{Y^*, K^*\}$, to minimize the discounted sum

$$J(Y) = \sum_{i=0}^{K-1} e^{-i\delta T} C(Y) = \frac{1 - e^{-\delta K T}}{1 - e^{-\delta T}} C(Y) = \frac{1 - e^{-\delta K \frac{Y}{Q}}}{1 - e^{-\delta \frac{Y}{Q}}} C(Y). \tag{7}$$

Assume initially that the capacity of all the landfills has to be totally depleted under the solution (this issue is explicitly analyzed below), so that no excess of capacity is to be built. As a consequence, K and Y must satisfy the following equality constraint:

$$KY = \tau Q \tag{8}$$

and, using such a constraint, the problem consists of finding the value of Y that minimizes

$$J(Y) = \sum_{i=0}^{\frac{\tau Q}{Y}-1} e^{-i\delta\frac{Y}{Q}} C(Y) = \left(1-e^{-\delta\tau}\right)\frac{C(Y)}{1-e^{-\delta\frac{Y}{Q}}}. \tag{9}$$

In this case, we can not apply any continuous optimization technique, as we face an integer optimization problem because of (8) and the fact of K being an integer variable. Using (8), we can also formulate the problem as that of finding the integer positive value of $K \in \{K_{\min}, K_{\min}+1, \ldots, K_{\max}\}$ minimizing

$$\frac{1-e^{-\delta\tau}}{1-e^{-\frac{\delta\tau}{K}}} C\left(\frac{\tau Q}{K}\right), \tag{10}$$

$K_{\min}$ and $K_{\max}$ representing the minimum and maximum feasible values for K, given by

$$K_{\min} = \begin{cases} \frac{\tau Q}{\bar{Y}} & \text{if } \frac{\tau Q}{\bar{Y}} \text{ is an integer,} \\ Int\left(\frac{\tau Q}{\bar{Y}}+1\right) & \text{otherwise,} \end{cases} ;\ K_{\max} = Int\left(\frac{\tau Q}{\underline{Y}}\right)$$

$Int(\xi)$ denoting the integer part of ξ.

As the time horizon τ is now a parameter of the problem, we can perform now some sensitivity analysis to study its effect on the solution. The figures 1.a. and 1.b. show the optimal values K^* and Y^* for different values of τ, in an example with the linear cost function $C(Y) = a + bY$ and the following parameter values: $a = 1000$, $b = 5$, $Q = 10$, $\delta = 0.05$, $\underline{Y} = 90$, $\bar{Y} = 350$. Given the small size of the problem, it can be solved merely by computing the value of the objective function (10) for different values of K. Define

$$K^*(\tau) = \arg\min_{\{K\}} \frac{1-e^{-\delta\tau}}{1-e^{-\frac{\delta\tau}{K}}} C\left(\frac{\tau Q}{K}\right), \qquad Y^*(\tau) = \frac{\tau Q}{K^*}.$$

Note that the solution, as a function of τ, is piecewise continuous. Let τ_1 be a value of the time horizon belonging to the interior of an interval of continuity. Let $K^*(\tau_1)$ and $Y^*(\tau_1)$ denote the optimal number and the optimal capacity of landfills corresponding to τ_1. Assume that, from τ_1, τ suffers a "small" increment of size $\Delta\tau$, implying an increment in the amount of generated waste equal to $Q \cdot \Delta\tau$. This increment leads to keeping the number of landfills constant and increasing the individual capacity Y^* in the quantity

$(Q \cdot \Delta\tau)/K^*$. Nevertheless, if τ suffers a "large enough" increment, say up to the value τ_2, it becomes worthwhile to construct an additional landfill, instead of further increasing the individual capacity, so that $K^*(\tau_2) = K^*(\tau_1) + 1$. The availability of a new landfill allows the capacity of each one to be reduced, in such a way that $Y^*(\tau_2) = \dfrac{Q\tau_2}{K^*(\tau_2)} < \dfrac{Q\tau_2}{K^*(\tau_1)}$. If, from τ_2 on, the time horizon slightly increases again, Y^* increases while keeping K^* constant, until a new threshold value is reached. Consequently, as a function of τ, K^* has a stair shape and Y^* has a sawtooth shape. The larger the fixed construction cost (as measured by the parameter a) and the smaller the marginal cost (as measured by b), the larger the amplitude of such continuity intervals.

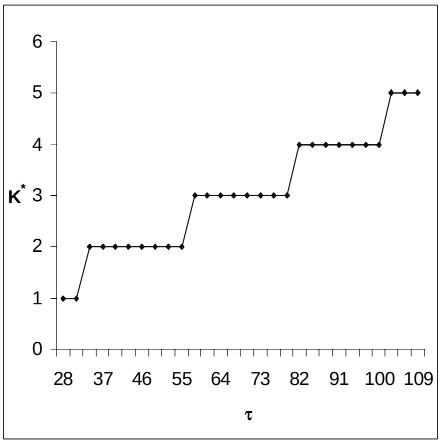

Figure 1.a. Optimal number of landfills

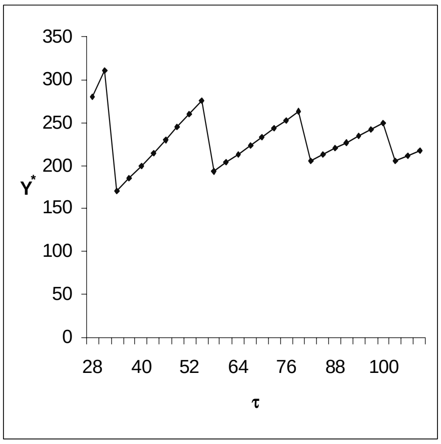

Figure 1.b. Optimal capacity

Capacity exhaustion

An analytical issue that arises specifically in the finite horizon setting is that of the optimal exhaustion of landfill capacity. The results shown up to now have been obtained under the assumption that no excess of capacity is to be installed, or equivalently, that the landfills are constructed in such a way that their whole capacity is exhausted. Even if this assumption is intuitive and economically reasonable, it does not always hold. As is shown below, it may be optimal, under some special circumstances, to build an excess of capacity that would be optimally under-used.

It is straightforward to conclude that it is optimal to exhaust the capacity of the landfills $0, 1, \ldots, K-2$. Otherwise, the solution could be improved merely by exhausting such capacities so that the building costs of subsequent landfills would be delayed. It is not trivial to obtain the same conclusion for the $K-1^{th}$ landfill.

In order to illustrate the opposite possibility, consider the following example: assume a linear setup costs function $C(Y) = a + bY$ and the following parameter values: $a = 145$, $b = 10^{-5}$, $\delta = 0.1$, $Q = 40$, $\tau = 200$. The solution of the problem under the assumption of capacity exhaustion is $K^* = 2$, $Y^* = \frac{\tau Q}{2} = 4000$, so that $T = 100$, and the value of the objective function is $J(Y^*) = C(Y^*)\left[1 + e^{-\delta \frac{Y^*}{Q}}\right] = 145.047$. Now, suppose that the capacity exhaustion constraint (8) becomes an inequality constraint to guarantee that the whole amount of waste is landfilled:

$$YK \geq Q\tau. \tag{11}$$

We obtain $\tilde{K} = 2$, $\tilde{Y} = 4200$, where $\left(\tilde{K}, \tilde{Y}\right)$ denotes the solution to the problem with the constraint (11) instead of (8). Note that the first landfill is exhausted at $T = Y/Q = 105$ and when the end of the planning horizon τ is reached, the capacity of the second landfill is not totally depleted, but there is an excess of constructed capacity equal to 400. Nevertheless, despite the excess of capacity installed, the discounted cost of the solution does not increase with respect to that of (K^*, Y^*), but it slightly decreases to 145.046.

From the Kuhn-Tucker conditions, we know that the derivative of the objective function evaluated at $\tilde{Y}$ is

$$J'(Y) = C'(Y)\left[1 + e^{-\delta \frac{Y}{Q}}\right] - C(Y)\frac{\delta}{Q}e^{-\delta \frac{Y}{Q}} \geq 0. \tag{12}$$

If $J'(Y^*) \geq 0$, the solution to both problems coincides, given that, from Y^* on, increasing Y increases the discounted cost and decreasing Y is not feasible. However, if $J'(Y^*) < 0$, it is worthwhile increasing the value of Y because the higher construction cost is overcompensated by the delay in the building of the second landfill (even if the capacity of such a landfill will not get exhausted). In this case, the constraint (11) is not binding and it is optimal to increase Y until $J'\left(\tilde{Y}\right) = 0$. Using (12), the condition for this event to happen can be expressed as

$$C'(Y^*)\left(1 + e^{\delta \frac{Y^*}{Q}}\right) < \frac{\delta}{Q}C(Y^*). \tag{13}$$

Note that the left hand side measures the marginal cost of increasing the capacity which, in this case, is given by the (present value of the) increase in the construction cost of both landfills, as measured by the first derivative of C, properly discounted for the second landfill. The right hand side measures the marginal gain of increasing the capacity, which is given by the decrease in the present value of the construction cost of the second landfill, due to the fact that it will be constructed later. Rearranging (13) we have the following alternative condition, where the terms directly depending on the technical structure of the cost function are grouped in the left hand side:

$$\frac{C'(Y^*)}{C(Y^*)} < \frac{\delta}{Q}\frac{1}{e^{\delta\frac{Y^*}{Q}}+1}. \tag{14}$$

The following proposition provides the equivalent condition for any number off landfills.

Proposition 1 *In a solution for problem (9), for an arbitrary value of $K \geq 2$, it is optimal to build and excess of capacity (i.e. $Y^*K > Q\tau$) if and only if the following condition holds:*

$$\frac{C'(Y^*)}{C(Y^*)} < \frac{\delta}{Q}\frac{1}{1-e^{-K\delta\frac{Y^*}{Q}}}\left[\frac{e^{-\delta\frac{Y^*}{Q}}-e^{-K\delta\frac{Y^*}{Q}}}{1-e^{-\delta\frac{Y^*}{Q}}}-(K-1)\,e^{-K\delta\frac{Y^*}{Q}}\right]_{\blacksquare} \tag{15}$$

Proof: see subsection 5.2$\blacksquare$

Note the economic meaning of both (14) and (15): such conditions hold if the marginal cost at Y^* (as measured by the first derivative of C) is very "small" with respect to the total cost (specifically, below the threshold provided for each condition). In such a situation, a marginal increase of Y generates a small increase in the cost that is overcompensated by the gain obtained from the delay of future costs.

3 A dynamic approach: variable capacity

We can think of some situations where it is optimal for every single landfill to have a different capacity. A possible reason for this to occur is each landfill having a different cost function, due to the fact that every landfill is constructed in a different place with some specific land characteristics. Furthermore, apart from the building costs, waste treatment also implies some operating costs (which basically come from the collection, transportation and processing processes) that could differ from one landfill to another.

Assume that a planner has to build a sequence of landfills with (perhaps different) capacities $\{Y_0, Y_1, \dots\}$, where the setup cost of landfill i is given by the increasing and convex cost function $C_i(Y_i)$. While the landfill i is being used he has to pay the instantaneous operating cost, given by the linear function $h_i(Q(t)) = \phi_i Q(t)$, where ϕ_i is the cost of transporting and processing one unit of waste and $Q(t)$ is the quantity of waste generated at instant t, assumed to be constant: $Q(t) = Q\ \forall\ t$. A landfill constructed at the instant T_i with a capacity Y_i will last until $T_{i+1} = T_i + \frac{Y_i}{Q}$.

From a mathematical point of view, note that, unlike section 2, we do not now have a static optimization problem, but a dynamic one. This problem has a particular structure which incorporates some continuous time and some discrete time elements. On the one hand, the time variable t is continuous, waste is generated in continuous time and the processing costs $h_i(Q(t))$ happen in continuous time. The variables T_i, which refer to time, can take any real value, as corresponds to a continuous time Optimal Control model. On the other hand, the construction costs happen at a finite number of times, as in discrete time Optimal Control problems. The problem consists of finding a sequence of capacities $\{Y_0, Y_1, ...\}$, in order to minimize the function

$$\sum_{i=0}^{\infty} e^{-\delta T_i} C_i(Y_i) + \sum_{i=0}^{\infty} \left[\int_{T_i}^{T_{i+1}} e^{-\delta t} h_i(Q(t))\, dt \right] \equiv \sum_{i=0}^{\infty} e^{-\delta T_i} \left[C_i(Y_i) + \int_{T_i}^{T_{i+1}} e^{-\delta(t-T_i)} \phi_i Q dt \right] \tag{P}$$

subject to[3] $T_0 = 0$ and $T_{i+1} = T_i + \dfrac{Y_i}{Q}$ $(\forall\ i = 0, 1, 2, \dots)$. Note that (P) can be regarded as a discrete time Optimal Control problem, where the "discrete time" is not given by the chronological time t, but by the landfill index $i = 0, 1, \dots, K-1$, and $T_{i+1} = T_i + \dfrac{Y_i}{Q}$ is the state equation.

This problem is conceptually similar to that of exploiting a sequence of deposits of a natural resource, as studied in Herfindahl 1967, Weitzman 1976, Hartwick 1978, or Hartwick, Kemp and Long 1986, where the role of extraction cost is played by the operating costs in our problem. Anyway, there are two important differences: first, in our case, the capacity depletion rate, analogous to a resource extraction rate, can not be decided because it is given by the exogenous generation of waste. Second, the initial landfill capacity (analogous to the initial resource stock) is not given in our problem, as it is in natural resource extraction models, but it is a decision variable.

(P) can also be expressed as[4]

$$\min_{\{Y_0, Y_1, \dots\}} C_0(Y_0) + \frac{Q\phi_0}{\delta}\left[1 - e^{-\delta \frac{Y_0}{Q}}\right] + \sum_{i=1}^{\infty} e^{-\frac{\delta}{Q}\sum_{j=0}^{i-1} Y_j} \left[C_i(Y_i) + \frac{Q\phi_i}{\delta}\left(1 - e^{-\delta \frac{Y_i}{Q}}\right)\right]. \tag{16}$$

Proposition 2 *In an interior solution to problem (P), for two consecutive landfills, k and $k+1$ ($k = 0, 1, \dots, K-2$), the following Optimal Capacity Condition holds:*

$$C_k'(Y_k) = e^{-\frac{\delta}{Q} Y_k} \left[C_{k+1}'(Y_{k+1}) + \frac{\delta}{Q} C_{k+1}(Y_{k+1}) + \Delta\phi_k \right] = e^{-\delta(T_{k+1}-T_k)} \left[C_{k+1}'(Y_{k+1}) + \frac{\delta}{Q} C_{k+1}(Y_{k+1}) + \Delta\phi_k \right] \tag{17}$$

where $\Delta\phi_k = \phi_{k+1} - \phi_k$ is the unit processing cost increment from landfill k to landfill $k+1$∎

Proof: see subsection 5.3■

The second order conditions hold if the objective function in (16) is a convex function in the variables Y_0, Y_1,...

Condition (17) is a nonlinear first order difference equation which represents the relation between the optimal capacity of two consecutive landfills. In order to interpret this condition economically, think of a situation in which $\Delta\phi_k = 0\ \forall k$ and $\delta = 0$, so that the unit processing cost is identical for all the landfills and there is no time discount. Then (17) takes the form

$$C_k'(Y_k) = C_{k+1}'(Y_{k+1}), \tag{18}$$

which can be interpreted as a non-arbitrage condition: if $C_k'(Y_k) < (>)\, C_{k+1}'(Y_{k+1})$, then total cost could be reduced by reducing Y_{k+1} (Y_k) and increasing Y_k (Y_{k+1}). Condition (18) establishes the impossibility of reducing the total cost by transferring some capacity from one landfill to another one. With a strictly positive discount rate and different unit processing costs, the relevant equation is (17), which is still a non-arbitrage condition, but now the marginal effect of transferring capacity from one landfill to another has two additional components: the delay of future construction costs (the larger Y_k, the later landfill $k+1$ will be necessary) and the difference between the processing costs borne on both landfills. The greater is the expected cost increment $\Delta\phi_k$, the greater is the value of the right hand side of (17). In order to maintain the equality, the left hand side has to be greater too. Given that C_k is assumed to be a convex function, and therefore $C_k'(Y_k)$ is nondecreasing with Y_k, it follows that, the larger $\Delta\phi_k$, the larger the optimal capacity of landfill k. This conclusion is reasonable from an economic point of view: if future landfills are subject to large processing cost increments, it is optimal to increase the capacity of the present landfill in order to extend its lifetime and to delay future processing costs associated with the next landfills.

As mentioned at the beginning of this section, the differences in the building and processing cost functions justify the possibility of the landfills having different capacities. Note furthermore that, even if the cost functions are common for all the landfills (i.e. $C_0(Y) = C_1(Y) = \cdots = C(Y)$ and $h_0(Q) = h_1(Q) = \cdots = h(Q) = \phi Q$) it may be optimal for every landfill have a different capacity. This can be seen merely by noting that the suitable version of the difference equation (17) for this case,

$$C'(Y_k) = e^{-\frac{\delta}{Q}Y_k}\left[C'(Y_{k+1}) + \frac{\delta}{Q}C(Y_{k+1})\right] = e^{-\delta(T_{k+1}-T_k)}\left[C'(Y_{k+1}) + \frac{\delta}{Q}C(Y_{k+1})\right], \tag{19}$$

generates a sequence of capacities $\{Y_0, Y_1, \ldots, Y_k, \ldots\}$, in which, in general, $Y_k \neq Y_{k+1}$. This argument can be further stressed by noting that such a sequence of optimal capacities does not even converge to a stable steady state, as shown in the following subsection.

3.1 Instability of the steady state

Equation (19) implicitly defines the optimal value of Y_{k+1} as a function of Y_k

$$Y_{k+1} = \Psi(Y_k) \tag{20}$$

the solution of which provides the optimal sequence of capacities $\{Y_0, Y_1, \ldots, Y_k, \ldots\}$. We next investigate the dynamic behavior of such an optimal sequence, and specifically, the existence and stability of a steady state. Define a steady state of the solution for problem (P) as an equilibrium point for equation (20), i.e. a value Y^* such that $Y^* = \Psi(Y^*)$. For the notion of steady state to make sense in this context, we refer to the situation in which all the landfills have the same cost functions. If the cost function of all the landfills were different, no general statements could be made about a steady state, because all depends on the specific forms of $C_i(\cdot)$ and ϕ_i. By imposing the condition $Y_k = Y_{k+1} = Y^*$ in (19), we obtain the steady state condition

$$C'(Y^*) = e^{-\frac{\delta}{Q}Y^*}\left[C'(Y^*) + \frac{\delta}{Q}C(Y^*)\right] \Leftrightarrow C'(Y^*) = \frac{\delta}{\left(e^{\delta\frac{Y^*}{Q}} - 1\right)Q}C(Y^*), \tag{21}$$

which coincides with the OCC for the problem with standard capacity in section 2 (equation (3)). The existence of a feasible steady state with $Y^* > 0$ (or equivalently the existence of a solution for (21)) depends on the form of the function C and the value of the parameters δ and Q.

Assume (21) has a solution, so that a steady state exists. The following relevant question is that of its stability in order to determine if, under the optimal solution, the sequence of capacities converges to such an equilibrium point.

Define the function $\Upsilon(Y_{k+1}, Y_k)$ as

$$\Upsilon(Y_{k+1}, Y_k) = e^{-\frac{\delta}{Q}Y_k}\left[C'(Y_{k+1}) + \frac{\delta}{Q}C(Y_{k+1})\right] - C'(Y_k),$$

so that a steady state can be redefined as a value of Y^* that satisfies $\Upsilon(Y^*, Y^*) = 0$. We can make a first-order Taylor approximation around the steady state to obtain

$$\frac{\partial\Upsilon(Y^*, Y^*)}{\partial Y_{k+1}}(Y_{k+1} - Y^*) + \frac{\partial\Upsilon(Y^*, Y^*)}{\partial Y_k}(Y_k - Y^*) \simeq 0,$$

or using the expression for Υ,

$$\begin{aligned} & e^{-\frac{\delta}{Q}Y^*}\left[C''(Y^*) + \tfrac{\delta}{Q}C'(Y^*)\right](Y_{k+1} - Y^*) - \\ & -\left\{\tfrac{\delta}{Q}e^{-\frac{\delta}{Q}Y^*}\left[C'(Y^*) + \tfrac{\delta}{Q}C(Y^*)\right] + C''(Y^*)\right\}(Y_k - Y^*) \simeq 0, \end{aligned} \tag{22}$$

which is an (approximate) liner difference equation whose characteristic equation is

$$e^{-\frac{\delta}{Q}Y^*}\left[C''(Y^*) + \frac{\delta}{Q}C'(Y^*)\right]\lambda - \left\{\frac{\delta}{Q}e^{-\frac{\delta}{Q}Y^*}\left[C'(Y^*) + \frac{\delta}{Q}C(Y^*)\right] + C''(Y^*)\right\} = 0.$$

The only solution to such an equation is

$$\lambda = \frac{\frac{\delta}{Q}\left[C'(Y^*) + \frac{\delta}{Q}C(Y^*)\right] + e^{\frac{\delta}{Q}Y^*}C''(Y^*)}{\left[C''(Y^*) + \frac{\delta}{Q}C'(Y^*)\right]}. \tag{23}$$

The stability condition of (22) is $|\lambda| < 1$. If C is an increasing and convex function, then $\lambda > 0$, and the stability condition reduces to $\lambda < 1$ or, using (23) and rearranging,

$$C(Y^*) < \frac{Q^2}{\delta^2}\left[1 - e^{\frac{\delta}{Q}Y^*}\right]C''(Y^*). \tag{24}$$

For any nonnegative value of Y^* we have $\left[1 - e^{\frac{\delta}{Q}Y^*}\right] \leq 0$ and, (C being a convex function) the right hand side of (24) is nonpositive. Given that, by definition, $C(Y^*) > 0$, there is no solution for (24) and so we have proved the following proposition:

Proposition 3 *Given the assumptions made on the properties of $C(Y)$, equation (20) do not have any positive stable steady state.*

The latter proposition prevents us from finding a stable steady state for problem (P), so that, when an infinite horizon approach is used, the minimum and maximum capacity constraints $\underline{Y} \leq Y \leq \overline{Y}$ come into play to prevent the sequence of capacities going to infinity or minus infinity. A typical infinite-time optimal capacity pattern will show an exponentially increasing trend up to a point where the maximum capacity $\overline{Y}$ is reached, and a constant value equal to $\overline{Y}$ from that moment on, or an exponentially decreasing trend up to a point where the minimum capacity $\underline{Y}$ is reached and a constant value equal to $\underline{Y}$ from that moment on.

3.2 Finite horizon approach

In practice, it can be difficult to find a solution for problem (P) with infinite horizon. The result proved above concerning the inexistence of a stable steady state is a further difficulty because we can not make general statements about the long term behavior of the difference equation (19). In this section we study the finite horizon version of the same problem. To simplify the discussion, we will focus on a case in which the processing cost functions are the same for all the landfills: $h_0(Q) = h_1(Q) = \cdots = h(Q) = \phi Q$. Similar results obtain for a more general case with $\phi_i \neq \phi_j$ $(\forall\ i \neq j)$. Let τ denote the (finite) time horizon. The planner has to decide the number (K) and the capacity of the landfills to be constructed, denoted by $\{Y_0, Y_1, ...Y_{K-1}\}$ to minimize[5]

$$\sum_{i=0}^{K-1} e^{-\delta T_i} C_i(Y_i) \tag{25}$$

subject to $T_0 = 0$, $T_K = \tau$ and $T_{i+1} = T_i + \frac{Y_i}{Q}$, (for $i = 0, 1, 2, ..., K-1$), where we have assumed that the capacity of landfills is exhausted under the optimal solution[6].

Because K is a decision variable, (P) is a free time horizon problem. The easiest way to solve it consists of finding the solution for all possible values of K, and choosing that which provides the minimum total cost. K can take any integer value from the set $\{K_{\min}, K_{\min}+1, ..., K_{\max}-1, K_{\max}\}$, where

$$K_{\min} = \begin{cases} \dfrac{\tau Q}{\overline{Y}} & \text{if } \dfrac{\tau Q}{\overline{Y}} \text{ is an integer,} \\ Int\left(\dfrac{\tau Q}{\overline{Y}} + 1\right) & \text{otherwise,} \end{cases} \Bigg| \; ; \quad K_{\max} = Int\left(\frac{\tau Q}{\underline{Y}}\right),$$

$Int(\xi)$ denoting the integer part of ξ. Henceforth, $K_{\max} - K_{\min} + 1$ discrete time Optimal Control problems have to be solved. Let $\hat{C}_K$ be the optimal discounted cost which can be obtained by constructing K landfills. The optimal value of K is given by $K^* = \underset{\{K = K_{\min}, \cdots, K_{\max}\}}{\arg\min} \hat{C}_K$. We need to solve $K_{\max} - K_{\min} + 1$ discrete time optimization problems. Let us focus on the solution of each of those problems. Assume $K_{\min} = 1$.

$\underline{K=1}$: If a single landfill is to be built, then the trivial solution consists of $Y_0 = Q\tau$ and the value of the objective function is $\hat{C}_1 = C_0(Q\tau)$.

$\underline{K=2}$: When two landfills are to be constructed, we have an optimal control problem with two periods consisting of $\underset{\{Y_0,Y_1\}}{Min}\, C_0(Y_o) + e^{-\delta T_1} C_1(Y_1)$ subject to $T_1 = \frac{Y_0}{Q}$ and $T_2 = T_1 + \frac{Y_1}{Q} = \tau$. The solution can be obtained by Dynamic Programming or by the Lagrange method. Let us illustrate the solution by Dynamic Programming. The two-period problem is divided in two one-period problems and we begin with the second one:

- **i=1:** $\underset{\{Y_1\}}{Min} J_1 = e^{-\delta T_1} C_1(Y_1)$ subject to $T_2 = T_1 + \dfrac{Y_1}{Q} = \tau$, where T_1 is taken as given. The constraint of this problem states that the only feasible value for Y_1 is $Y_1 = (\tau - T_1) Q$, the value function for this period is $J_1 = e^{-\delta T_1} C_1(Y_1) = e^{-\delta T_1} C_1((\tau - T_1) Q)$.

- **i=0:** $\underset{\{Y_0\}}{Min} J_0 = C_0(Y_0) + J_1 = C_0(Y_0) + e^{-\delta T_1} C_1((\tau - T_1) Q)$subject to $T_1 = \dfrac{Y_0}{Q}$. Using this restriction, the objective function of this period can be expressed as $J_0 = C_0(Y_0) + e^{-\delta \frac{Y_0}{Q}} C_1(\tau.Q - Y_0)$ and the corresponding first order condition is

$$C_0'(Y_0) = e^{-\delta \frac{Y_0}{Q}} C_1'(\tau.Q - Y_0) + \frac{\delta}{Q} e^{-\delta \frac{Y_0}{Q}} C_1(\tau.Q - Y_0). \tag{26}$$

  Let us pay some further attention to the condition (26). The left hand side represents the marginal cost, and the right hand side the marginal gain, of increasing the capacity of the first landfill. The marginal gain has two components: the first one represents the (discounted) saving due to the fact that the second landfill needs a smaller capacity, and smaller construction costs. The second one measures the effect of delaying the instant of paying the cost $C_1(Y_1)$. By solving the equation (26) we find the optimal value of Y_0 and $Y_1 = Q\tau - Y_0$, and the optimal value of the objective function $\hat{C}_2 = C_0(Y_o) + e^{-\delta \frac{Y_0}{Q}} C_1(Y_1)$.

Any value of K: For every possible value of K, we have an optimal control problem that can be solved either by Dynamic Programming (as illustrated for the case $K = 2$), or by the Lagrange method, as we will illustrate right now: Using recursively the formula $T_{i+1} = T_i + \dfrac{Y_i}{Q}$, we have $T_i = \frac{1}{Q}\sum_{j=0}^{i-1} Y_j$, for $i = 1, 2, ...K$, and substituting this expression in (25), the problem can be stated as $\underset{\{Y_0,Y_1,\ldots,Y_{K-1}\}}{Min} C_0(Y_o) +$

$\sum_{i=1}^{K-1} e^{-\frac{\delta}{Q}\sum_{j=0}^{i-1} Y_j} C_i(Y_i)$ subject to $Q.\tau = Y_0 + Y_1 + ... + Y_{K-1}$. The Lagrangian is defined as

$$\mathcal{L} = C_0(Y_o) + \sum_{i=1}^{K-1} e^{-\frac{\delta}{Q}\sum_{j=0}^{i-1} Y_j} C_i(Y_i) + \lambda(Q.\tau - Y_0 - Y_1 - ... - Y_{K-1}).$$

The first order conditions for an interior solution are:

$$\begin{aligned} C'(Y_0) - \frac{\delta}{Q}\sum_{i=1}^{K-1} e^{-\frac{\delta}{Q}\sum_{j=0}^{i-1} Y_j} C_i(Y_i) - \lambda &= 0, \\ e^{-\frac{\delta}{Q}\sum_{j=0}^{k-1} Y_j} C'_k(Y_k) - \frac{\delta}{Q}\sum_{i=k+1}^{K-1} e^{-\frac{\delta}{Q}\sum_{j=0}^{i-1} Y_j} C_i(Y_i) - \lambda &= 0 \quad k = 1, \ldots, K-2, \\ e^{-\frac{\delta}{Q}\sum_{j=0}^{K-2} Y_j} C'_{K-1}(Y_{K-1}) - \lambda &= 0. \end{aligned}$$

The multiplier λ measures the increment in the optimal discounted cost caused by an increment in the whole amount of waste $\tau.Q$, (due to a change in Q or τ). Solving all the first order equations for λ we have the following relation among the optimal capacity of all the landfills:

$$\begin{aligned} \lambda &= C'_0(Y_0) - \frac{\delta}{Q}\sum_{i=1}^{K-1} e^{-\frac{\delta}{Q}\sum_{j=0}^{i-1} Y_j} C_i(Y_i) = e^{-\frac{\delta}{Q}Y_0} C'_1(Y_1) - \frac{\delta}{Q}\sum_{i=2}^{K-1} e^{-\frac{\delta}{Q}\sum_{j=0}^{i-1} Y_j} C_i(Y_i) \\ &= \cdots = e^{-\frac{\delta}{Q}\sum_{j=0}^{K-3} Y_j} C'_{K-2}(Y_{K-2}) - \frac{\delta}{Q} e^{-\frac{\delta}{Q}\sum_{j=0}^{K-2} Y_j} C_{K-1}(Y_{K-1}) = e^{-\frac{\delta}{Q}\sum_{j=0}^{K-2} Y_j} C'_{K-1}(Y_{K-1}) \end{aligned}$$

and, operating with the first order conditions corresponding to the landfills k and $k+1$, we obtain the same difference equation (17) relating the capacity of any two consecutive landfills in the infinite horizon case, so that, proposition (2) also holds for the finite horizon case.

Using these equations and the constraint $Y_0 + Y_1 + ... + Y_{K-1} = Q.\tau$, we obtain the optimal values of $Y_0, Y_1, \ldots, Y_{K-1}$ and, substituting in the objective function, we have the minimum cost with K landfills, denoted by $\hat{C}_K$. After solving all the $K_{\max} - K_{\min} + 1$ optimization problems and obtaining the values $\hat{C}_1$, $\hat{C}_2$,...,$\hat{C}_{\bar{K}}$, we select $K^* = \underset{\{K=1,\ldots,\bar{K}\}}{\arg\min}\ \hat{C}_K$ and have the solution to (25).

3.3 Example: linear cost function

Assume that the cost function (common for all the landfills) is $C(Y) = a + bY$, with $a, b > 0$. Substituting in (19) and rearranging, we obtain the difference equation

$$Y_{k+1} = \frac{Q}{\delta}\left(e^{\frac{\delta}{Q}Y_k} - 1\right) - \frac{a}{b}. \tag{27}$$

Given some numerical values for the parameters a, b, δ, Q, τ, we use the following algorithm:

1. Solve the problem for $K_{\min}$ by the following steps:

   (a) Assume interior solution: $\underline{Y} \leq Y_i \leq \bar{Y}$, for $i = 0, 1, \ldots, Y_{K_{\min}-1}$.

   (b) Determine the optimal sequence $Y_0, Y_1, \ldots, Y_{K_{\min}-1}$, by the bisection method (the idea is to find that sequence which satisfies (27) and the constraint $Y_0 + \cdots + Y_{K_{\min}-1} = Q\tau$):

      i. Select two extreme initial values for Y_0: Y_0^a and Y_0^b, so that the sequence of capacities generated by the equation (27) using Y_0^a and Y_0^b as initial conditions satisfy $\sum_{i=0}^{K_{\min}-1} Y_i^a < Q\tau$ and $\sum_{i=0}^{K_{\min}-1} Y_i^b > Q\tau$. For example, $Y_0^a = 0$, $Y_0^b = Q\tau$.

      ii. Compute $Y_0^m = \dfrac{Y_0^a + Y_0^b}{2}$.

      iii. Using (27), generate the sequence $Y_1, \ldots, Y_{K_{\min}-1}$ using Y_0^m as the initial condition.

      iv. If $\sum_{i=0}^{K_{\min}-1} Y_i < Q\tau$, the solution generated in (iii) is unfeasible. Assign $Y_0^a = Y_0^m$, keeping Y_0^b unchanged and go to step ii.

      v. If $\sum_{i=0}^{K_{\min}-1} Y_i > Q\tau$, the solution is feasible but not optimal. Assign $Y_0^b = Y_0^m$, keeping Y_0^a unchanged and go to step ii.

      vi. When we get a sequence satisfying $\sum_{i=0}^{K_{\min}-1} Y_i = Q\tau$, go to (c).

   (c) Check that the obtained solution is interior, so that all the capacities satisfy $\underline{Y} \leq Y_i \leq \bar{Y}$ (this holds in all the discussed cases). Store the value of the objective function $\hat{C}_{K_{\min}}$.

2. Repeat step 1 for all the possible values of $K = K_{\min} + 1, \ldots, K_{\max}$ and compute all the values of the objective function $\hat{C}_K$.

3. Compare the optimal value of the objective function for different values of K and select $K^* = \underset{\{K=K_{\min},\ldots,K_{\max}\}}{\arg\min} \hat{C}_K$.

The figure 2 shows the solution with the following benchmark parameter values:

$$\begin{aligned} a &= 10000, & Q &= 1000, & \underline{Y} &= 5000, & \delta &= 0.05, \\ b &= 1, & \tau &= 50, & \bar{Y} &= 50000, \end{aligned} \tag{28}$$

so that $K_{\min} = 1$, $K_{\max} = 10$. In this example, the optimal number of landfills is $K^* = 3$ and the sequence of capacities is slightly decreasing. The increasing or decreasing character of the solution depends on the specific cost function and the parameter values, and no general statements can be made. To illustrate this point, figure 3 shows the solution with $a = 20000$ keeping the rest of the parameter at their values given above. In this situation, the optimal number of landfills is $K^* = 2$ and $Y_1 > Y_0$, so that the sequence of capacities is increasing.

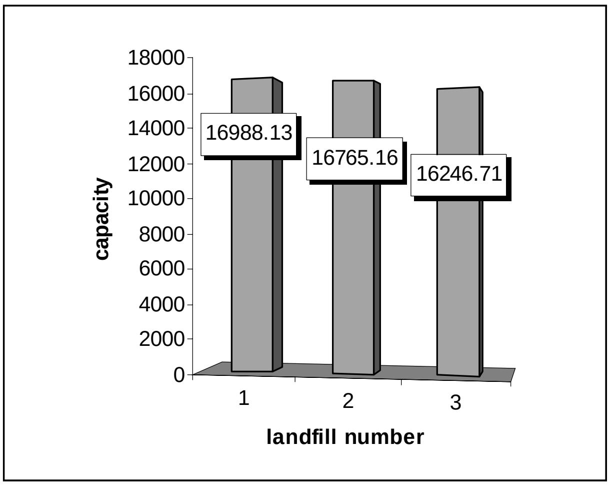

Figure 2. Solution with $C(Y) = a + bY$

$a = 10000, b = 1, \delta = 0.05, Q = 1000,$

$\underline{Y} = 5000,\ \bar{Y} = 50000$

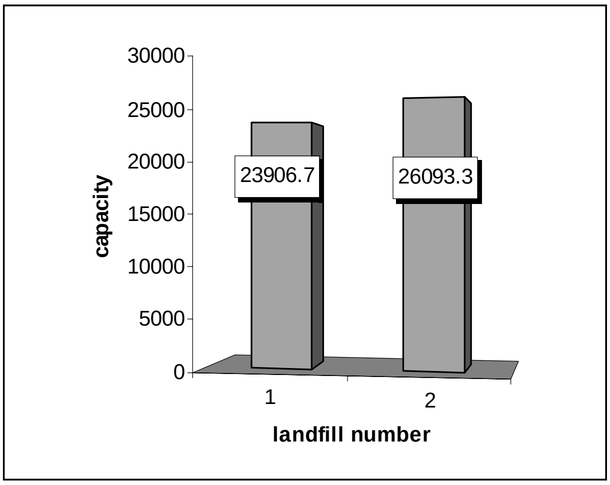

Figure 3. Solution with $C(Y) = a + bY$

$a = 20000, b = 1, \delta = 0.05, Q = 1000,$

$K_{\min} = 1, K_{\max} = 10$

We now perform some comparative statics analysis to illustrate the effect of the parameters on the solution. Specifically, from the benchmark values given at (28), we change the value of a single parameter each time and study the effect on the optimal number of landfills K^*, and the optimal average capacity $\tilde{Y}^* = \frac{1}{K}\sum_{i=0}^{K-1} Y_i^*$.

The following table summarizes the sensitivity analysis exercises. We have carried out some additional proofs with different parameter combinations, but the qualitative results do not change, so the information displayed here is enough to have a good idea about the properties of the solution.

Parameter	**Benchmark**	**Minimum**	**Maximum**	**Step**
a	10000	1000	20000	100
b	1	0.2	10	0.2
δ	0.05	0.01	0.3	0.01
Q	1000	400	10000	50
τ	50	20	150	1

The figures 4 and 5 show the results: the first column represents the variable K^* (vertical axis) as a function of the parameters a, b, δ, Q and τ (horizontal axis). The second column shows the variable $\tilde{Y}^*$ as a function of the same parameters. Note the similarity of these results with those obtained in the example 2.1. Increasing the parameter a (fixed construction cost) makes it optimal to build a smaller number of (larger in average) landfills. The contrary happens when b (marginal cost) increases: it is optimal to build more (smaller on average) landfills. Increasing the rate of discount makes it optimal to build more (smaller) landfills in order to delay costs.

Increasing Q or τ makes the whole amount of waste increase, so that a larger total capacity is needed. The graphics in the figure 5 show how this increase of capacity is optimally achieved. "Small" increments of τ or Q cause the optimal number of landfills to keep constant and increase the average capacity up to a point that the increment is large enough to build a new landfill. As a consequence, the variable $K^*(\tilde{Y}^*)$, as a function of Q and τ, have a stair (sawtooth) shape.

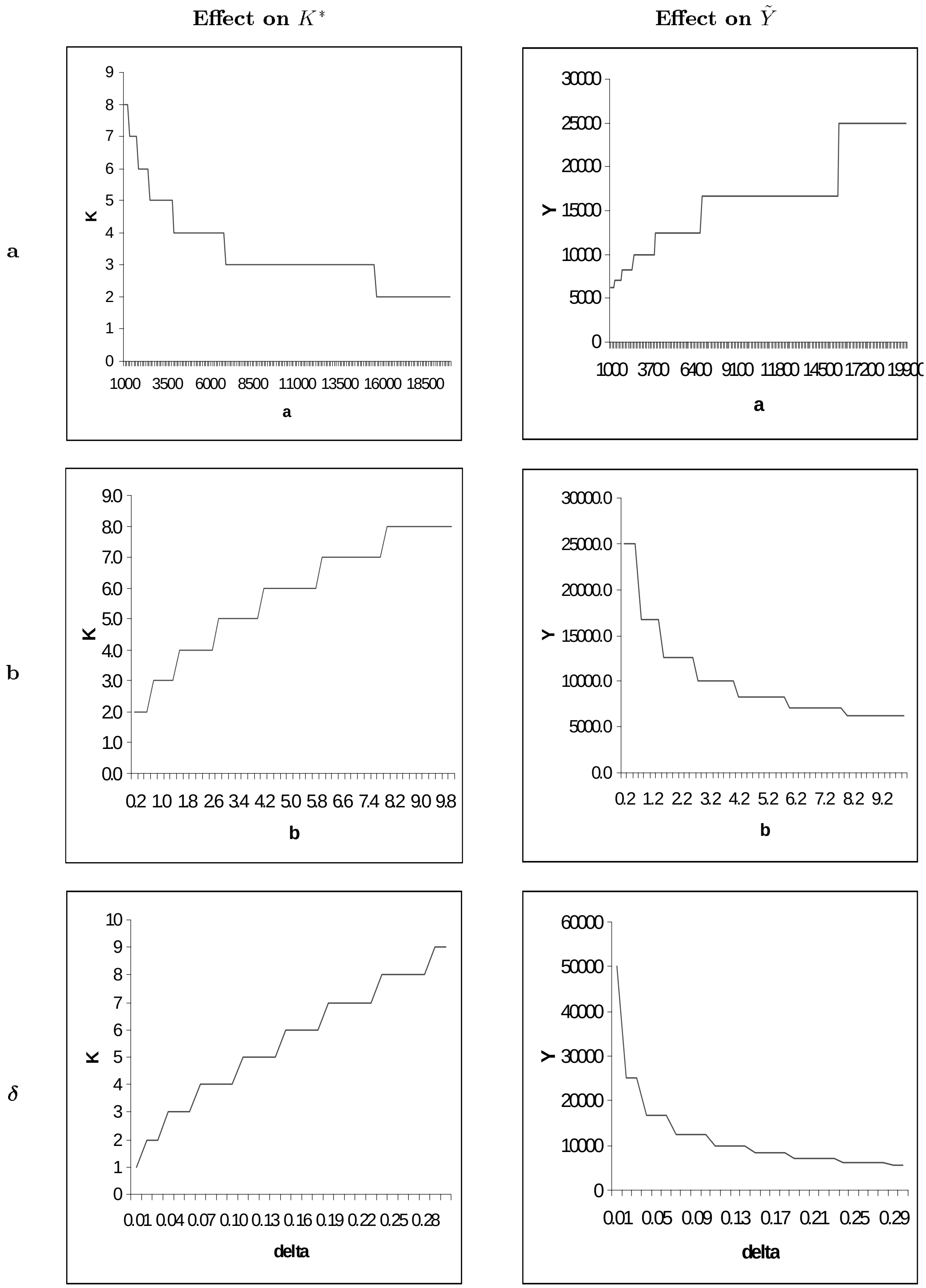

Figure 4. Effect of different parameters with $C(Y) = a + bY$

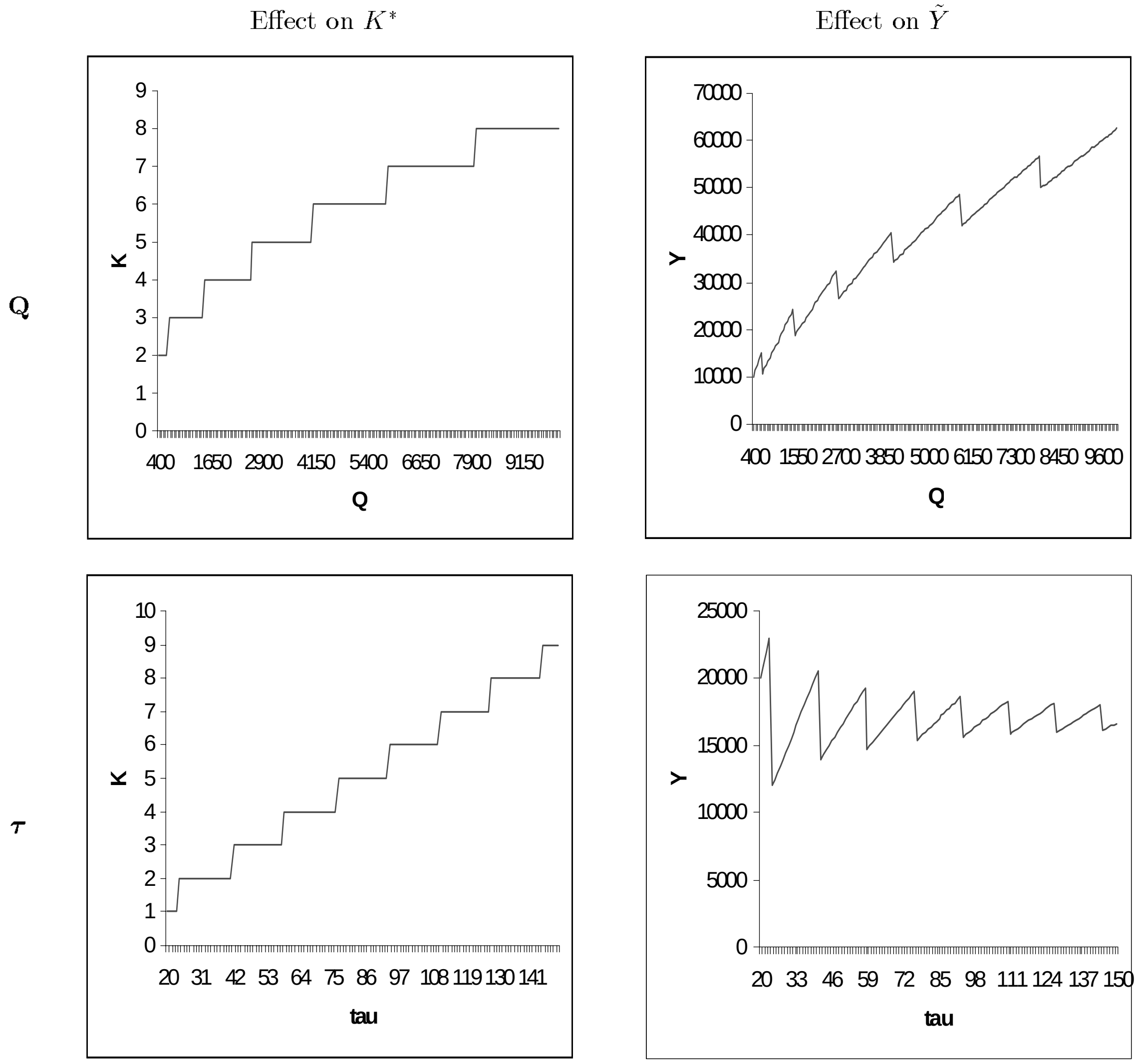

Figure 5. Effect of different parameters with $C(Y) = a + bY$

4 Conclusions and further research

The optimal capacity of a sequence of landfills, which is usually taken as given in most economic articles, has been studied in the present paper within a dynamic framework. The basic dynamic nature of the problem has been pointed out and several features of the solution have been explored. In an interior solution, the optimal capacity is determined according to the so-called *Optimal Capacity Condition*, which states the equality between the marginal cost and marginal gain of increasing the capacity. The marginal gain comes from all the discounted cost saving attached to future landfills that can be achieved by increasing the capacity of the present landfill.

If all the landfills are to have the same capacity and the time horizon is finite, we have shown that the optimal number of landfills, as a function of the time horizon, is stair shaped and the optimal capacity has a sawtooth shape. In the finite horizon case, if the marginal cost is small enough with respect to total cost, it may be optimal to construct an excess of capacity that is unexhausted under the optimal solution.

Relaxing the assumption of a constant capacity for all the landfills, a class of Optimal Control problems, sharing some continuous time and some discrete time features, has been stated and solved to study the properties of an optimal sequence of landfill capacities. The methodological way to address these problems consists of disregarding the temporal nature of the switching time variable, which becomes the state variable of the problem, and the time-variable role is played by the landfill index.

Proposition 2 provides the proper version of the *Optimal Capacity Condition* for the dynamic problem, which is a nonlinear first order difference equation relating the capacity of two consecutive landfills. This equation can be interpreted as a non-arbitrage condition establishing the impossibility of reducing the total cost by transferring some capacity form one landfill to another one. In the particular case where the discount rate is zero and the unit processing cost are always the same, such a condition collapses to an equality between the marginal cost of any two consecutive landfills.

An instability result is shown for the infinite horizon case under the usual assumptions on the cost function. From equation (24) we can conclude that a necessary condition for a stable steady state to exist is that the cost function has to be strictly concave.

We have provided some further insights for the solution to the finite horizon case using both Dynamic Programming and the Lagrange method. We have also provided a numerical illustration when the cost function is linear. We offer an algorithm to solve the problem in this case and some sensitivity analysis results concerning the effects of the parameters on the solution. The results show that the optimal number of landfills depends negatively on parameter a (fixed construction cost) and positively on the rest of the parameters of the model. The (average) optimal capacity of landfills depends positively on a, negatively on b (marginal construction cost) and the discount rate, and shows a sawtooth pattern with respect to the instantaneous waste generation rate and the time horizon.

Some future research lines include the endogenous decision of the amount of waste to be landfilled (by using some alternative waste treatment apart from landfilling, such as incineration and recycling) and the joint decisions of capacity and location of a sequence of landfills. Note the dynamic interaction between both decisions: once a landfill is built, depending of its capacity, there is a given land area around it which can not be used for building future landfills, so that the feasible set becomes smaller.

The decision problems discussed in this article share a common structure that involves splitting a time horizon of planning into some subintervals the length of which has to be decided. In each of the subintervals some costs, the amount of which depends on the decision variables, have to be borne. This dynamic structure arising from the optimal capacity decision resembles other economic dynamic problems that, up to our knowledge, have not been addressed from this perspective. Take, as an example, a consumer's decision about the purchase of a durable good, for example, a computer: purchasing a last-generation computer implies a larger cost but is likely to have a longer lifetime, while a cheaper computer will become obsolete sooner. An Optimal Capacity Condition (similar to the one proposed in this article for landfill management) seems to fit quite well with the computer purchasing dynamic policy of a consumer, as well as the infrastructure policy of a firm or a public agency.

Notes

[1]The generation of waste can be affected, to some extent, by policy variables such as waste taxes or recycling incentives. As far as the capacity of landfills do not seem to be one of those variables, we take the flow of waste as exogenously given information when making the capacity decisions.

[2]Using the expressions for $C(Y)$, $C'(Y)$ and $C''(Y)$, and substituting (5) in (4), we have the following expression for the second order condition:

$$\frac{\delta}{Q}\left(e^{-\delta\frac{Y}{Q}}+e^{-2\delta\frac{Y}{Q}}\right)(a+bY)\geq 2e^{-\delta\frac{Y}{Q}}\left(1-e^{-\delta\frac{Y}{Q}}\right)b.$$

Multiplying both sides by $e^{\delta\frac{Y}{Q}}$ and rearranging, we have $\dfrac{e^{\delta\frac{Y}{Q}}\left(1+e^{-\delta\frac{Y}{Q}}\right)(a+bY)}{e^{\delta\frac{Y}{Q}}-1}\geq\dfrac{2bQ}{\delta}$. Using (5) to substitute $e^{\delta\frac{Y}{Q}}-1$ and simplifying, we obtain $e^{\delta\frac{Y}{Q}}\geq 1$, that taking logarithms becomes $\frac{\delta}{Q}Y\geq 0$. Given that δ, $Q>0$, the second order condition holds for any $Y\geq 0$.

[3]As we will focus on interior solutions, the minimum and maximum capacity constraints $\underline{Y}\leq Y_i\leq\overline{Y}$ will not be explicitly taken into account.

[4]Using recursively the formula $T_{i+1}=T_i+\frac{Y_i}{Q}$, we have $T_i=\frac{1}{Q}\sum_{j=0}^{i-1}Y_j$, $(i=1,2,\dots)$.

Solving the integral in the objective function of (P) and using the latter equation, we have

$$\int_{T_i}^{T_{i+1}}e^{-\delta(t-T_i)}\phi_i Q dt=\frac{Q\phi_i}{\delta}\left[1-e^{-\delta(T_{i+1}-T_i)}\right]=\frac{Q\phi_i}{\delta}\left[1-e^{-\delta\frac{Y_i}{Q}}\right].$$

[5]Provided that $\phi_0=\phi_1=\cdots=\phi_{K-1}=\phi$, we have that the aggregated processing cost

$$\sum_{i=0}^{K-1}\left[\int_{T_i}^{T_{i+1}}e^{-\delta t}\phi Q dt\right]=\int_0^{T_K}e^{-\delta t}\phi Q dt=\frac{Q\phi}{\delta}\left[1-e^{-\delta T_K}\right]$$

is a constant and need not be considered when solving the optimization problem.

[6]If such an assumption were relaxed, the constraint $T_K=\tau$ would become $T_K\geq\tau$. It is strightforward to conclude that it is never optimal to under-exploit the capacity of the landfills $i=0,1,\dots,K-2$. As for the landfill, $K-1$, a situation with $Y_{K-1}^*>\underline{Y}$ and $T_K^*>\tau$ can not be optimal because the total cost could be reduced just by reducing the value of Y_{K-1}^* until $Y_{K-1}^*=\underline{Y}$ or $T_K=\tau$ holds, and the only possibility of optimally keeping the $K-1^{th}$ landfill unexhausted is $Y_{K-1}^*=\underline{Y}$. As we focus on interior solutions, we will not pay explicit attention to this possibility.

5 Appendix: mathematical results

5.1 Sensitivity analysis of example 2.1

Derivating (5) with respect to a, b, Q and δ, and rearranging,

$$\frac{\partial Y}{\partial a}=\frac{1}{b\left[e^{\delta\frac{Y}{Q}}-1\right]},\qquad \frac{\partial Y}{\partial b}=\frac{Y}{b\left[e^{\delta\frac{Y}{Q}}-1\right]}-\frac{Q}{b\delta},$$

$$\frac{\partial Y}{\partial Q}=\frac{-1}{\delta}-\frac{Y}{Q}\frac{e^{\delta\frac{Y}{Q}}}{1-e^{\delta\frac{Y}{Q}}}.\qquad \frac{\partial Y}{\partial \delta}=\frac{Q}{\delta^2}+\frac{Ye^{\delta\frac{Y}{Q}}}{\delta\left(1-e^{\delta\frac{Y}{Q}}\right)},$$

or using (5) to substitute $e^{\delta\frac{Y}{Q}}-1=\frac{\delta}{bQ}(a+bY)$, and rearranging,

$$\begin{aligned}&\frac{\partial Y}{\partial a}=\frac{Q}{\delta\left[a+bY\right]}>0, &&\frac{\partial Y}{\partial b}=\frac{-Qa}{b\delta\left(a+bY\right)}<0,\\ &\frac{\partial Y}{\partial Q}=\frac{Y}{Q}-\frac{1}{\delta}\frac{a}{(a+bY)}\lesseqgtr 0, &&\frac{\partial Y}{\partial \delta}=\frac{Q}{\delta^2}\left[\frac{a}{a+bY}\right]-\frac{Y}{\delta}\lesseqgtr 0,\end{aligned}\tag{29}$$

Derivating the expression for $\frac{\partial Y}{\partial Q}$ with respect to a and b and rearranging to obtain

$$\frac{\partial^2 Y}{\partial Q\partial a}=\frac{\partial Y}{\partial a}\left[\frac{1}{Q}+\frac{ab}{\delta\left(a+bY\right)^2}\right]-\frac{bY}{\delta\left(a+bY\right)^2},\qquad \frac{\partial^2 Y}{\partial Q\partial b}=\frac{\partial Y}{\partial b}\left[\frac{1}{Q}+\frac{ab}{\delta\left(a+bY\right)^2}\right]+\frac{aY}{\delta\left(a+bY\right)^2},$$

or, using (29) to substitute $\frac{\partial Y}{\partial a}$ and $\frac{\partial Y}{\partial b}$, and rearranging,

$$\frac{\partial^2 Y}{\partial Q\partial a}=\frac{\delta a^2+\delta abY+abQ}{\delta^2\left(a+bY\right)^3}>0,\qquad \frac{\partial^2 Y}{\partial Q\partial b}=\frac{-a^2\left(\delta a+\delta bY+bQ\right)}{\delta^2 b\left(a+bY\right)^3}<0.$$

5.2 Proof of proposition 1

Derivating (7) with respect to Y,

$$J'(Y)=\frac{1-e^{-K\delta\frac{Y}{Q}}}{1-e^{-\delta\frac{Y}{Q}}}C'(Y)+\frac{\frac{K\delta}{Q}\left(e^{-K\delta\frac{Y}{Q}}-e^{-(K+1)\delta\frac{Y}{Q}}\right)-\frac{\delta}{Q}\left(e^{-\delta\frac{Y}{Q}}-e^{-(K+1)\delta\frac{Y}{Q}}\right)}{\left(1-e^{-\delta\frac{Y}{Q}}\right)^2}C(Y),$$

from which we have the following condition for $J'(Y)$ to be negative:

$$J'(Y)<0\Longleftrightarrow\frac{C'(Y)}{C(Y)}<\frac{\delta}{Q}\frac{\left(e^{-\delta\frac{Y}{Q}}-e^{-(K+1)\delta\frac{Y}{Q}}\right)-K\left(e^{-K\delta\frac{Y}{Q}}-e^{-(K+1)\delta\frac{Y}{Q}}\right)}{\left(1-e^{-K\delta\frac{Y}{Q}}\right)\left(1-e^{-\delta\frac{Y}{Q}}\right)},$$

which can also be expressed as

$$\frac{C'(Y)}{C(Y)}<\frac{\delta}{Q}\cdot\frac{1}{1-e^{-K\delta\frac{Y}{Q}}}\cdot\frac{e^{-\delta\frac{Y}{Q}}-e^{-K\delta\frac{Y}{Q}}-(K-1)\,e^{-K\delta\frac{Y}{Q}}+(K-1)\,e^{-(K+1)\delta\frac{Y}{Q}}}{1-e^{-\delta\frac{Y}{Q}}}$$

and using $\frac{e^{-K\delta\frac{Y}{Q}}-e^{-(K+1)\delta\frac{Y}{Q}}}{1-e^{-\delta\frac{Y}{Q}}}=e^{-K\delta\frac{Y}{Q}}$, (15) obtains.

5.3 Proof of proposition 2

Derivating (16), we have the first order conditions

$$C_0'(Y_0)+\phi_0 e^{-\delta\frac{Y_0}{Q}}-\frac{\delta}{Q}\sum_{i=1}^{\infty}e^{-\frac{\delta}{Q}\sum_{j=0}^{i-1}Y_j}\left[C_i(Y_i)+\frac{Q\phi_i}{\delta}\left(1-e^{-\delta\frac{Y_i}{Q}}\right)\right]=0,$$

$$e^{-\frac{\delta}{Q}\sum_{j=0}^{k-1}Y_j}\left[C_k'(Y_k)+\phi_k e^{-\delta\frac{Y_k}{Q}}\right]-\frac{\delta}{Q}\sum_{i=k+1}^{\infty}e^{-\frac{\delta}{Q}\sum_{j=0}^{i-1}Y_j}\left[C_i(Y_i)+\frac{Q\phi_i}{\delta}\left(1-e^{-\delta\frac{Y_i}{Q}}\right)\right]=0\quad k=1,2\ldots$$

Multiplying both sides by $e^{\frac{\delta}{Q}\sum_{j=0}^{k-1}Y_j}$ (for $k=1,2,\ldots$) these conditions can be expressed as

$$C_k'(Y_k)+\phi_k e^{-\delta\frac{Y_k}{Q}}-\frac{\delta}{Q}\sum_{i=k+1}^{\infty}e^{-\frac{\delta}{Q}\sum_{j=k}^{i-1}Y_j}\left[C_i(Y_i)+\frac{Q\phi_i}{\delta}\left(1-e^{-\delta\frac{Y_i}{Q}}\right)\right]=0,\qquad k=0,1,2,\ldots.$$

or, rearranging,

$$C_k'(Y_k)+\phi_k e^{-\delta\frac{Y_k}{Q}}=e^{-\frac{\delta}{Q}Y_k}\frac{\delta}{Q}\left[C_{k+1}(Y_{k+1})+\frac{Q\phi_{k+1}}{\delta}\left(1-e^{-\delta\frac{Y_{k+1}}{Q}}\right)\right]$$

$$+\frac{\delta}{Q}\sum_{i=k+2}^{\infty}e^{-\frac{\delta}{Q}\sum_{j=k+1}^{i-1}Y_j}\left[C_i(Y_i)+\frac{Q\phi_i}{\delta}\left(1-e^{-\delta\frac{Y_i}{Q}}\right)\right]$$

and, taking into account the condition for the landfill $k+1$,

$$C_{k+1}'(Y_{k+1})+\phi_{k+1}e^{-\delta\frac{Y_{k+1}}{Q}}=\frac{\delta}{Q}\sum_{i=k+2}^{\infty}e^{-\frac{\delta}{Q}\sum_{j=k+1}^{i-1}Y_j}\left[C_i(Y_i)+\frac{Q\phi_i}{\delta}\left(1-e^{-\delta\frac{Y_i}{Q}}\right)\right],$$

to substitute $\frac{\delta}{Q}\sum_{i=k+2}^{\infty}(\cdot)$ and rearranging, (17) follows.

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