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A Bargaining set based on External and Internal Stability and Endogenous Coalition Formation

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Abstract

A new bargaining set based on notions of both internal and external stability is developed in the context of endogenous coalition formation. It allows to make an explicit distinction between within-group and outside-group deviation options. This type of distinction is not present in current bargaining sets. For the class of weighted majority games, the outcomes in the bargaining set containing a minimal winning coalition are characterized. Furthermore, it is shown that the bargaining set of any homogeneous weighted majority game contains an outcome for which the underlying coalition structure consists of a minimal winning coalition and its complement. The paper also introduces a new class of games called cooperation externalities games. For a symmetric cooperation externalities game conditions are provided such that every outcome in the bargaining set supports the same coalition structure. This coalition structure consists of one coalition of all players with an externality parameter higher than one and a collection of singleton coalitions, one for every player with a cooperation externality parameter lower than one.

JEL Classification: C71, C78

Keywords: Bargaining set, endogenous coalition formation, internal and external stability

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1 Introduction

Economic entities, such as medical practices, insurance groups, research teams and coalitional governments, involve agents who generate value by cooperating in groups. In some situations the groups that actually form will partition the whole population into smaller groups, while in others the population as a whole will form one cooperating group. Studying this endogenous formation of groups and predicting which groups will break up or will be stable is a captivating area of research. In politics it can predict which governments can be stable. In organizational science it can predict which researchers can be grouped together or alternatively should work alone. The value generated by a coalition in most cases cannot be traced back to the individual efforts. This brings about an additional question of how the group value should be translated into individual payoffs.

These two questions, of coalition formation and of value allocation, are interdependent and require a simultaneous answer as argued by Maschler (1992). They were addressed simultaneously in the seminal work of Aumann and Maschler (1964) where an outcome of a cooperative game consists of a coalition structure, *i.e.*, a partition of the player set into coalitions, and a payoff vector which divides the value of each coalition in the partition among its members. To analyze the stability properties of an outcome Aumann and Maschler (1964) introduce the *Maschler bargaining set*[4]. The Maschler bargaining set is the set of outcomes which survive a specific bargaining process among all players. In this bargaining process over a given outcome, players put forward "objections" and "counterobjections" against other members of the same coalition in the coalition structure of the outcome. An objection consists of a new coalition, of which the objecting player is a member and the player against whom the objection has been raised is not, such that all members of the new coalition can obtain higher payoffs than what is allocated to them in the proposed outcome. The player against whom the objection has been raised can launch a counterobjection. A counterobjection consists of a coalition and a payoff vector such that the coalition members can obtain at least as high a payoff as in the original outcome and those of them who also participate in the coalition used in the objection can get at least as much as they would have obtained if the objection had been executed. The player who launches the counterobjection must be a member of the coalition used in the counterobjection, while the player who has raised the objection must be excluded from it. The bargaining set contains those outcomes for which each objection can be countered.

An early work by Peleg (1967) shows that any coalition structure is stable for a

[4]Aumann and Maschler (1964) introduce several definitions of bargaining sets and study in depth only one of them, which is not the *Maschler bargaining set*. The Maschler bargaining set gains popularity and is studied in more detail in later works, *e.g.*, Davis and Maschler (1967).

coalitional game with a non-empty imputation set in terms of the Maschler bargaining set, *i.e.*, there is a payoff vector which allocates the value of each coalition in the coalition structure among its members such that the coalition structure and this payoff vector constitute an outcome in the Maschler bargaining set. This finding precludes the use of the Maschler bargaining set in analyzing endogenous coalition formation. Zhou (1994) offers a new bargaining set which has the desirable property that in this setting it does not support all possible coalition structures. A more recent work by Morelli and Montero (2003) introduces another solution concept which selects "more desirable" outcomes out of those selected in the Zhou's bargaining set.

A common aspect of these bargaining sets is that they treat the deviation possibilities within a coalition structure element and between coalition structure elements in a symmetric way. This, in our opinion, is a serious limitation since in many economic situations transaction costs and institutional arrangements will require to make a distinction between the two. When considering a deviation within a group, all subsets of this group should be taken as a possible threat point against the group.[5] However, when considering a deviation involving more than one coalition structure element, our new bargaining set only allows a player to join an already formed group. As a motivation one can think of prohibitively high transaction costs in terms of licensing requirements, which make it impossible that new groups are formed based on subgroups of distinct coalition structure elements. The different treatment of internal and external stability distinguishes our bargaining set from the previously studied bargaining sets. Below we offer two examples that illustrate the difference between internal and external objections in an endogenous coalition formation setting.

Consider a parliament of representatives of four parties and a seat distribution such that no party can form a government on its own. Suppose that there is one big party and three small ones such that the big party with any of the three small parties can form a government, and so can the three small parties together. Consider an outcome in which a government is formed by the big party and one of the small parties. Our bargaining set predicts that the allocation of government value is different if the opposition parties act together or separately. In the first case any of the government parties may threaten to split off the government by joining the opposition to form a new government. In the second case, such a threat is only available to the big party.

Investigating the outcomes in our bargaining set in the general setting of weighted majority games is the first application that we offer. We show that in any weighted majority game, the minimal winning coalition formed by the players with the highest weights and all other players acting alone leads to a coalition

[5] This is also the case in all other existing bargaining sets.

structure that is supported by an outcome in our bargaining set.[6] This implies that a coalitional government based on the biggest parties in the parliament when the opposition parties do not cooperate is stable. However, in practice one also observes situations of united opposition. This raises the question whether in any weighted majority game, there is a stable partition comprised of a minimal winning coalition and its complement. We answer this question negatively for the general case but for the case of homogeneous weighted majority games the answer is positive.

Now consider a different setting of a group of researchers who have the same research capabilities and only differ in a cooperation externality parameter. Some researchers experience positive spillovers when working in teams and carry positive externality, while others tend to free-ride when they are in a team and thus carry negative externalities. Consider a coalition structure consisting of teams of researchers. For a coalition structure to be internally stable, there should not be an internal objection of a researcher against another researcher member of the same team, which the latter researcher cannot counter. An internal objection in a coalition structure element is analogous to an objection in the Maschler bargaining set of the coalition-restricted cooperative game. A valid counterobjection in our setting has an additional requirement over the counterobjection defined in the Maschler bargaining set: the subset of the team used in the objection and the one used in the counterobjection have at least one member in common. This type of modification was originally introduced by Zhou (1994) and it tailors the bargaining set to select coalition structures with higher total partition value. In addition, a researcher may raise an external objection against another researcher of her team by threatening to join another team in the coalition structure. Such an external objection can be countered if the researcher against whom it has been raised is at least as desirable to the outside team as the researcher who launches the objection. A coalition structure and a payoff vector such that for any internal objection there is an internal counterobjection and for every external objection there is an external counterobjection will constitute an outcome in our bargaining set.

This example illustrates the second application that is studied. In this application researchers will differ not only in the direction of externality, but also in the degree of positive or negative externalities that they cause. In the symmetric case in which each researcher has either a fixed "negative" (less than 1) or "positive" externality (higher than 1), the negative externalities weakly dominate the positive ones, and in which there are at least two players who have an externality parameter higher than 1, we find that the unique coalition structure supported by the bargai-

[6]Burani and Zwicker (2003) show that the same type of coalitional structure is both core and Nash stable in the setting of hedonic coalition formation games with descending separable preferences.

ning set is the one in which the "cooperative" researchers form a coalition, while the "non-cooperative" researchers are singletons. [7]

In addition to the distinction between internal and external deviations, we introduce two types of coalitional rationality conditions, splitting-proofness, which is a weak form of the coalitional rationality condition present in the bargaining set studied in depth by Aumann and Maschler (1964), and merging-proofness. These conditions require that total payoffs do not increase if a coalition structure element is split in two or if two coalition structure elements merge.

The remainder of the paper is organized in the following way. In Section 2 we formally introduce our bargaining set. Section 3 and Section 4 consider applications to weighted majority games and games with cooperation externalities, respectively.

2 The Bargaining Set

We first give some basic notions. Let $N = \{1, 2, \ldots, n\}$ be a finite set of players. Players can form coalitions $S \subseteq N$. The set of all possible coalitions is denoted by 2^N. A value function $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, provides the value each coalition generates by cooperation of its members. In addition and without loss of generality we assume $v(\{i\}) = 0$ for all $i \in N$. The pair (N, v) is a *coalitional game*. A partition of N into non-empty coalitions is called a *coalition structure*. The set of all possible coalition structures of N is denoted by $\mathcal{P}$.

An outcome of a coalitional game is represented by a *payoff configuration*. A payoff configuration is a pair $(\mathtt{P}, x)$ where $\mathtt{P} \in \mathcal{P}$ is a coalition structure of N and $x \in \mathbb{R}^N$ is an efficient payoff vector for $\mathtt{P}$, *i.e.*, $x(S) = v(S)$ for all $S \in \mathtt{P}$, where $x(S) := \sum_{i \in S} x_i$. A payoff configuration $(\mathtt{P}, x)$ of a coalitional game (N, v) is *individually rational* if $x_i \geq 0$ for all $i \in N$.

For the sake of completeness and comparison we first present the Maschler and Zhou bargaining sets.

For the definition of the Maschler bargaining set we follow Maschler (1992). Let $(\mathtt{P}, x)$ be an individually rational payoff configuration of a coalitional game (N, v). Let players k and l be two distinct members of some coalition $S \in \mathtt{P}$. The pair (T, y) with $T \in 2^N$ and $y \in \mathbb{R}^T$ is called an *objection* of k against l in $(\mathtt{P}, x)$ if $k \in T$, $l \notin T$, $y(T) = v(T)$, $y_i > x_i$ for all $i \in T$. The pair (Q, z) with $z \in \mathbb{R}^Q$ is called a *counterobjection* to the above objection (T, y) in $(\mathtt{P}, x)$ if $l \in Q$, $k \notin Q$, $z(Q) = v(Q)$, $z_i \geq x_i$ for all $i \in Q$, and $z_i \geq y_i$ for all $i \in Q \cap T$.

The Maschler bargaining set $\mathcal{M}(v)$ consists of those individually rational payoff configurations for which each objection can be countered.

[7] This result is also reminiscent of the one described in Burani and Zwicker (2003).

The bargaining set introduced in Zhou (1994) differs from the Maschler bargaining set in the definitions of both an objection and a counterobjection. Let $(\mathtt{P}, x)$ be a payoff configuration of a coalitional game (N, v). The pair (T, y) with $y \in \mathbb{R}^T$ is called a $\mathcal{Z}$*-objection* of coalition $T \in 2^N$ in payoff configuration $(\mathtt{P}, x)$ if $y(T) = v(T)$ and $y_i > x_i$ for all $i \in T$. The pair (Q, z) with $z \in \mathbb{R}^Q$ is called a $\mathcal{Z}$ *-counterobjection* to the above objection (T, y) in $(\mathtt{P}, x)$ if $z(Q) = v(Q)$, $Q \setminus T \neq \emptyset$, $T \setminus Q \neq \emptyset$, $T \cap Q \neq \emptyset$, $z_i \geq x_i$ for all $i \in Q$, and $z_i \geq y_i$ for all $i \in Q \cap T$.

The Zhou bargaining set $\mathcal{Z}(v)$ consists of those payoff configurations for which each $\mathcal{Z}$-objection can be countered.

Note that the above condition will imply that any element of $\mathcal{Z}(v)$ is individually rational.

Our new bargaining set will combine separate notions of internal stability and external stability. First we present the internal bargaining set. The following two conditions reflect the notions of internal coalitional rationality and stability against internal deviations, respectively.

Definition 2.1 *Let* (N, v) *be a coalitional game. A coalition structure* $\mathtt{P}$ *is* splitting-proof *if for all* $S \in \mathtt{P}$*, and all disjoint coalitions* $T, Q \subseteq S$*, such that* $T \cup Q = S$*,* $v(S) \geq v(T) + v(Q)$*.*

Let $(\mathtt{P}, x)$ be a payoff configuration of a coalitional game (N, v) and let k and l be two distinct members of the same coalition $S \in \mathtt{P}$. The pair (T, y) with $T \subseteq S \setminus \{l\}$, $k \in T$ and $y \in \mathbb{R}^T$ is called an *internal objection* of k against l in $(\mathtt{P}, x)$ if $y(T) = v(T)$ and $y_i > x_i$ for all $i \in T$. The pair (Q, z) with $z \in \mathbb{R}^Q$ is called an *internal counterobjection* to the above internal objection (T, y) in $(\mathtt{P}, x)$ if $z(Q) = v(Q)$, $l \in Q$, $Q \subseteq S \setminus \{k\}$, $Q \cap T \neq \emptyset$, $z_i \geq x_i$ for all $i \in Q$, and $z_i \geq y_i$ for all $i \in Q \cap T$. We say that an internal objection (T, y) of player k against l in $(\mathtt{P}, x)$ is *justified* if there is no internal counterobjection.

Definition 2.2 *Let* (N, v) *be a coalitional game. A payoff configuration* $(\mathtt{P}, x)$ *is in the internal bargaining set* $\mathcal{B}^I(v)$ *if*

(i) $\mathtt{P}$ *is splitting-proof,* *and*

(ii) there is no justified internal objection.

A payoff configuration $(\{N\}, x)$ of the internal bargaining set is in the Maschler bargaining set as well: the definition of objection is the same and that of counterobjection imposes the additional requirement that the coalition used in the objection and the coalition used in the counterobjection are not disjoint. Moreover, $\{N\}$ is required to be splitting-proof. For payoff configurations comprising other coalition structures than $\{N\}$, there is no general relation since we require the coalitions used both in the objection and the counterobjection to be subsets of one specific coalition structure element.

Compared to the Zhou bargaining set we have the following differences. First, any coalition is allowed to object in the Zhou bargaining set; in our bargaining set the coalition T used in the objection is a subset of one coalition structure element and, moreover, must exclude at least one player. On the other hand, countering an objection is easier in the definition of Zhou. We require the counterobjection to be launched by the player against whom the objection has been raised, whereas in the Zhou approach a counterobjection can be launched by any player. Furthermore, we require that the coalition used in the counterobjection must be a subset of the same coalition structure element of which T is a subset, whereas any coalition which has a non-empty intersection with T can be used to counterobject in Zhou's framework. Given the differences it is easy to see that a payoff configuration $(\{N\}, x)$ of the internal bargaining set is in the Zhou bargaining set as well: any $\mathcal{Z}$-objection can be translated into an internal objection, and the corresponding internal counterobjection can also be used as a $\mathcal{Z}$-counterobjection. However, the reverse does not necessarily hold. The next example clearly illustrates the differences.

Example 2.3 Let $N = \{1, 2, 3\}$, $v(\{i\}) = 0$ for all $i \in N$, $v(\{2, 3\}) = 0$, $v(\{1, 2\}) = v(\{1, 3\}) = 20$, and $v(N) = 21$.

Consider the payoff configuration $(\{1, 2, 3\}, (7, 7, 7))$. It is in the Zhou bargaining set: any $\mathcal{Z}$-objection $(\{1, 2\}, y)$ can be countered using $\mathcal{Z}$-counterobjection $(\{1, 3\}, z)$ with $z_1 = y_1$. Similarly, $\mathcal{Z}$-objections using coalition $\{1, 3\}$ can be countered using coalition $\{1, 2\}$. However, it is not in the internal bargaining set: player 1 has a justified internal objection $(\{1, 2\}, (10, 10))$ against player 3.

Consider the payoff configuration $(\{\{1, 2\}, \{3\}\}, (10, 10, 0))$. It is in the internal bargaining set because there are no internal objections. However, it is not in the Zhou bargaining set: the pair $(\{1, 2, 3\}, (10\frac{1}{3}, 10\frac{1}{3}, \frac{1}{3}))$ constitutes a $\mathcal{Z}$ -objection that cannot be countered. ♦

The following result is immediate.

Proposition 2.4 *Let* (N, v) *be a coalitional game. Then*

(i) $(\langle N\rangle^{8}, 0) \in \mathcal{B}^I(v)$ *;*

(ii) $(\mathrm{P}, x) \in \mathcal{B}^I(v)$ *implies that* (P, x) *is individually rational.*

Next we present the external bargaining set. Similar to the internal bargaining set, the external bargaining set is based on two notions which reflect external coalitional rationality and stability against external deviations, respectively.

[8] $\langle S\rangle := \{\{i\}_{i\in S}\}$ for all $S \in 2^N$.

Definition 2.5 *Let* (N, v) *be a coalitional game. A coalition structure* $\mathtt{P}$ *is* merging-proof *if for all* $S, T \in \mathtt{P}$ *we have* $v(S) + v(T) \geq v(S \cup T)$.

Let $(\mathtt{P}, x)$ be a payoff configuration of a coalitional game (N, v) and let k and l be two distinct members of the same coalition $S \in \mathtt{P}$. The pair $(T \cup \{k\}, y)$ with $y \in \mathbb{R}^{T\cup\{k\}}$ is called an *external objection* of k against l in $(\mathtt{P}, x)$ if $T \in \mathtt{P}$, $T \neq S$, $y(T \cup \{k\}) = v(T \cup \{k\})$ and $y_i > x_i$ for all $i \in T \cup \{k\}$. The pair $(T \cup \{l\}, z)$ with $z \in \mathbb{R}^{T\cup\{l\}}$ is called an *external counterobjection* to the above objection $(T \cup \{k\}, y)$ in $(\mathtt{P}, x)$ if $z(T \cup \{l\}) = v(T \cup \{l\})$, $z_k \geq x_k$, and $z_i \geq y_i$ for all $i \in T$. We say that an external objection (T, y) of player k against l in $(\mathtt{P}, x)$ is *justified* if there is no external counterobjection.

Definition 2.6 *Let* (N, v) *be a coalitional game. A payoff configuration* $(\mathtt{P}, x)$ *is in the external bargaining set* $\mathcal{B}^E(v)$ *if*

(i) $\mathtt{P}$ *is merging-proof,* *and*

(ii) there is no justified external objection.

Since no external objections can be launched against the grand coalition, we find the following result.

Proposition 2.7 *For a coalitional game* (N, v) *any payoff configuration* $(\{N\}, x)$ *lies within* $\mathcal{B}^E(v)$.

Note that the payoff configurations in the external bargaining set are not necessarily individually rational.

The new bargaining set $\mathcal{B}(v)$ consists of all payoff configurations that are both in the internal bargaining set and in the external bargaining set.

Definition 2.8 *For a coalitional game* (N, v) *the bargaining set* $\mathcal{B}(v)$ *is given by*

$$\mathcal{B}(v) = \mathcal{B}^I(v) \cap \mathcal{B}^E(v).$$

We say that the coalition structure $\mathtt{P}$ is *stable* if there is a payoff vector $x \in \mathbb{R}^N$ such that the payoff configuration $(\mathtt{P}, x)$ is an element of $\mathcal{B}(v)$.

Unfortunately, the bargaining set can be empty.

Example 2.9 Consider the coalitional game (N, v) given by $N = \{1, 2, 3, 4, 5, 6, 7\}$ with $v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = v(\{5, 6\}) = v(\{5, 7\}) = v(\{6, 7\}) = 20$, $v(\{1, 2, 3\}) = v(\{5, 6, 7\}) = 21$, $v(\{1, 2, 3, 4\}) = v(\{4, 5, 6, 7\}) = 30$, $v(S) = 0$ for $|S| = 1$, and $v(S) < 0$, otherwise. For this game $\mathcal{B}(v) = \emptyset$.

First note that no coalition with a negative value can be a coalition structure element of a payoff configuration which belongs to our bargaining set since such a payoff configuration will not be individually rational.

Suppose $(\mathtt{P}, x) \in \mathcal{B}(v)$ with $\mathtt{P} = \{\{1, 2, 3, 4\}, \{5, 6, 7\}\}$. First suppose $x_4 = 0$. Then player 4 has a justified external objection $(\{4, 5, 6, 7\}, y)$ against any other member of the coalition $\{1, 2, 3, 4\}$. Next suppose $x_4 > 0$. Then it must be the case that at least one of the other players, 1, 2, or 3, receives a payoff of strictly less than 10, *e.g.*, player 1; and that the sum of the payoffs of two of these players must be strictly less than 20, *e.g.*, $x_1 + x_2 < 20$. Player 1 can then launch a justified internal objection against player 4 using coalition $\{1, 2\}$ and a payoff vector that gives to both players more than what is allocated to them. Hence $(\mathtt{P}, x) \notin \mathcal{B}(v)$.

Next consider the coalition structure $\{\{1, 2, 3\}, \{4\}, \{5, 6, 7\}\}$. It is not stable since it is not merging-proof: $v(\{1, 2, 3, 4\}) > v(\{1, 2, 3\}) + v(\{4\})$.

In a similar fashion it can be shown that no payoff configuration belongs to the bargaining set. ♦

However, the bargaining set is not empty in a coalitional game with three players.

Proposition 2.10 *Let* (N, v) *be a coalitional game with* $N = \{1, 2, 3\}$. *Let* $\mathtt{P}$ *be such that* $\sum_{S \in \mathtt{P}} v(S)$ *is maximized. Then there exists a payoff vector* $x \in \mathbb{R}^N$ *such that* $(\mathtt{P}, x) \in \mathcal{B}(v)$.

Proof. First note that $\mathtt{P}$ is splitting-proof and merging-proof. Recall that $\mathcal{M}(v)$ is nonempty for all coalition structures. So there is a payoff vector $x \in \mathbb{R}^3$ such that $(P, x) \in \mathcal{M}(v)$. We will show that $(P, x) \in \mathcal{B}(v)$.

Let $\mathtt{P} = \{N\}$. By individual rationality of $\mathcal{M}(v)$, if a player i has an objection against another player j at x it must be using coalition $\{i, k\}$. Because $\{N\}$ is the coalition structure with maximum total value, an objection of i against j can only exist if $x_j > 0$. A counterobjection in the Maschler sense must then use coalition $\{j, k\}$, which is also a valid counterobjection in the sense of $\mathcal{B}(v)$.

Similar arguments can be used in case $\mathtt{P} = \{\{i, j\}, \{k\}\}$. The case $\mathtt{P} = \langle N \rangle$ is straightforward since objections are not possible. ■

3 Monotonic Proper Simple Games

First we provide some basic definitions. A coalitional game (N, v) is called *simple* if $v(\emptyset) = 0$, $v(N) = 1$, and $v(S) \in \{0, 1\}$, otherwise. A simple game is monotonic if $v(T) = 1$ whenever $v(S) = 1$ for some $S \subseteq T$. We denote the set of *winning coalitions* in a simple game by $\mathcal{W} := \{S \in 2^N \mid v(S) = 1\}$ and by $\mathcal{W}^m := \{S \in 2^N \mid v(S) = 1 \text{ and } v(T) = 0 \text{ for all } T \subsetneq S\}$ the set of *minimal winning coalitions*. A simple game is *proper* if for all $S, T \in \mathcal{W}$ it holds that $S \cap T \neq \emptyset$.

We can establish the following result with respect to monotonic proper simple games.

Theorem 3.1 *Let (N, v) be a monotonic proper simple game. Then*

$$(\{N\}, x) \in \mathcal{B}(v) \Longleftrightarrow (\{N\}, x) \in \mathcal{M}(v).$$

Proof. We have already seen that for general coalitional games $(\{N\}, x) \in \mathcal{B}(v)$ implies $(\{N\}, x) \in \mathcal{M}(v)$. To show the converse, consider a payoff configuration $(\{N\}, x) \in \mathcal{M}(v)$. Proposition 2.7 gives that $(\{N\}, x) \in \mathcal{B}^E(v)$. Next we show that $(\{N\}, x) \in \mathcal{B}^I(v)$. In all proper simple games $\{N\}$ is splitting-proof. Furthermore, every internal objection can be countered. Let (T, y) be an internal objection of player k against player l. Any internal objection is also a Maschler objection. Since $(\{N\}, x) \in \mathcal{M}(v)$, there exists a counterobjection (Q, z) in the Maschler sense. We have to find a counterobjection $(\tilde{Q}, \tilde{z})$ in the new sense. If $Q \cap T \neq \emptyset$, we can take $(\tilde{Q}, \tilde{z}) = (Q, z)$. So assume $Q \cap T = \emptyset$. Since $(\{N\}, x)$ is individually rational, it follows that $T \in \mathcal{W}$. Then by properness $v(Q) = 0$, and by individual rationality $z_i = x_i = 0$ for all $i \in Q$. Since $v(T) > x(T)$, there must be a player $j \in N \setminus T$ such that $x_j > 0$. Player k can launch an objection against player j by (T, y) as well. Let (Q', z') be the counterobjection of player j in the Maschler sense. Then, $Q' \in \mathcal{W}$ and by properness $Q' \cap T \neq \emptyset$. Monotonicity implies that $Q' \cup \{l\} \in \mathcal{W}$. So we can take $\tilde{Q} = Q' \cup \{l\}$ and $\tilde{z}$ such that $\tilde{z}_i = z'_i$ for all $i \in Q'$ and $\tilde{z}_l = x_l = 0$. ■

Note that Theorem 3.1 implies that the bargaining set is non-empty for all monotonic proper simple games.

Next we provide a characterization of the payoff configurations in the bargaining set of a weighted majority game that contain a minimal winning coalition as a coalition structure element . A *weighted majority game* (N, v) is a simple game for which there exists a vector of weights $w \in \mathbb{R}^N_+$ and a threshold $q \in \mathbb{R}_+$ with $0 \leq q \leq w(N)$ such that $S \in \mathcal{W}$ if and only if $w(S) \geq q$. The pair $(q; w)$ is called a *representation* of weighted majority game (N, v). Let $S \in \mathcal{W}^m$ and let $\mathtt{P} \in \mathcal{P}$ with $S \in \mathtt{P}$. We denote the set of members of S who can make another coalition in $\mathtt{P}$ winning by joining it $E(S, \mathtt{P})$, *i.e.*,

$$E(S, \mathtt{P}) = \{i \in S \mid v(T \cup \{i\}) = 1 \text{ for some } T \in \mathtt{P} \setminus \{S\}\}.$$

Further we denote the indicator vector of a coalition $S \in 2^N$ by e_S, *i.e.*, $e_S(i) = 1$ if $i \in S$ and $e_S(i) = 0$ if $i \in N \setminus S$.

Theorem 3.2 *Let (N, v) be a proper weighted majority game. Let $S \in \mathcal{W}^m$ and let $\mathtt{P} \in \mathcal{P}$ be a coalition structure with $S \in \mathtt{P}$. Then $\{(\mathtt{P}, x) \mid x \in \mathbb{R}^N\} \cap \mathcal{B}(v)$ equals:*

(i) $\{(\mathtt{P}, x) \mid x \in \mathbb{R}^N_+, x(S) = 1, x(N \setminus S) = 0\}$ *if* $E(S, \mathtt{P}) = \emptyset$,

(ii) $\{(\mathtt{P}, e_i)\}$ *if* $E(S, \mathtt{P}) = \{i\}$,

(iii) $\emptyset$ *if* $E(S, \mathtt{P}) \subsetneq S$ *and* $|E(S, \mathtt{P})| \geq 2$,

(iv) $\{(\mathtt{P}, \frac{1}{|S|} e_S)\}$ *if* $E(S, \mathtt{P}) = S$ *and* $|E(S, \mathtt{P})| \geq 2$.

Proof. Observe that $\mathtt{P}$ is robust against merging and splitting. Let $(\mathtt{P}, x)$ be a payoff configuration in the bargaining set. It is immediate that $x \in \mathbb{R}^N_+$, $x(S) = 1$, and $x(N \setminus S) = 0$. Clearly, there are no internal objections. Moreover, external objections can be made only by members of $E(S, \mathtt{P})$ who are allocated less than one.

If $E(S, \mathtt{P}) = \emptyset$, there are no further requirements for $(\mathtt{P}, x)$ being an element of the bargaining set. Hence we are in case (*i*).

If $E(S, \mathtt{P}) = \{i\}$ and furthermore $E(S, \mathtt{P}) \subsetneq S$, then player i has a justified external objection against any player in $S \setminus E(S, \mathtt{P})$ unless x_i equals 1. If $E(S, \mathtt{P}) = S = \{i\}$, the only feasible payoff configuration is $(\mathtt{P}, e_i)$. We are in case (*ii*).

If $E(S, \mathtt{P}) \subsetneq S$ and $|E(S, \mathtt{P})| \geq 2$, then there are at least two distinct players i and j in $E(S, \mathtt{P})$ who have justified external objections against any player in $S \setminus E(S, \mathtt{P})$, unless both i and j have payoffs equal to 1. Since we have established above that $x \in \mathbb{R}^N_+$ and $x(S) = 1$, $x_i = x_j = 1$ is not feasible. This gives case (*iii*).

If $E(S, \mathtt{P}) = S$ and $|E(S, \mathtt{P})| \geq 2$, let $i \in S$ and let $(T \cup \{i\}, y)$ be an objection against another player j in S. Clearly, $y(T)$ equals $1 - x_i - \varepsilon$ for some $\varepsilon > 0$. A counterobjection exists if $1 - x_i - \varepsilon + x_j \leq 1$, *i.e.*, $x_j \leq x_i + \varepsilon$. From this argument, we may conclude that $x_i = x_j$ for all $i, j \in S$. Hence we are in case (*iv*). ■

The following example illustrates the above result.

Example 3.3 Let $(6; 3, 2, 2, 2, 1)$ be a representation of a proper weighted majority game.

There are three types of minimal winning coalitions: a coalition formed by all players with weight 2; coalitions formed by the player with weight 3 and two players with weight 2; and coalitions formed by the player with weight 3, a player with weight 2 and the player with weight 1. Depending on the type of the minimal winning coalition and the coalitions formed by the players outside the minimal winning coalition, there are six types of coalition structures that contain a minimal winning coalition. We will discuss an example of each of the six types.

First, consider the minimal winning coalition $S_1 = \{2,3,4\}$ and the coalition structure $\mathtt{P}_1 = \{\{2,3,4\},\{1\},\{5\}\}$. Clearly, $E(S_1,\mathtt{P}_1) = \emptyset$ and we are in case (*i*) of Theorem 3.2. The payoff configurations in the set $\{(\mathtt{P}_1, x) \mid x \in \mathbb{R}^N_+, x(S_1) = 1, x(N \setminus S_1) = 0\}$ are elements of the bargaining set.

The other coalition structure that contains the same minimal winning coalition is $\mathtt{P}'_1 = (\{\{2,3,4\},\{1,5\}\})$. Since $E(S_1,\mathtt{P}'_1) = S_1$, we are in case (*iv*) of Theorem 3.2. The payoff configuration $(\mathtt{P}'_1,(0,\frac{1}{3},\frac{1}{3},\frac{1}{3},0)$ is in the bargaining set: internal objections are not possible and any external objection can be countered.

Next consider the minimal winning coalition $S_2 = \{\{1,2,3\}$ and the coalition structure $\mathtt{P}_2 = \{\{1,2,3\},\{4\},\{5\}\}$. Since $E(S_2,\mathtt{P}_2) = \emptyset$, we are in case (*i*) of Theorem 3.2

Furthermore, consider the coalition structure $\mathtt{P}'_2 = \{\{1,2,3\},\{4,5\}\}$. Here $E(S_2,\mathtt{P}'_2) = \{1\}$. Clearly, we are in case (*ii*) of Theorem 3.2. The payoff configuration $(\mathtt{P}'_2,(1,0,0,0,0,0)$ is in the bargaining set since internal and external objections are not possible.

Last consider the minimal winning coalition $S_3 = \{1,2,5\}$ and the coalition structure $\mathtt{P}_3 = \{\{1,2,5\},\{3\},\{4\}\}$. Since $E(S_3,\mathtt{P}_3) = \emptyset$, we are in case (*i*) of Theorem 3.2.

Consider the coalition structure $\mathtt{P}'_3 = \{\{1,2,5\},\{3,4\}\}$. In this coalition structure $E(S_3,\mathtt{P}'_3) = \{1,2\}$. Since $|E(S_3,\mathtt{P}'_3)| = 2$ and $E(S_3,\mathtt{P}'_3) \subsetneq S_3$, we are in case (*iii*) of Theorem 3.2. There is no payoff configuration in the bargaining set pertaining to $\mathtt{P}_3$ since either player 1 or player 2 can have a justified external objection against player 5. ♦

The next result concerns types of stable coalition structures of any proper weighted majority game.

Theorem 3.4 *Let $(q;w)$ with $w_1 \geq \ldots \geq w_n$ be a representation of a proper weighted majority game (N,v). Then the coalition structure $\{\{1,2,\ldots,k\},\{k+1\}, \{k+2\},\ldots,\{n\}\}$ with k such that $\{1,2,\ldots,k\} \in \mathcal{W}^m$ is stable.*

Proof. Consider the coalition structure $\mathtt{P} = \{\{1,2,\ldots,k\},\{k+1\},\{k+2\},\ldots,\{n\}\}$ with k such that $\{1,2,\ldots,k\} \in \mathcal{W}^m$. First, assume $w_1 + w_{k+1} < q$. Then the payoff configuration $(\mathtt{P},(1,0,\ldots,0))$ is an element of the bargaining set: no player can raise an objection. Next, assume $w_1 + w_{k+1} \geq q$. Hence, $w_1 + w_2 \geq q$ and therefore $k \leq 2$. If $k = 1$ or if both $k = 2$ and $w_2 + w_{k+1} < q$, $(\mathtt{P},(1,0,\ldots,0)) \in \mathcal{B}(v)$. If $k = 2$ and $w_2 + w_{k+1} \geq q$, then $(\mathtt{P},(\frac{1}{2},\frac{1}{2},0,\ldots,0))$ is an element of the bargaining set. Hence $\mathtt{P}$ is stable. ■

Theorem 3.4 implies that the coalition structure containing the minimal winning coalition formed by the players with the highest weights and the rest of the players as singletons is stable in any proper weighed majority game.

In the remainder of this section we address the question whether for any proper weighted majority game, there is a stable coalition structure consisting of a minimal winning coalition and its complement. The following example illustrates that this is not true for the general class of proper weighted majority games.

Example 3.5 Let $(11; 4, 4, 2, 2, 2, 1, 1, 1, 1, 1, 1)$ be a representation of a proper weighted majority game.

We provide the main arguments why coalition structures containing a minimal winning coalition and its complement are not stable. We distinguish between minimal winning coalitions with weight 11 and minimal winning coalitions with weight 12.

A minimal winning coalition with weight 11 always contains a player with weight 1 and at least two players with weights higher than 1. At least one of these two players with weight higher than 1 has a justified external objection against the player with weight 1. An example is the payoff configuration $(\{\{1, 3, 4, 5, 6\}, \{2, 7, 8, 9, 10\}\}, (1, 0, \ldots, 0))$: player 3 has a justified objection against player 6.

A minimal winning coalition with weight 12 consists of all players with weight 4 and two players with weight 2. At least one the players with weight 4 has a justified external objection against a player with weight 2. For instance, consider the payoff configuration $(\{\{1, 2, 3, 4\}, \{5, 6, 7, 8, 9, 10, 11\}\}, (1, 0, \ldots, 0))$ where player 2 has a justified external objection against player 3. ♦

There is however a stable coalition structure consisting of a minimal winning coalition and its complement for all proper and homogenous weighted majority games. A proper weighted majority game (N, v) is *homogeneous* if there exists a representation $(q; w)$ such that $w(S) = q$ for all $S \in \mathcal{W}^m$.

Theorem 3.6 *Let (N, v) be a proper homogeneous weighted majority game. Then there exists a minimal winning coalition S such that the coalition structure $\{S, N \setminus S\}$ is stable.*

Proof. Let $(q; w)$ be a representation of (N, v). For ease of exposition all players $i \in N$ for whom $w_i + w(N) - q \geq q$ are called *strong* and all others *weak*. Homogeneity implies that strong players in a minimal winning coalition S are exactly those players in the set $E(S, \{S, N \setminus S\})$.

If there are no strong players, it is easily derived that any coalition structure containing a minimal winning coalition is stable.

So we may assume there are strong players. If the set of strong players is winning, following the same line of argument as in the proof of case (*iv*) in Theorem 3.2, there exists a minimal winning coalition S consisting of strong players only, and, $(\{S, N \setminus S\}, \frac{1}{s} e_S) \in \mathcal{B}(v)$. If the set of strong players is losing, take a minimal winning coalition $T \in \mathcal{W}^m$ which contains all strong players. Take any strong player $i \in T$. Then the coalition $(N \setminus T) \cup \{i\}$ is winning. This coalition contains a minimal winning coalition S with $i \in S$. Again using the same argument as in the proof of case (*ii*) of Theorem 3.2, one can check that $(\{S, N \setminus S\}, e_i) \in \mathcal{B}(v)$. ■

Finally, it is easy to see that there is a stable coalition structure consisting of a minimal winning coalition and its complement for all strong weighted majority games. A weighted majority game (N, v) is *strong* if $S \notin W$ implies $N \setminus S \in W$. Since $E(S, \mathrm{P}) = S$ for all $S \in W^m$ the coalition structure $\{S, N \setminus S\}$ is stable for any minimal winning coalition S: either $|S| = 1$ and we are in case (*ii*) or $|S| \geq 2$ and we are in case (*iv*) of Theorem 3.2.

4 Cooperation Externalities Games

We now construct a special type of games, cooperation externalities games, that model settings in which the players only differ in a *cooperation externality* parameter. The cooperation externality parameter captures either negative externalities or positive externalities that a player carries in cooperation with others. If a player's marginal contribution to a group value is positive, we say that she carries *positive externality* with respect to that group. Conversely, if a player's contribution is negative, we say that she carries *negative externality* with respect to that group. If a player's marginal contribution to a group value is zero, then we say that the player is externality-free with respect to that group. A player may carry positive externality in one group and negative externality in another group depending on the composition of the group. Players may not only vary in the direction but also in the degree of cooperation externality they cause. Apart from their heterogeneity in the cooperation externalities, which becomes evident only when they are members of a group, players are homogeneous. These ideas are captured in the characteristic function of the coalitional game with cooperation externalities given below.

Let N be the set of players. The vector $\varepsilon \in \mathbb{R}^N_+$ with $0 \leq \varepsilon_i \leq 2$ for all $i \in N$ is the vector of cooperation externality parameters of the players. The *cooperation externalities game* (N, v_ε) is defined by

$$v(S) = |S|^{\bar{\varepsilon}(S)} - |S| \quad \text{with } \bar{\varepsilon}(S) := \tfrac{1}{|S|} \textstyle\sum_{i \in S} \varepsilon_i \quad \text{for all } S \in 2^N. \tag{1}$$

The first term captures the cooperation externalities. Note that all singletons have a zero value, hence, players are homogeneous in their individual productivity. Consider a coalition of more than one player. A coalition formed by players with $\varepsilon > 1$ has value higher than zero, while a coalition formed by players with $\varepsilon < 1$ has value lower than zero. A mixed coalition formed by players of both types may have either higher, equal, or lower value than zero, depending on whether the players with $\varepsilon > 1$ outweigh, balance, or have lower impact than the one of the players with $\varepsilon < 1$.

Example 4.1 Let $N = \{1, 2, 3, 4\}$ and $\varepsilon = (0.5, 1, 1, 1.5)$. Then the corresponding cooperation externalities game (N, v_ε) is given by $v_\varepsilon(\{1, 2\}) = v_\varepsilon(\{1, 3\}) = -0.318$, $v_\varepsilon(\{2, 4\}) = v_\varepsilon(\{3, 4\}) = 0.378$, $v_\varepsilon(\{1, 2, 3\}) = -0.502$, $v_\varepsilon(\{2, 3, 4\}) = 0.603$, and $v_\varepsilon(S) = 0$ otherwise.

The payoff configuration $(\{\{1\}, \{2, 3, 4\}\}, (0, 0.201, 0.201, 0.201))$ is an element of the bargaining set $\mathcal{B}(v_\varepsilon)$ since it does not allow for any (internal or external) objections. ♦

To facilitate the exposition below, we introduce the following additional notation. The set of players with a cooperation externality parameter at least one is denoted by $H := \{i \in N \mid \varepsilon_i \geq 1\}$; the set of players with a cooperation externality parameter lower than one is denoted by $L := \{i \in N \mid \varepsilon_i < 1\}$. The vector ε is called *symmetric* if there are real numbers ε_H and ε_L such that $\varepsilon_i = \varepsilon_H$ for all $i \in H$ and $\varepsilon_i = \varepsilon_L$ for all $i \in L$. A cooperation externalities game is called *symmetric* if the underlying vector ε is symmetric.

Theorem 4.2 *Let (N, v_ε) be a symmetric cooperation externalities game with an externality parameter $\varepsilon_H \in [1, 2]$ for players in H and $\varepsilon_L \in [0, 1)$ for players in L. If $\varepsilon_H + \varepsilon_L \leq 2$, then the coalition structure $\{H, \langle L\rangle\}$ is stable. It is the unique stable coalition structure if additionally $\varepsilon_H > 1$ and $|H| \neq 1$.*

The proof requires three auxiliary results.

Lemma 4.3 *Let $s, t \in \mathbb{N}$ such that $s + t \geq 3$. Then*

$$(s + t)^{s-t} < s^s. \tag{2}$$

Proof. The following assertion is equivalent to (2)

$$\left(\frac{s + t}{s}\right)^s < (s + t)^t.$$

Newton's binomial formula gives that

$$(1 + \frac{t}{s})^s = \sum_{k=0}^{s} \binom{s}{k} (\frac{t}{s})^k < \sum_{k=0}^{s} \frac{t^k}{k!} < e^t.$$

Hence, for $s + t \geq 3$, (2) is valid. ■

Lemma 4.4 *Let $s, t \in [1, \infty)$ and $\alpha \in [0, \infty)$. Then*

$$(s+t)^\alpha - s^\alpha - t^\alpha \begin{cases} > 0 & \textit{for } \alpha \in (1, \infty) \\ = 0 & \textit{for } \alpha = 1 \\ < 0 & \textit{for } \alpha \in [0, 1) \end{cases} \tag{3}$$

Proof. Let $g : [1, \infty) \times [1, \infty) \longrightarrow \mathbb{R}$ be defined by

$$g(s, t) = (s + t)^\alpha - s^\alpha - t^\alpha.$$

The first order derivative with respect to s, yields

$$g'_s(s, t) = \alpha(s + t)^{\alpha-1} - \alpha s^{\alpha-1},$$

which is positive for $\alpha > 1$, zero for $\alpha = 1$ and negative for $\alpha < 1$. A similar analysis holds for the first order derivative with respect to t. Furthermore,

$$g(1, 1) = 2^\alpha - 2,$$

which is positive for $\alpha > 1$, zero for $\alpha = 1$ and negative for $\alpha < 1$. This completes the proof. ■

Lemma 4.5 *Let $i \in L$ and $S \subseteq N \setminus \{i\}$ such that $v_\varepsilon(S) \geq 0$. Then*

$$v_\varepsilon(S \cup \{i\}) \leq v_\varepsilon(S) + v_\varepsilon(\{i\}).$$

Proof. Denote $\bar{\varepsilon}(S)$ by ε. Because $v_\varepsilon(S) \geq 0$, we have that $\varepsilon \geq 1$. Define $g : [1, 2] \longrightarrow \mathbb{R}$ by

$$g(\varepsilon) = (s + 1)^{\frac{s\varepsilon+2-\varepsilon}{s+1}} - s^\varepsilon - 1.$$

Because $\varepsilon_L \leq 2 - \varepsilon$, we have $v_\varepsilon(S \cup \{i\}) - v_\varepsilon(S) - v_\varepsilon(\{i\}) \leq g(\varepsilon)$ for $\varepsilon \in [1, 2]$. Hence it suffices to show that $g(\varepsilon) \leq 0$. If $s = 1$, then $g(\varepsilon) = 0$. If $s \geq 2$, denote $(s + 1)^{\frac{s-1}{s+1}}$ by a. So $g(\varepsilon) = (s + 1)^{\frac{2}{s+1}} \cdot a^\varepsilon - s^\varepsilon - 1$. Because of Lemma 4.3, we have that $a < s$. Furthermore,

$$g'(\varepsilon) = \ln(a) \cdot (s + 1)^{\frac{2}{s+1}} \cdot a^\varepsilon - \ln(s) \cdot s^\varepsilon.$$

If $\varepsilon = 1$, this boils down to

$$\begin{aligned} g'(1) &= \ln(a) \cdot (s + 1)^{\frac{2}{s+1}} \cdot a - \ln(s) \cdot s \\ &= \ln(s + 1) \cdot \frac{s - 1}{s + 1} \cdot (s + 1)^{\frac{2}{s+1}} \cdot (s + 1)^{\frac{s-1}{s+1}} - \ln(s) \cdot s \\ &= \ln(s + 1) \cdot (s - 1) - \ln(s) \cdot s \\ &= \ln(s + 1)^{(s-1)} - \ln s^s, \end{aligned}$$

which is negative because of Lemma 4.3. Suppose ε is such that $g'(\varepsilon) < 0$. Then

$$\begin{aligned} g''(\varepsilon) &= \ln^2(a) \cdot (s+1)^{\frac{2}{s+1}} \cdot a^{\varepsilon} - \ln^2(s) \cdot s^{\varepsilon} \\ &< \ln(s) \cdot \ln(a) \cdot (s+1)^{\frac{2}{s+1}} \cdot a^{\varepsilon} - \ln^2(s) \cdot s^{\varepsilon} \\ &= \ln(s) \cdot g'(\varepsilon) \\ &< 0. \end{aligned}$$

Summarizing we find $g'(1) < 0$ and $g''(\varepsilon) < 0$ for all $\varepsilon \in [1,2]$ with $g'(\varepsilon) < 0$. This implies that $g'(\varepsilon) < 0$ for all $\varepsilon \in [1,2]$. Since $g(1) = 0$, this completes the proof. ■

Proof of Theorem 4.2. Let $\varepsilon_H + \varepsilon_L \leq 2$. Consider the payoff configuration $(\{H, \langle L \rangle\}, x)$ with $x \in \mathbb{R}^N$ given by $x := \frac{v(H)}{|H|} e_H$. We show that $(\{H, \langle L \rangle\}, x) \in \mathcal{B}(v_\varepsilon)$.[9,10] First, we show that there are no internal objections. Obviously, there are no internal objections in $\{i\}$ for all $i \in L$. Suppose there is an internal objection of player $k \in H$ against player $l \in H$ using coalition $T \subseteq H \setminus \{l\}$ with $k \in T$. Then $|T| \geq 2$, since $v_\varepsilon(\{k\}) = 0$. Player l can use coalition $Q = (T \setminus \{k\}) \cup \{l\}$ to counterobject since $v_\varepsilon(T)$ and $v_\varepsilon(Q)$ coincide.

Next, we show that H is splitting-proof and $\langle L \rangle$ is merging-proof. For the remainder of the proof we denote the cardinalities of the coalitions called S and T by s and t, respectively. First, consider $S, T \subseteq H$ such that $S \cap T = \emptyset$ and $S \cup T = H$. Then

$$v_\varepsilon(H) - v_\varepsilon(S) - v_\varepsilon(T) = (s+t)^{\varepsilon_H} - s - t - (s^{\varepsilon_H} + t^{\varepsilon_H} - s - t) \geq 0,$$

where the last inequality follows from $\varepsilon_H \geq 1$ and Lemma 4.4.

Next, consider $i, j \in L$. Then

$$v_\varepsilon(\{i, j\}) - v_\varepsilon(\{i\}) - v_\varepsilon(\{j\}) = (2)^{\varepsilon_L} - 2 < 0,$$

where the last inequality follows from $\varepsilon_L < 1$.

In order for $\{H, \langle L \rangle\}$ to be merging-proof, $v_\varepsilon(H \cup \{i\}) \leq v_\varepsilon(H) + v(\{i\})$ for all $i \in L$. This is the case because of lemma 4.5.

Finally, we show that there are no external objections by players in H. Since $v_\varepsilon(\{i, j\}) \leq 0$ for all $i \in H$ and $j \in L$, no player with ε_H can launch an external objection.

In order to show the part of the proposition concerning uniqueness, we need to prove that no coalition structure but $\{H, \langle L \rangle\}$ is stable under the additional conditions $\varepsilon_H > 1$ and $|H| \geq 2$. Let (P', x) be an individually rational payoff configuration of (N, v).

[9] In fact one can show that $(\{H, \langle L \rangle\}, x) \in \mathcal{B}(v_\varepsilon) \cap \mathcal{M}(v_\varepsilon) \cap \mathcal{Z}(v_\varepsilon)$.

[10] There are other payoff vectors pertaining to the coalition structure $\{\{H\}, \langle L \rangle\}$ that lead to a payoff configuration in our bargaining set, *e.g.*, each element of the Core of the H-restricted game and a zero payoff to all players in L.

Firstly, if there are coalitions $S, T \in \mathrm{P}'$ with $S, T \subset H$ and $S \neq T$, then P' is not merging-proof with respect to these coalitions: Lemma 4.4 gives

$$v_\varepsilon(S+T) - v_\varepsilon(S) - v_\varepsilon(T) = (s+t)^{\varepsilon_H} - s^{\varepsilon_H} - t^{\varepsilon_H} > 0.$$

Next, suppose there is a player $i \in L$ and a non-empty coalition $S \subseteq N \setminus \{i\}$ such that $S \cup \{i\} \in \mathrm{P}'$. Because x is individually rational, $v_\varepsilon(S \cup \{i\}) \geq 0$ and hence, $v_\varepsilon(S) \geq 0$ as well. Moreover, lemma 4.5 gives $v_\varepsilon(S \cup \{i\}) \leq v_\varepsilon(S) + v_\varepsilon(\{i\})$. For $S \cup \{i\}$ to be splitting proof, we need an equality. This is only the case when $s = 1$ and $\varepsilon_H = 2 - \varepsilon_L$. So let us focus on this case. Let $S = \{j\}$, and $\varepsilon_j = \varepsilon_H = 2 - \varepsilon_L$. We have $v_\varepsilon(S \cup \{i\}) = 0$, so $x_i = x_j = 0$ as well. Since $|H| \geq 2$, there must be another coalition $T \in \mathrm{P}'$ with $T \cap H \neq \emptyset$. Player j has a justified external objection using coalition $T \cup \{j\}$ against player i, since

$$v_\varepsilon(T \cup \{j\}) - v_\varepsilon(T) = (t+1)^{\frac{t\bar{\varepsilon}(T)+\varepsilon_H}{t+1}} - t^{\bar{\varepsilon}(T)} - 1 > 0.$$

The inequality follows from $1 \leq \varepsilon(T) \leq \varepsilon_H$ and Lemma 4.4. So, P' is not stable. ■

Lemma 4.5 and the proof of Theorem 4.2 show that in a symmetric externalities game (N, v_ε) with $\varepsilon_H + \varepsilon_L \leq 2$ and $\varepsilon_H > 1$, all players in H have positive externalities with respect to any coalition with a non-negative value, except for singleton coalitions consisting of a member of L. All players in L have negative externalities with respect to any coalition.

Next we will discuss the necessity of the various conditions provided in Theorem 4.2 for the stability of the coalition structure $\{H, \langle L \rangle\}$. The first example shows that when the cooperation externalities game is not symmetric, $\{H, \langle L \rangle\}$ may not be stable.

Example 4.6 Let $N = \{1,2,3,4,5,6,7,8,9,10\}$, $\varepsilon = (1.1,2,2,2,2,2,2,2,2,2)$ and $\tilde{H} := \{i \in N \mid \varepsilon_i = 2\}$. The value function can be calculated using Equation (1). Here we give just some of the values. $v_\varepsilon(\{i\}) = 0$ for all $i \in N$; $v_\varepsilon(N) = 71.283$, $v_\varepsilon(\tilde{H}) = 72$. Note that $H = N$. The coalition structure $\{N\}$ is not stable since it is not splitting-proof. ♦

The next example illustrates the necessity of the condition that $\varepsilon_H + \varepsilon_L \leq 2$.

Example 4.7 Let $N = \{1,2,3\}$, and $\varepsilon = (0.9, 0.9, 1.9)$. The value function is calculated using Equation (1) which yields $v_\varepsilon(\{1,2\}) = -0.134$, $v_\varepsilon(\{1,3\}) = v_\varepsilon(\{2,3\}) = 0.639$, $v_\varepsilon(\{1,2,3\}) = 0.876$ and $v_\varepsilon(S) = 0$, otherwise. The coalition structure $\{\{1\},\{2\},\{3\}\}$ is not stable since it is not merging proof. ♦

The next example shows that the condition $\varepsilon_H > 1$ is necessary to obtain uniqueness.

Example 4.8 Let $N = \{1, 2\}$ and $\varepsilon = (1, 1)$. So, $v(S) = 0$ for all $S \in 2^N$. Both coalition structures $\{N\}$ and $\{\{1\}, \{2\}\}$ are stable since $(\{N\}, (0, 0)) \in \mathcal{B}(v_\varepsilon)$ and $(\{\{1\}, \{2\}\}, (0, 0)) \in \mathcal{B}(v_\varepsilon)$. ♦

In addition, to guarantee uniqueness in Theorem 4.2, it is required that $|H| \neq 1$. The next example illustrates the necessity of this condition.

Example 4.9 Let $N = \{1, 2\}$ and $\varepsilon = (0.9, 1.1)$. Using equation (1), we obtain $v_\varepsilon(S) = 0$ for all $S \in 2^N$. Both coalition structures $\{N\}$ and $\{\{1\}, \{2\}\}$ are stable since $(\{N\}, (0, 0)) \in \mathcal{B}(v_\varepsilon)$ and $(\{\{1\}, \{2\}\}, (0, 0)) \in \mathcal{B}(v_\varepsilon)$. ♦

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