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TWO-DIMENSIONAL MINIMAX LATIN HYPERCUBE DESIGNS

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Two-dimensional minimax Latin hypercube designs

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Abstract

We investigate minimax Latin hypercube designs in two dimensions for several distance measures. For the ℓ^∞-distance we are able to construct minimax Latin hypercube designs of n points, and to determine the minimal covering radius, for all n. For the ℓ^1-distance we have a lower bound for the covering radius, and a construction of minimax Latin hypercube designs for (infinitely) many values of n. We conjecture that the obtained lower bound is attained, except for a few small (known) values of n. For the ℓ^2-distance we have generated minimax solutions up to $n = 27$ by an exhaustive search method. The latter Latin hypercube designs will be included in the website *www.spacefillingdesigns.nl*.

1 Introduction

The problem of determining minimax Latin hypercube designs originates from the field of deterministic computer simulations. To approximate a black box function on the square it needs to be evaluated at some of the points. When these evaluations are expensive (in time or costs) it is important to choose these design points in such a way that all evaluations give as much information, and that the entire square is well represented. The first is guaranteed by requiring that the design is non-collapsing, and even better, that it is a Latin hypercube design. Non-collapsing means that the projections of the design points onto the axes are distinct; in a Latin hypercube design these projections are equidistant. This prevents that if one of the input parameters has considerably less influence on the output than the other input parameter, then almost identical (and expensive) scenarios have been simulated. There are several ways to make sure that the entire square is well-represented by the design points. Here we consider the minimax criterion, that is, the design points should be chosen such that the maximal distance of any point in the square to the design (the covering radius) is minimal. Minimax designs have been investigated by Johnson et al [3] and John et al [2]; however, they do not consider Latin hypercube designs.

Other criteria, such as maximin, integrated mean square error (IMSE), and entropy have been considered also; see the book by Santner et al [6]. For maximin Latin hypercube designs in two dimensions we refer to [1].

More formally, a two-dimensional Latin hypercube design of n points is a set of n points $(x_i, y_i) \in \{0, 1, \ldots, n-1\}^2$ such that all x_i are distinct and all y_i are distinct.

The covering radius ρ of such a Latin hypercube design is the maximal distance of any point in the square $[0, n-1]^2$ to its closest design point. A minimax Latin hypercube design of n

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points is one with minimal covering radius. A point in the square that is at distance ρ from the design is called a remote site.

We investigate the problem of finding minimax Latin hypercube designs for the distance measures ℓ^∞, ℓ^1, and ℓ^2. For ℓ^∞ we are able to construct minimax Latin hypercube designs of n points, and to determine the minimal covering radius, for all n. For ℓ^1 we have a lower bound for the covering radius, and a construction of minimax Latin hypercube designs for (infinitely) many values of n. We conjecture that the obtained lower bound is attained, except for a few small (known) values of n. For the hardest case, ℓ^2, there seems to be no general construction possible. Here we have generated minimax solutions up to $n = 27$ by an exhaustive search method. The latter Latin hypercube designs will be included in the website *www.spacefillingdesigns.nl*.

2 ℓ^∞-Minimax Latin hypercube designs

The problem of arranging n points in the m-dimensional hypercube $[0, n-1]^m$ with minimal covering radius is easily solved for the ℓ^∞-distance. The minimal covering radius equals $\rho = \frac{n-1}{2\lfloor n^{1/m} \rfloor}$ and is attained, for example, by choosing at least the k^m points in $\{\frac{2i-1}{2k}(n-1) \mid i = 1, \ldots, k\}^m$, where $k = \lfloor n^{1/m} \rfloor$. That this covering radius is minimal can be shown by considering the $(k+1)^m$ points in $\{\frac{i}{k}(n-1) \mid i = 0, \ldots, k\}^m$, which are all mutually at least $\frac{n-1}{k}$ apart, and hence must be covered by $(k+1)^m > n$ distinct ℓ^∞-"circles" if $\rho < \frac{n-1}{2k}$, which is a contradiction. Although this result could not be found in the literature, it is most likely not new.

For the two-dimensional case that we consider in this paper, the minimal covering radius $\rho = \frac{n-1}{2\lfloor \sqrt{n} \rfloor}$ increases significantly if we restrict ourselves to Latin hypercube designs. In this case the minimal covering radius turns out to be $\rho = \min\{\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{2n+1} \rceil, \frac{1}{2} + \lceil -\frac{3}{4} + \frac{1}{4}\sqrt{8n+9} \rceil\}$. We shall first show that this number is indeed a lower bound for the covering radius of a Latin hypercube design. After that we shall give constructions attaining this lower bound.

Lemma 1. *Let $n \geq 2$. A Latin hypercube design of n points in two dimensions has covering ℓ^∞-radius ρ at least* $\min\{\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{2n+1} \rceil, \frac{1}{2} + \lceil -\frac{3}{4} + \frac{1}{4}\sqrt{8n+9} \rceil\}$.

Proof. Consider a Latin hypercube design of n points in two dimensions, as subset of $\{0, \ldots, n-1\}^2$, with covering radius ρ. We remark first that the covering radius ρ is either an integer or half an integer. Suppose first that ρ is an integer. Then the points on the left boundary ($x = 0$) of the square $[0, n-1]^2$ can only be covered by the $\rho+1$ design points with x-coordinates $0, 1, \ldots, \rho$. Each such design point can only cover a part of the left boundary of length at most 2ρ, which implies that $n - 1 \leq 2\rho(\rho + 1)$. However, if equality is attained, then the y-coordinates of the $\rho + 1$ design points with x-coordinates $0, 1, \ldots, \rho$ must form the set $\{\rho, 3\rho, 5\rho, \ldots, n - 1 - \rho\}$. Similarly it follows that in this case the y-coordinates of the $\rho+1$ design points with x-coordinates $n - 1 - \rho, n - \rho, \ldots, n - 1$ must form this same set (consider the right boundary $x = n - 1$), which is a contradiction (since $n - 1 \neq \rho$). Thus we may conclude that $n \leq 2\rho(\rho + 1)$, if ρ is an integer; and in this case ρ is at least $\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{2n+1} \rceil$.

Suppose next that ρ is not an integer, but half an integer. Now the points on the left boundary ($x = 0$) of the square can only be covered by the $\rho + \frac{1}{2}$ design points with x-coordinates $0, 1, \ldots, \rho - \frac{1}{2}$. Each such design point can only cover a part of the left boundary of length at most 2ρ; moreover, the points that cover the corner points cover at most $2\rho - \frac{1}{2}$. This implies that $n - 1 \leq 2\rho(\rho + \frac{1}{2}) - 1$. However, similar as before equality gives a contradiction, which implies that $n \leq \rho(2\rho + 1) - 1$ if ρ is not an integer. Thus in that case we can deduce that ρ is at least $\frac{1}{2} + \lceil -\frac{3}{4} + \frac{1}{4}\sqrt{8n+9} \rceil$, which finishes the proof. □

To show that the above lower bound is attained we proceed as follows. First we consider the case where ρ is an integer, and construct a partial Latin hypercube design of $\rho^2 + 4\rho$ points with covering radius ρ for the square $[0, n-1]^2$, where $n = 2\rho^2 + 2\rho$.

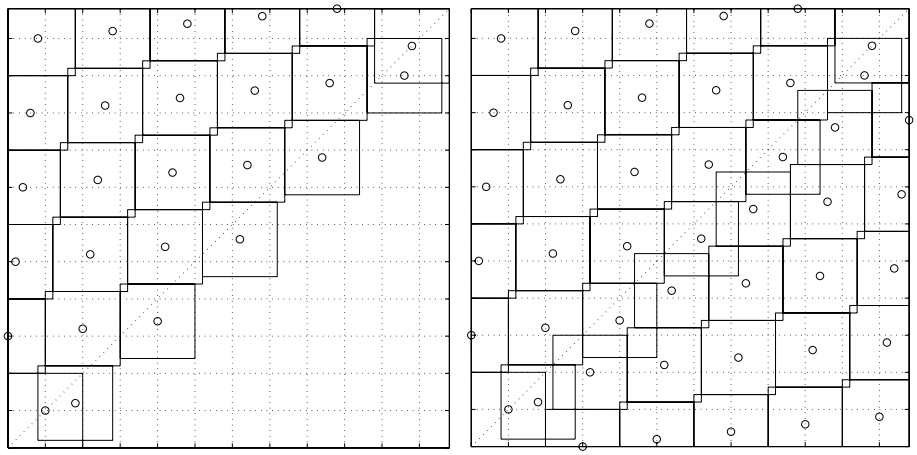

Figure 1: Partial LHDs D_u and D; $\rho = 5$

Construction 1. *Let $\rho \geq 2$ be an integer, and let $n = 2\rho^2 + 2\rho$. Let $D_u = \{(2i\rho + j, (2j+3)\rho + i) | i = 0, \ldots, \rho; j = i-2, \ldots, \rho-1; (i,j) \neq (0,-2), (0,-1), (\rho, \rho-1)\} \cup \{(\rho,\rho), (n-1-\rho, n-1-\rho)\}$, and let $D_l = \{(x,y) | (y,x) \in D_u, x > y\}$. Then $D = D_u \cup D_l$ is a partial Latin hypercube design of $\rho^2 + 4\rho$ points with covering radius ρ for the square $[0, n-1]^2$.*

Proof. For the sake of readability we only give a brief sketch of the proof, skipping the technicalities. The ℓ^∞-circles (squares) with radius ρ centered at the points in D_u cover the upper left half of the square (all points (x,y) with $y \geq x$); see Figure 1. All x-values in D_u are distinct, and so are all y-values. Moreover, one can show that the x-values in D_u are distinct from the y-values in D_u, except for the values ρ and $n-1-\rho$. This implies that by reflecting D_u in the line $y = x$, and omitting the copies of (ρ,ρ) and $(n-1-\rho, n-1-\rho)$, one gets a partial Latin hypercube design covering the entire square. Clearly, one can also remove the reflections of the points $(x,y) \in D_u$ with $x > y$, since these reflections end up in the upper left half, and therefore cover nothing in the right lower half that is not already covered by the points in D_u. We thus obtain the partial Latin hypercube D that covers the entire square with covering radius ρ; see also Figure 1. □

From Construction 1 we now construct Latin hypercube designs of m points with covering radius (integer) ρ for $\rho^2 + 4\rho \leq m \leq n = 2\rho^2 + 2\rho$. This can be done by first extending the partial Latin hypercube design D by $m - \rho^2 - 4\rho$ points having x and y-values that do not yet occur (thus obtaining a partial Latin hypercube design of m points). An example of this first step is given by the Latin hypercube design of 60 points ($m = n$) with covering radius $\rho = 5$ in Figure 2. Note that one can add the points "randomly"; however, one may also assign the points while using a second optimization criterion.

Secondly, we compress the partial Latin hypercube design of m points in the square $[0, n-1]^2$ into a Latin hypercube design of m points, by mapping all m x-values in the partial Latin hypercube design to $\{0, 1, \ldots, m-1\}$ by the (unique) increasing map, and doing the same for the y-values. The result of this second step is illustrated by the Latin hypercube design of 45 points ($m = \rho^2 + 4\rho$) with covering radius $\rho = 5$ in Figure 2. It is clear that both adding points and compressing do not increase the covering radius.

For ρ not integer, but half an integer, we have a similar construction.

Construction 2. *Let $\rho \geq \frac{3}{2}$ be such that $\rho - \frac{1}{2}$ is an integer, and let $n = \rho(2\rho+1) - 1$. Let $D_u = \{(2i\rho + j, (2j+3)\rho + i - \frac{1}{2}) | i = 0, \ldots, \rho - \frac{1}{2}; j = i-2, \ldots, \rho - \frac{3}{2}; (i,j) \neq (0,-2), (0,-1), (\rho - \frac{1}{2}, \rho - \frac{3}{2})\} \cup \{(\rho - \frac{1}{2}, \rho - \frac{1}{2}), (n - \frac{1}{2} - \rho, n - \frac{1}{2} - \rho)\}$, and let $D_l = \{(x,y) | (y,x) \in D_u, x > y\}$.*

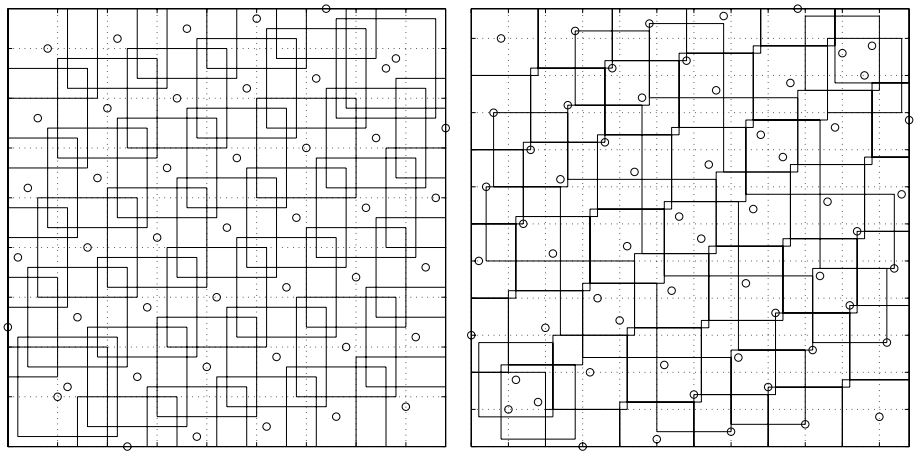

Figure 2: ℓ^∞-Minimax LHDs of 45 and 60 points; $\rho = 5$

Then $D = D_u \cup D_l$ is a partial Latin hypercube design of $(\rho - \frac{1}{2})^2 + 4(\rho - \frac{1}{2})$ points with covering radius ρ for the square $[0, n-1]^2$.

Similar as before, adding points and compressing gives Latin hypercube designs of m points with covering radius ρ for $(\rho - \frac{1}{2})^2 + 4(\rho - \frac{1}{2}) \leq m \leq n = \rho(2\rho + 1) - 1$. Examples are given in Figure 3 for $\rho = 4.5$.

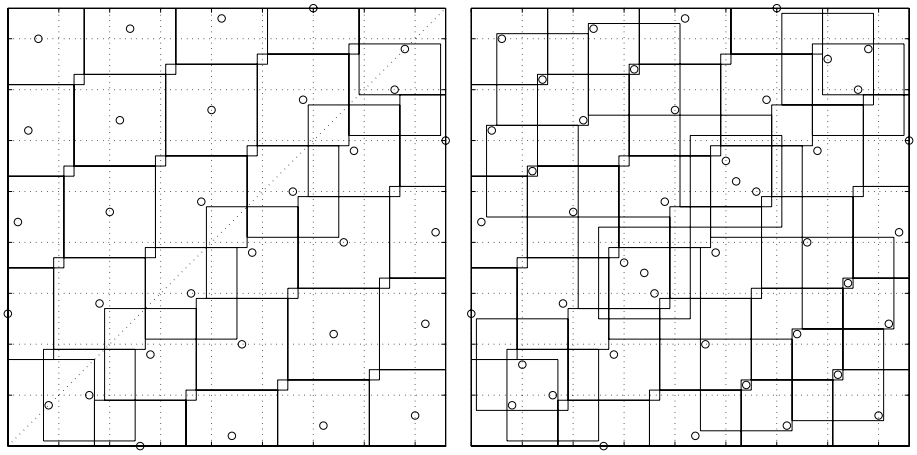

Figure 3: A partial and an ℓ^∞-minimax LHD of 44 points; $\rho = 4.5$

We can now derive that the lower bound of Lemma 1 is attained.

Proposition 1. *Let $n \geq 2$. A minimax Latin hypercube design of n points in two dimensions has covering ℓ^∞-radius* $\min\{\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{2n+1} \rceil, \frac{1}{2} + \lceil -\frac{3}{4} + \frac{1}{4}\sqrt{8n+9} \rceil\}$.

Proof. We have constructed Latin hypercube designs of n points with covering radius integer ρ for $\rho^2 + 4\rho \leq n \leq 2\rho^2 + 2\rho$, and with covering radius half integer ρ for $(\rho - \frac{1}{2})^2 + 4(\rho - \frac{1}{2}) \leq n \leq \rho(2\rho + 1) - 1$. One can show that these constructions thus comprise Latin hypercube designs attaining the lower bound of Lemma 1 for all n except $n = 2, 3, 4$ ($\rho = 1$), $6 \leq n \leq 11$ ($\rho = 2$), $15 \leq n \leq 20$ ($\rho = 3$), and $28 \leq n \leq 31$ ($\rho = 4$). The Latin hypercube designs corresponding to these exceptions can however be obtained by taking the minimax Latin hypercube designs of $\rho^2 + 4\rho$ points and covering radius ρ, and subsequently removing a point and compressing, in

such a way that the covering radius does not increase, and by repeating this until the required number of points is obtained. We claim that this is possible in the required cases if the right points for removal are chosen. □

3 ℓ^1-Minimax Latin hypercube designs

For the ℓ^1-distance the situation is more complicated. A few examples of (unrestricted) designs covering the square with minimal covering radius are given by Johnson et al [3]. The one on 7 points turns out to be a Latin hypercube design. For such Latin hypercube designs we have the following lower bound on the covering radius.

Lemma 2. *Let $n \geq 2$. A Latin hypercube design of n points in two dimensions has covering ℓ^1-radius ρ at least* $\min\{\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{4n-3}\rceil, -\frac{1}{2} + \lceil\sqrt{n}\rceil\}$.

Proof. Consider a Latin hypercube design of n points in two dimensions, as subset of $\{0,\ldots,n-1\}^2$, with covering radius ρ. As in the previous section we remark that the covering radius ρ is either an integer or half an integer. Suppose first that ρ is an integer. Again we consider the left boundary ($x=0$) of the square $[0,n-1]^2$. Here it can only be covered by the ρ design points with x-coordinates $0,1,\ldots,\rho-1$. Such a design point with x-coordinate i can only cover a part of the left boundary of length at most $2(\rho-i)$, which implies that $n-1 \leq \sum_{i=0}^{\rho-1} 2(\rho-i) = \rho(\rho+1)$. Thus if ρ is an integer, ρ is at least $\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{4n-3}\rceil$.

Suppose next that ρ is not an integer, but half an integer. Now the points on the left boundary ($x=0$) of the square can only be covered by the $\rho+\frac{1}{2}$ design points with x-coordinates $0,1,\ldots,\rho-\frac{1}{2}$. Also here the design point with x-coordinate i can only cover a part of the left boundary of length at most $2(\rho-i)$, whereas if it covers a corner point, then it covers at most $2(\rho-i)-\frac{1}{2}$. This implies that $n-1 \leq \sum_{i=0}^{\rho-\frac{1}{2}} 2(\rho-i)-1 = \rho^2+\rho-\frac{3}{4}$, and hence that $n \leq (\rho+\frac{1}{2})^2$. Thus if ρ is half an integer, but not an integer, then $\rho \geq -\frac{1}{2}+\lceil\sqrt{n}\rceil$, which finishes the proof. □

It turns out that this lower bound is not tight for $n = 3, 4, 9$, and 16. It easy to check that the minimax Latin hypercube design of 3 points has covering radius 1.5, while the one of 4 points has covering radius 2. By computer we checked that the ones of 9 and 16 points have covering radius 3 and 4, respectively. We conjecture that for all other values of n the obtained lower bound is attained. We are able to prove this for the values of $n \neq 3$ for which the lower bound on the covering radius is integer. This will follow from the following construction.

Construction 3. *Let $\rho \geq 2$ be an integer, and let $n = \rho^2+\rho+1$. Let $x_{ij} = (\rho+1)i + j$ and $y_{ij} = \rho + (\rho-1)i + (2\rho-1)j$ for any i and j. Let*

$$
\begin{aligned}
D_0 &= \{(x_{ij}, y_{ij}) \mid i = 0,\ldots,\rho; j = \lceil\tfrac{-\rho-(\rho-1)i}{2\rho-1}\rceil,\ldots,\lfloor\tfrac{\rho^2-(\rho-1)i}{2\rho-1}\rfloor\},\\
D_1 &= \{(-x_{ij}, y_{ij}) \mid i = -1; j = \lceil\tfrac{-\rho-(\rho-1)i}{2\rho-1}\rceil+2,\ldots,\lfloor\tfrac{\rho^2-(\rho-1)i}{2\rho-1}\rfloor\},\\
D_2 &= \{(2(n-1)-x_{ij}, y_{ij}) \mid i = \rho+1; j = \lceil\tfrac{-\rho-(\rho-1)i}{2\rho-1}\rceil,\ldots,\lfloor\tfrac{\rho^2-(\rho-1)i}{2\rho-1}\rfloor-2\},\\
D_3 &= \{(x_{ij}, -y_{ij}) \mid i = 3 \leq i \leq \rho; i \text{ odd}; j = \lceil\tfrac{-\rho-(\rho-1)i}{2\rho-1}\rceil-1\},\\
D_4 &= \{(x_{ij}, 2(n-1)-y_{ij}) \mid 0 \leq i \leq \rho-3; \rho-i \text{ odd}; j = \lfloor\tfrac{\rho^2-(\rho-1)i}{2\rho-1}\rfloor+1\}.
\end{aligned}
$$

Then $D = D_0 \cup D_1 \cup D_2 \cup D_3 \cup D_4$ is a partial Latin hypercube design of $\lfloor\frac{1}{2}\rho^2\rfloor + 3\rho - 1$ points with covering radius ρ for the square $[0,n-1]^2$.

Proof. As before, we only sketch the proof, and skip the technical details. Consider the points (x_{ij}, y_{ij}) where (i,j) ranges as in the sets $D_h, h = 0,\ldots,4$. Then the ℓ^1-circles (diamonds) with radius ρ around these points cover the square $[0,n-1]^2$; see the left picture in Figure 4 for the case $\rho = 5$. The points with (i,j) ranging as in D_0 lie in the square, the other points lie outside

the square. After "folding" the plane along the four boundaries of the square, one obtains the partial Latin hypercube design D, and it covers the square with covering radius ρ; see the right picture in Figure 4. Note that for odd ρ, one point (in the upper left corner) from D_1 coincides with a point in D_4, and one point (in the lower right corner) from D_2 coincides with a point in D_3. □

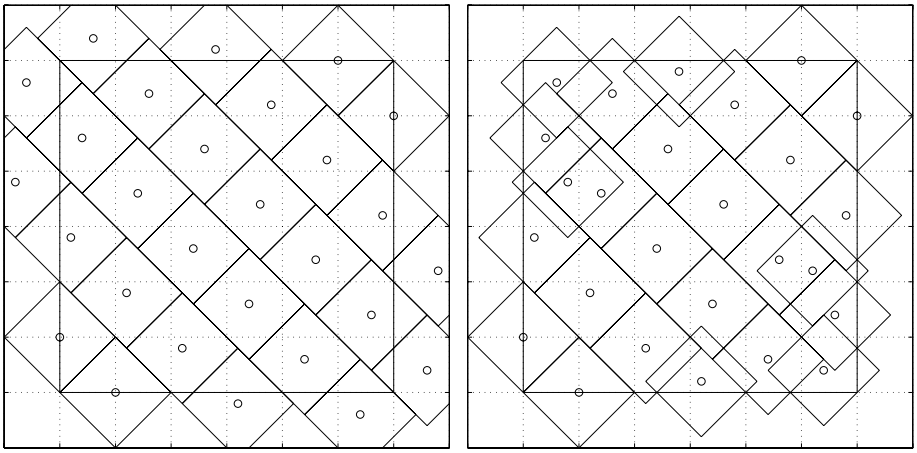

Figure 4: Cover and partial LHD; $\rho = 5$

As in the case of ℓ^∞ we can use Construction 3 to obtain Latin hypercube designs of n points and covering radius ρ with $\lfloor \frac{1}{2}\rho^2 \rfloor + 3\rho - 1 \leq n \leq \rho^2 + \rho + 1$ for ρ integer, by adding points and compressing. In Figure 5 the obtained Latin hypercube designs for extremal n in the case $\rho = 5$ are given.

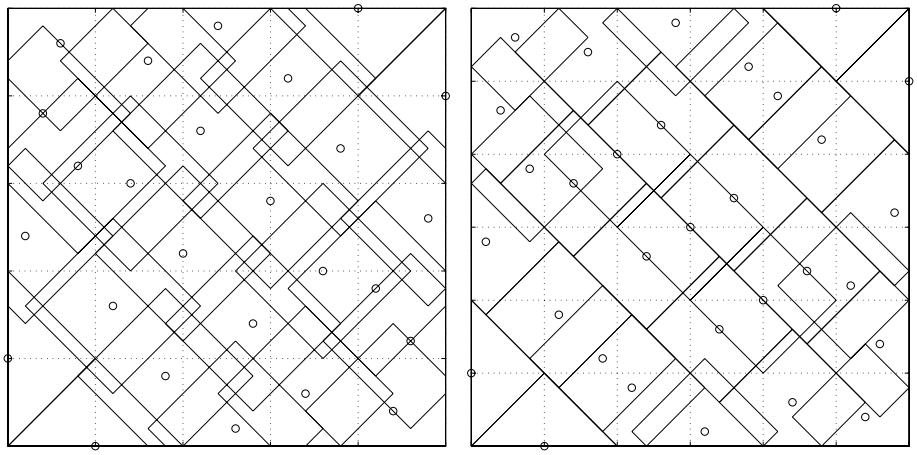

Figure 5: ℓ^1-Minimax LHDs of 26 and 31 points; $\rho = 5$

The above construction settles the problem for integer covering radius. In fact, we have the following upper bound on the covering radius.

Proposition 2. *Let $n \geq 4$. A minimax Latin hypercube design of n points in two dimensions has covering ℓ^1-radius ρ at most $\lceil -\frac{1}{2} + \frac{1}{2}\sqrt{4n-3} \rceil$.*

Proof. We have constructed Latin hypercube designs of n points with covering radius integer ρ for $\lfloor \frac{1}{2}\rho^2 \rfloor + 3\rho - 1 \leq n \leq \rho^2 + \rho + 1$. Thus it follows that this construction gives Latin hypercube

designs attaining the stated upper bound for all n except $n = 4, 5, 6$ ($\rho = 2$), $8 \leq n \leq 11$ ($\rho = 3$), $14 \leq n \leq 18$ ($\rho = 4$), $22 \leq n \leq 25$ ($\rho = 5$), and $32 \leq n \leq 34$ ($\rho = 6$). However, the Latin hypercube designs corresponding to these exceptions are easily constructed. □

Unfortunately we have no general construction for half integer covering radius. In Figure 6 we give Latin hypercube designs for maximal n with covering radii 2.5, 3.5, 4.5, and 5.5, respectively. We were also able to construct a Latin hypercube design of 49 points with covering radius 6.5. This, and Proposition 2 support the conjecture that the lower bound of Lemma 2 is attained for all n, except for $n = 3, 4, 9$, and 16.

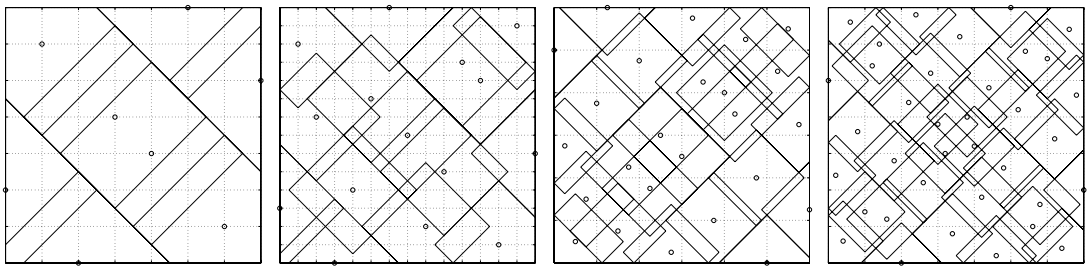

Figure 6: ℓ^1-Minimax LHDs of $n = 8, 15, 25, 36$ points; $\rho = -\frac{1}{2} + \lceil\sqrt{n}\rceil$

4 ℓ^2-Minimax Latin hypercube designs

The situation is even more complicated for the ℓ^2-distance. There seems to be no general pattern for the optimal Latin hypercube designs, as there was in the cases of the ℓ^∞ and ℓ^1-distance. For unrestricted minimax designs (i.e., circle coverings of the square) the situation is similar; cf. [4].

By computer we have been able to determine all minimax Latin hypercube designs with n points for $n \leq 26, n \neq 23$; see Table 1. For $n = 23$ and $n = 27$ we obtained a partial list of minimax Latin hypercube designs. In the table, ρ denotes the minimal covering radius and # the number of non-isomorphic (under the action of the symmetry group of the square) minimax designs of n points; these numbers are split according to the symmetries of the designs. Here D_2 stands for the dihedral group of order 4; designs with this symmetry group are invariant under reflections in the diagonals, and rotation over 180 degrees. Designs that have the cyclic group C_4 as symmetry group are invariant under rotations over 90, 180, and 270 degrees, while designs with symmetry group D_1 are invariant under a reflection in one of the diagonals, and those with symmetry group C_2 are invariant under a rotation over 180 degrees. The remaining designs have no symmetries, and are listed under the trivial group **1**. Note that the full symmetry group D_4 (of order 8) of the square cannot be the symmetry group of a Latin hypercube design.

In our search method we started by enumerating all possibilities for the points near the boundary of the square such that all boundary points are covered - within some distance ρ - by these points. We were careful to check that isomorphic copies (under the action of the symmetry group of the square) were removed on the way. The initial value for (the aimed to be covering radius) ρ for n points was based on the covering radius for $n - 1$ points. If no partial Latin hypercube designs covering the boundary were found then ρ was increased a bit, and the above was repeated. For each obtained partial Latin hypercube design we then added the remaining points one by one, with increasing x-value. After adding the point with smallest missing x-value, say X, it was checked whether (a discrete subset of) the line $x = X+1-\lceil\rho\rceil$ was covered - within distance ρ - by the design points. If not, we backtracked; if so, we added the next point. Once

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15
ρ	1	$\frac{5}{4}$	$\sqrt{2}$	$\frac{5}{3}$	2	2	$\frac{17}{8}$	$\sqrt{5}$	$\sqrt{5}$	$\frac{5}{26}\sqrt{170}$	$\frac{5}{2}$	$\frac{1}{3}\sqrt{65}$	$\frac{29}{10}$	3
$\approx$	1	1.25	1.414	1.667	2	2	2.125	2.236	2.236	2.507	2.5	2.687	2.9	3
#	1	1	1	2	1	1	1	22	1	5	1	1	3	199
D_2	1	0	1	1	0	1	1	0	0	0	0	0	2	4
C_4	0	0	0	1	0	0	0	2	0	0	0	1	0	0
D_1	0	1	0	0	1	0	0	5	0	0	0	0	0	41
C_2	0	0	0	0	0	0	0	1	1	0	1	0	1	4
1	0	0	0	0	0	0	0	14	0	5	0	0	0	150

n	16	17	18	19	20	21	22	23	24	25	26	27
ρ	3	$\frac{37}{12}$	$\sqrt{10}$	$\frac{17}{46}\sqrt{74}$	$\frac{10}{3}$	$\frac{17}{5}$	$\frac{25}{7}$	$\sqrt{13}$	$\sqrt{13}$	$\frac{15}{4}$	$\frac{1}{3}\sqrt{130}$	4
$\approx$	3	3.083	3.162	3.179	3.333	3.4	3.571	3.606	3.606	3.75	3.801	4
#	10	4	404	1	11	8	111	≥ 500	8	325	7	≥ 119025
D_2	4	1	0	0	0	0	0	0	0	0	0	0
C_4	0	0	0	0	2	1	0	0	0	3	0	0
D_1	4	1	0	0	2	0	0	0	0	0	0	1907
C_2	1	1	34	0	2	2	10	9	6	0	1	297
1	1	1	370	1	5	5	101	≥ 491	2	322	6	≥ 116821

Table 1: Minimal ℓ^2-covering radius ρ for Latin hypercube designs of n points

a full Latin hypercube design was obtained, we computed its covering radius by using Voronoi diagrams (cf. [5]). In this way the best designs were determined, say with minimal covering radius ρ'. If this covering radius turned out the be larger than the initial value ρ, the search was repeated after resetting $\rho = \rho'$. If not then ρ' was the minimal covering radius, and all minimax designs had been determined. Finally, we checked on isomorphism of the minimax designs.

Surprisingly the resulting sequence ρ is not monotone. The covering radius for $n = 11$ is $\frac{5}{26}\sqrt{170} \approx 2.507$, which is larger than the covering radius 2.5 for $n = 12$.

Examples for all values of n from 3 up to 27 are given in Figures 7, 8, and 9. In these, asterisks (*) are used to indicate the remote sites, i.e., the points of the square that are at extremal distance ρ from the design. If more than one minimax Latin hypercube design of n points exists, we give an example with largest possible symmetry group.

For $n = 5$ we give the example with symmetry group C_4. The other example

$$\{(0,0),(1,3),(2,2),(3,1),(4,4)\}$$

has symmetry group D_2. For $n = 9$ we give the example with symmetry group C_4 with fewest remote sites (4). The other design with symmetry group C_4

$$\{(0,2),(1,5),(2,8),(3,1),(4,4),(5,7),(6,0),(7,3),(8,6)\}$$

has 8 remote sites. For $n = 11$ we give the "periodic" design. This is however the example with the most (6) remote sites; the design

$$\{(0,2),(1,8),(2,6),(3,4),(4,0),(5,10),(6,7),(7,3),(8,1),(9,9),(10,5)\}$$

has only one remote site. For $n = 23$ a complete search has not yet finished. We have classified all minimax Latin hypercube designs with a non-trivial symmetry group however. For $n = 27$ we determined all minimax designs with a non-trivial symmetry group; we give an example with symmetry group D_1 in Figure 9. A (very) partial search for the minimax designs with a trivial

group indicated that these occur in great numbers; we obtained 116,821 of those after searching only a small part of the entire search space.

Finally, a complete search showed that it is impossible to cover only the left boundary of the square by a partial Latin hypercube design with covering radius 4 for $n = 28$. Thus, for $n > 27$ the minimal covering radius is larger than 4.

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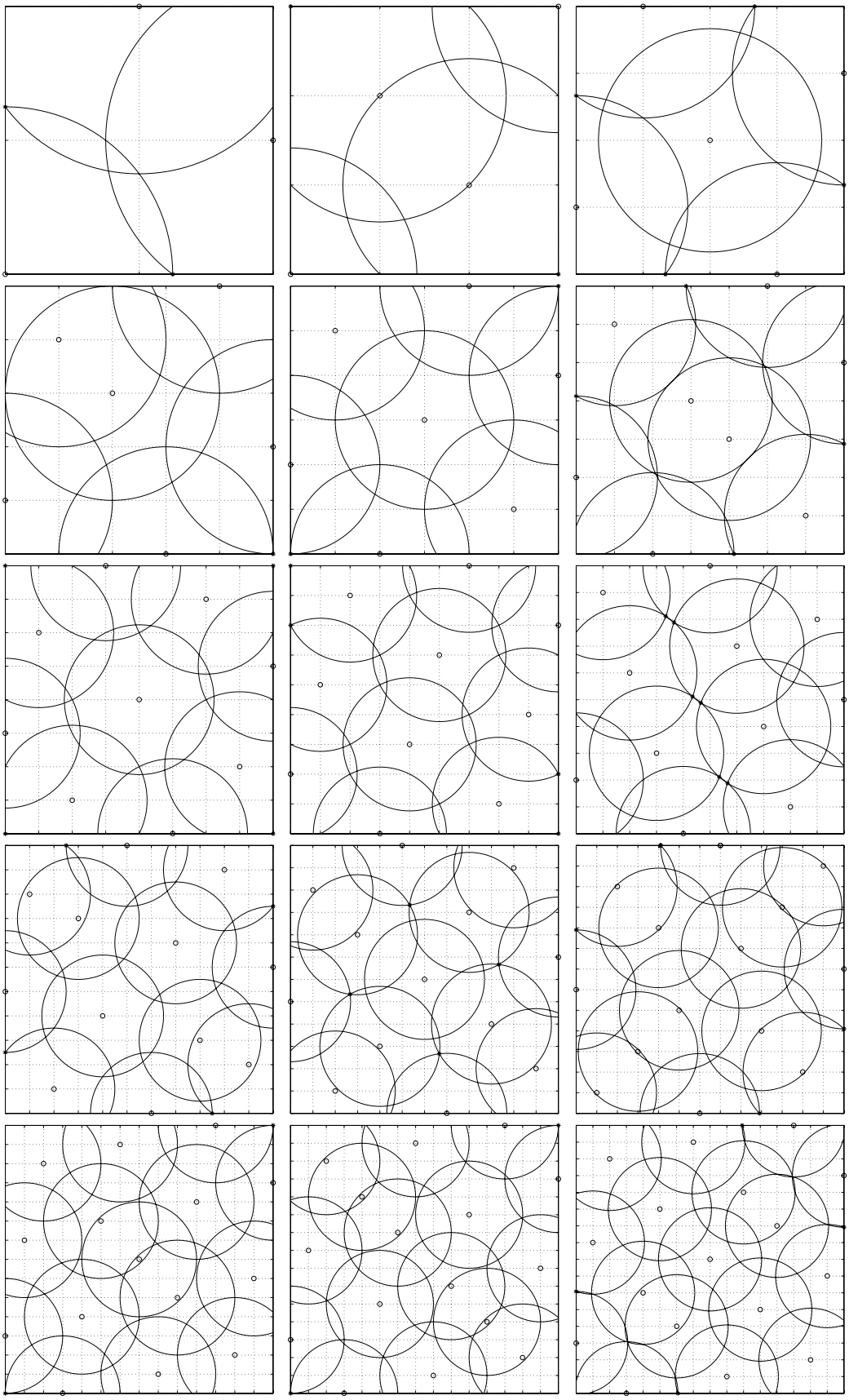

Figure 7: ℓ^2-Minimax LHDs of $3, 4, \ldots, 17$ points

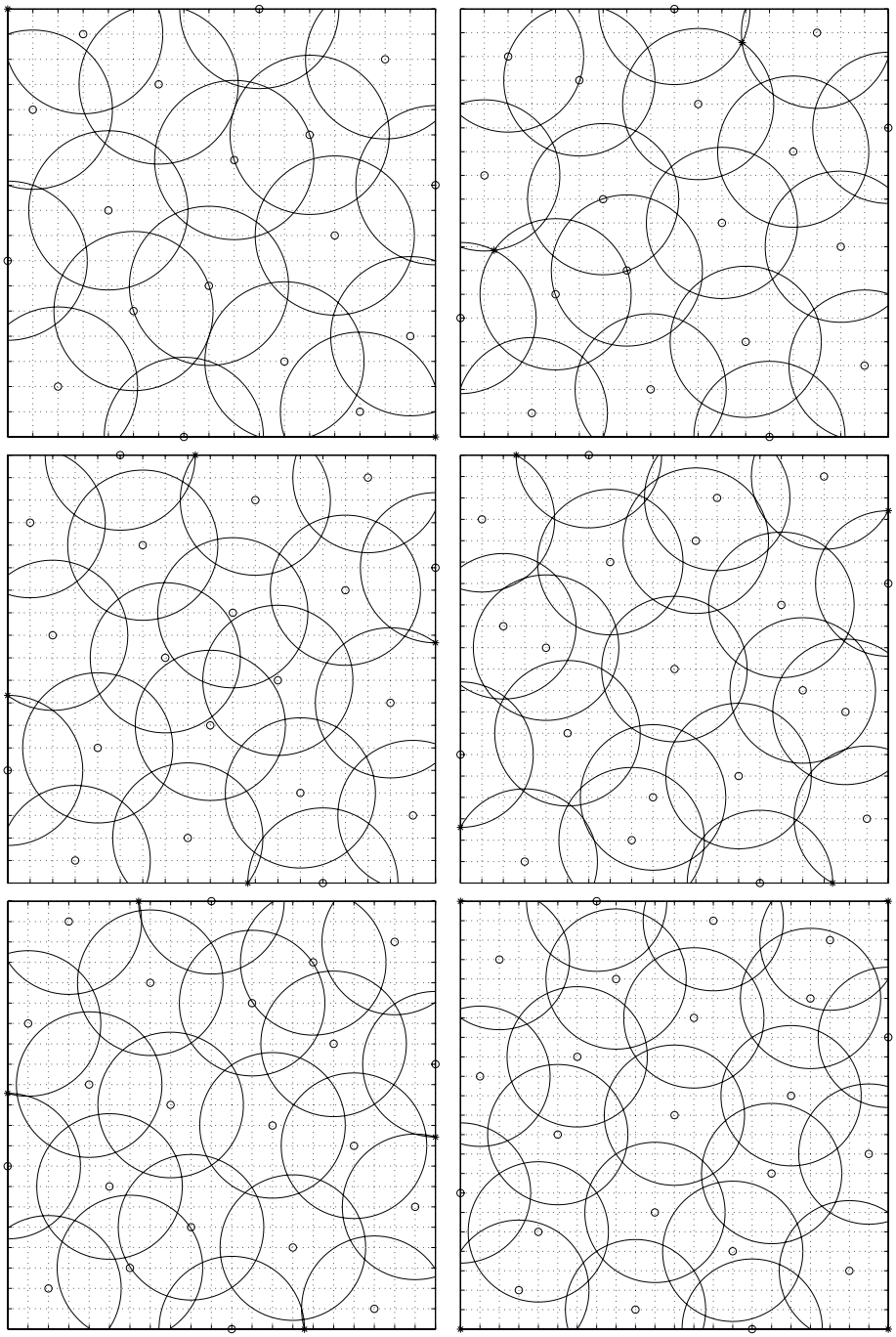

Figure 8: ℓ^2-Minimax LHDs of $18, 19, \ldots, 23$ points

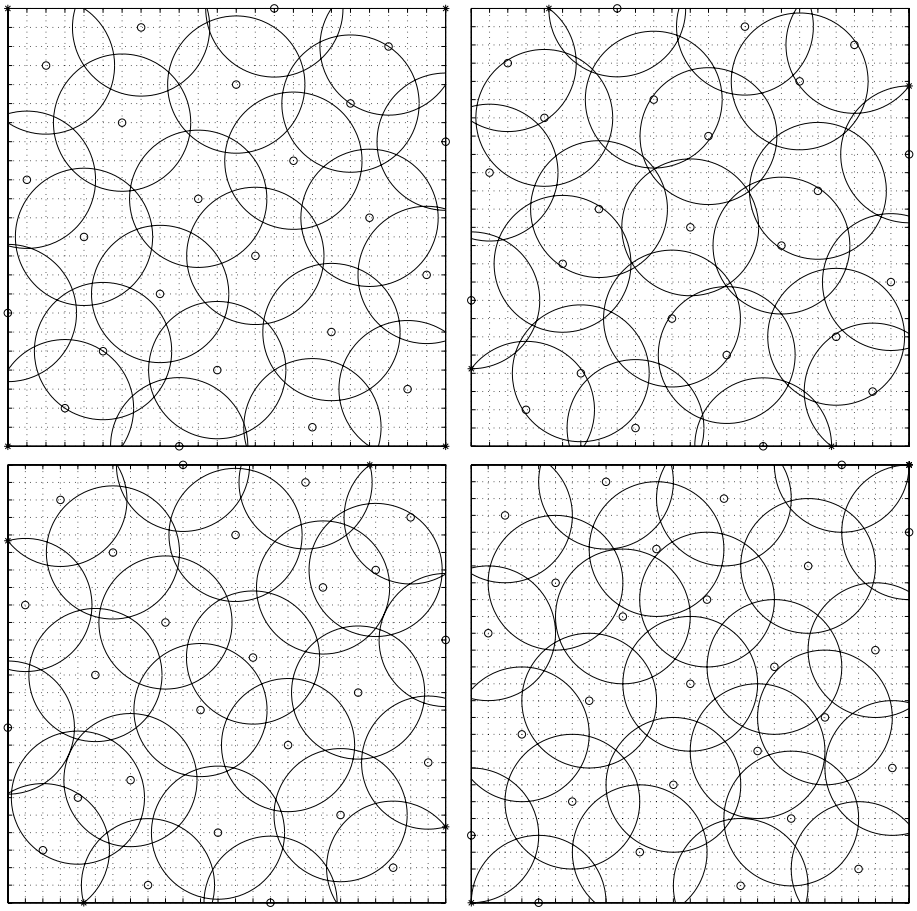

Figure 9: ℓ^2-Minimax LHDs of $24, 25, 26, 27$ points