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**EQUILIBRIUM PRICE DISPERSION IN A MATCHING
MODEL WITH DIVISIBLE MONEY**

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Discussion paper

Equilibrium Price Dispersion in a Matching Model with Divisible Money

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Abstract

The main purpose of this paper is to show that, for any given parameter values, an equilibrium with dispersed prices (two-price equilibrium) exists in a simple matching model with divisible money presented by Green and Zhou (1998). We also show that our two-price equilibrium is unique in certain environments. Moreover, the welfare effect of price dispersion is analysed.

Keywords: Price Dispersion, Matching Model, Divisible Money.

Journal of Economic Literature Classification Number: D51, E40.

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1 Introduction

The main purpose of this paper is to show that, for any given parameter values, an equilibrium with dispersed prices exists in a matching model with divisible money presented by Green and Zhou (1998), referred to below as GZ. Special attentions will be paid to the uniqueness of the equilibrium and the welfare effect of price dispersion.

An influential paper by Kiyotaki and Wright (1989) has analyzed the search model of money in which both goods and money are indivisible and consumers can hold just one unit of them. Subsequently, Shi (1995, 1997), Trejos and Wright (1995), and GZ have extended the model to allow for divisible money and/or goods. In models with divisibility, there potentially exists an equilibrium with dispersed prices; different sellers charge different prices in an equilibrium. Indeed, Camera and Corbae (1999), Soller-Curtis and Wright (2000), and Matsui and Shimizu (2001) have succeeded in presenting models with dispersed price equilibria. In this paper, we show that equilibrium price dispersion occurs in the environment of GZ.

In search models of money, a random matching takes place and a potential buyer and a potential seller meet. Suppose that one of them presents a take-it-or-leave-it offer. If she knows the other's characteristics, such as his money holding, then she would extract all gains from trade. Suppose she offers a price (and quantity of consumption good) depending on the other's characteristics, then equilibrium price dispersion obviously occurs. Camera and Corbae (1999) have indeed presented such a model; the buyer, who knows the amount of the seller's money holding, offers a price (and quantity of consumption good). Of course, in their model, there only exists an equilibrium with dispersed prices.

Soller-Curtis and Wright (2000) have successfully proved the existence of equilibria with dispersed prices without assuming sellers' knowledge on buyers' characteristics. Their model, however, contains exogenous preference shocks, i.e., the buyer's characteristics are determined exogenously and stochastically. In the equilibrium, the seller is indifferent between a low price with a high probability of success in trading and a high price with a low probability of success in trading.

Introducing market places into Green and Zhou's model, Matsui and Shimizu (2001) have proved the existence of equilibria with dispersed prices without assuming exogenous shocks or sellers' knowledge on buyers' characteristics. However, the market places play a crucial role for the existence of a two-price equilibrium. In this paper, we will prove the existence of a two-price equilibrium in the original model.

GZ have shown that there is a continuum of single price equilibria in their model. Our existence result has a remarkable contrast to theirs. That is, confining our attention to geometric distributions of money holdings, it is shown that there is the unique p - $2p$ equilibrium in which the sellers offer prices p or $2p$.

As for the welfare property of our equilibrium, it also has a remarkable contrast to that of single price equilibria. Even if the proportion of the agents with positive money holdings is arbitrarily small, there exists a single price equilibrium. Thus the probability of success in trading can be arbitrarily small, and therefore the arbitrarily small welfare will be. While in our two-price equilibrium, the proportion of the agents with positive money holdings is uniquely determined for given parameter values and it is a certain positive value larger than $1/2$. Thus the probability of success in trading is a certain positive value.

The plan of this paper is as follows. Section 2 presents GZ's model and introduce new notations in order to investigate equilibrium price dispersion. Section 3 is devoted to the definition of our equilibrium concept and to the proof of its existence. Section 4 presents an environment in which our equilibrium is unique. Section 5 analyzes the welfare effect of price dispersion.

2 Green and Zhou's Model

In this section, we present GZ's model.

In the economy, there are infinitely lived agents with a nonatomic mass of measure one. There are $k \geq 3$ types of agents and each type i , $i = 1, \dots, k$, has equally $1/k$ mass. There are $k + 1$ goods. The first k goods are indivisible and immediately perishable, and good i is consumed by type i agents. The remaining good is a perfectly divisible and perfectly durable fiat-money object with an exogenously given total nominal stock $M > 0$. Agents can hold any amount of fiat money. A type i agent can costlessly produce one unit of good $i + 1$ at any time for $i = 1, \dots, k - 1$. (An agent of type k produces good 1.) She consumes only good i and derives instantaneous utility $u > 0$. All agents have common discount rate $\gamma > 0$ and maximize their discounted expected utility of the stream of their consumption.

Time is continuous starting from period 0. Agents meet pair wise randomly according to a Poisson process with parameter $\mu > 0$. Since the consumption goods are perishable and there is no double coincidence of wants, all trade should involve fiat money as a medium of exchange. Thus consumption goods cannot be used

as commodity money. Each agent is characterized by her type and the amount of money she holds. We assume that a partner's type is observable, but not her money holding, and that an agent knows the distribution of money holdings of the economy. Transactions occur according to a seller-posting-price protocol as follows. When a type i agent who has fiat money (potential buyer) meets a type $i - 1$ agent (potential seller) who can produce the buyer's desired consumption good, the seller posts an offer first, then the buyer decides to accept or reject it. Transaction occurs if and only if the offer is accepted and the buyer pays the offered price.

We will focus on stationary equilibrium where the strategy that agents with an identical money holding and an identical type take is time-invariant. Therefore, we will hereafter discuss a generic type i .

Let $\eta \in \mathbb{R}_+$ denote an agent's money holding. A strategy of type i agent is defined as a set of two correspondences, an offer strategy $\omega(\eta) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and a reservation price strategy $\rho(\eta) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. The former is a set of prices that a type i agent with money holding η offers when she meets a potential buyer. A seller with money holding η offers one of the elements in $\omega(\eta)$. It will be shown that, by the perfectness condition, ρ gives the maximum price that a buyer is willing to defray for the consumption good, and so it becomes a function rather than a correspondence. Of course, since the reservation price cannot exceed the buyer's money holdings, ρ should satisfy the following feasibility condition:

$$(1) \quad \rho(\eta) \leq \eta.$$

GZ only consider stationary distributions of money holdings. However, we consider stationary distributions of offer prices and reservation prices as well as money holdings, since we allow agents with the same money holding for taking different strategies. That is, for a money holding η , an offer price o , and a reservation price r , $H(\eta, o, r)$ denotes a stationary distribution defined on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$. From H , the stationary distribution of offer prices, Ω , and the stationary distribution of reservation prices, R , are defined as follows.

$$(2) \quad \Omega(x) = H\{(\eta, o, r) | o \leq x\}$$

$$(3) \quad R(x) = H\{(\eta, o, r) | r < x\}.$$

We define R to be continuous from the left.

Let $\mathcal{V} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a value function. That is $\mathcal{V}(\eta)$ is the maximum value of discounted utility achievable by the agent's current money holding η . At every

moment, a type i agent with money holding η meets a type $i-1$ agent with probability μ/k . Transaction does not occur and money holding does not change if the partner's offer x exceeds the type i 's reservation price r . If partner's offer price x is not more than reservation price r , then transaction occurs and the type i agent derives utility u from consumption and enters in the next trading opportunity with money holding $\eta-x$. The probability that type i with money holding η meets a type $i+1$ agent is also μ/k . Transaction does not occur if the type i 's offer o is greater than the partner's reservation price. If type i 's offer o does not exceed the partner's reservation price, then transaction occurs and faces the next matching opportunity with money holding $\eta+o$. Then, using γ , μ , Ω , and R , the Bellman equation for $\mathcal{V}(\eta)$ is given by ¹

$$(4) \quad \mathcal{V}(\eta) = \frac{\mu}{k\gamma + 2\mu} \left(\max_{r \in [0, \eta]} \left\{ \int_0^r (u + \mathcal{V}(\eta - x)) d\Omega(x) + (1 - \Omega(r)) \mathcal{V}(\eta) \right\} + \max_{o \in \mathbb{R}_+} \left\{ R(o) \mathcal{V}(\eta) + (1 - R(o)) \mathcal{V}(\eta + o) \right\} \right).$$

Some remarks on $\mathcal{V}(\eta)$ as follows. $\mathcal{V}(\eta)$ is nonnegative, since an agent can always choose $r = 0$, i.e., she can always refrain from purchase. $\mathcal{V}(\eta)$ is bounded above, since consumption opportunities occur with $1/\mu$ intervals on average and the utility should be discounted.

In terms of $\mathcal{V}(\eta)$, it is optimal to accept offer o if $u + \mathcal{V}(\eta - o) \geq \mathcal{V}(\eta)$. The same condition in terms of reservation price ρ is $\rho(\eta) \geq o$. Then the perfectness condition with respect to reservation price is as follows:

$$(5) \quad \rho(\eta) = \max\{r \in [0, \eta] \mid u + \mathcal{V}(\eta - r) \geq \mathcal{V}(\eta)\}.$$

That is, type i 's reservation price is her full value for good $i+1$, and thus it is a function of η .

The economy is stationary if H is an initial stationary distribution of the process induced by the optimal trading strategy (ω, ρ) . Now we define the stationary equilibrium grounded on the above. We adopt stationary perfect Bayesian Nash equilibrium as our equilibrium concept.

DEFINITION 1 $\langle H, R, \Omega, \omega, \rho, \mathcal{V} \rangle$ is said to be a stationary equilibrium if

1. H is stationary under trading strategies ω and ρ , and the distribution of offer prices Ω and that of reservation prices R are derived from H by (2) and (3),
and

¹As for the details of derivation, see GZ.

2. given the distributions H , R and Ω , the reservation price strategy ρ and the offer strategy ω satisfy the feasibility condition (1) and the perfectness condition (5), respectively, and the value function \mathcal{V} , together with ρ and ω , solves the Bellman equation (4). Therefore,

$$(6) \quad \mathcal{V}(\eta) = \frac{1}{\phi + 2} \left(\left(\int_0^{\rho(\eta)} (u + \mathcal{V}(\eta - x)) d\Omega(x) + (1 - \Omega(\rho(\eta))) \mathcal{V}(\eta) \right) + \left(R(\omega(\eta)) \mathcal{V}(\eta) + (1 - R(\omega(\eta))) (\mathcal{V}(\eta + \omega(\eta))) \right) \right),$$

holds, where $\phi = k\gamma/\mu$.

3 Two-Price Equilibrium

3.1 The Equilibrium

In what follows, we focus on a stationary distribution H such that its support is the set $\{(np, \bar{n}p, \bar{n}p) | n, \bar{n}, \bar{n} = 0, 1, 2, \dots\}$ for some $p > 0$. Thus H can be expressed by $\bar{h}(n, s, t)$, the measure of the set of agents with a money holding np , an offer price sp , and a reservation price tp . Of course, \bar{h} satisfies

$$(7) \quad \sum_n \sum_s \sum_t \bar{h}(n, s, t) = 1$$

$$(8) \quad \bar{h}(n, s, t) \geq 0$$

$$\bar{h}(n, s, t) > 0 \quad \text{only if}$$

$$(9) \quad sp \in \omega(np) \text{ and } \rho(np) = tp.$$

To begin with, we define the concept of a p -2 p equilibrium (two price equilibrium) of which existence we are going to show.

DEFINITION 2 $\langle H, R, \Omega, \omega, \rho, \mathcal{V} \rangle$ is said to be a p -2 p equilibrium if, for some $p > 0$,

1. $\langle H, R, \Omega, \omega, \rho, \mathcal{V} \rangle$ satisfies 1 and 2 of Definition 1, $\omega(np) \subset \{p, 2p\}$, $n = 0, 1, \dots$, and

2. \bar{h} satisfies

$$(10) \quad \exists n, t \in \{0, 1, 2, \dots\}, \quad \bar{h}(n, 1, t) > 0$$

$$(11) \quad \exists n', t' \in \{0, 1, 2, \dots\}, \quad \bar{h}(n', 2, t') > 0$$

$$(12) \quad \exists n'', s \in \{0, 1, 2, \dots\}, q \in \{2, 3, 4, \dots\}, \quad \bar{h}(n'', s, q) > 0.$$

From (10)–(12),

$$(13) \quad \exists n \in \{0, 1, \dots\}, \quad p \in \omega(np)$$

$$(14) \quad \exists n' \in \{0, 1, \dots\}, \quad 2p \in \omega(n'p)$$

$$(15) \quad \exists n'' \in \{0, 1, \dots\}, \quad \rho(n''p) \geq 2p$$

immediately follows. Thus (10)–(12) imply that there exist transactions both with price p and with price $2p$.

We are now ready to present our main theorem.

THEOREM 1 *For all $\phi > 0$, there exists a p - $2p$ equilibrium for some $p > 0$.*

First, in the next subsection, we specify a strategy which is shown to be a p - $2p$ equilibrium strategy. Then, in the following subsections, we find a stationary distribution and a value function consistent with the strategy, and show that they indeed constitute a p - $2p$ equilibrium.

3.2 The Strategy

We consider the strategy satisfying (i) and (ii) below. (i) For all money holdings $\eta = np, n = 0, 1, 2, \dots$, agents offer either p or $2p$. (ii) Agents with money holding p accept only offer p , and agents with money holding larger than or equal to $2p$ accept p and $2p$. That is

$$(16) \quad \omega(np) = \{p, 2p\}, \quad n = 0, 1, \dots,$$

$$(17) \quad \rho(0) = 0,$$

$$(18) \quad \rho(p) = p,$$

$$(19) \quad \rho(np) \geq 2p, \quad n \geq 2$$

hold. Note that we now just suppose that $\rho(np), n \geq 3$, is larger than or equal to $2p$, and it will be completely specified later.

The steps of the proof of Theorem 1 are as follows. First, in Subsection 3.3, we find a stationary distribution consistent with the above strategy. Then, in Subsection 3.4, using the fact that the values of the offer prices p and $2p$ are the same, we derive a simple relationship between $\mathcal{V}(np), n = 0, 1, \dots$. It is expressed as a homogeneous second order difference equation, and thus $\mathcal{V}(n), n = 2, 3, \dots$, can be expressed by $\mathcal{V}(0)$ and $\mathcal{V}(p)$. Then, in Subsection 3.5, we find values of unknowns, including $\mathcal{V}(0)$ and $\mathcal{V}(p)$, which solve the Bellman equation. Finally, in Subsection 3.6, we show that the agents have incentive to play the strategy.

3.3 A Stationary Distribution

In this subsection, we investigate a stationary distribution consistent with the strategy (16)–(19).

First, it is convenient to sum up the measures of the set of agents with reservation prices larger than or equal to $2p$, since no offer prices are larger than $2p$. Thus we define

$$(20) \quad h(n, s, t) = \begin{cases} \sum_q \bar{h}(n, s, q) & \text{if } n \geq 2 \text{ and } t = 2 \\ 0 & \text{if } n \geq 2 \text{ and } t > 2 \\ \bar{h}(n, s, t) & \text{otherwise.} \end{cases}$$

In the above definition, $h(n, s, 2)$ is equal to the measure of the set of agents with reservation prices larger than or equal to $2p$. We denote the measure of the agents with money holding np by

$$(21) \quad h(n) = \sum_s \sum_t h(n, s, t).$$

Next, we introduce some notations. Let m_1 be the proportion of the agents with positive money holdings, which corresponds to m in GZ, and m_2 be the proportion of the agents with $\rho(np) \geq 2$. Let z_1 be the proportion of the agents with offer price p and z_2 be that with offer price $2p$. Of course, $z_1 + z_2 = 1$ holds. Each of the above can be expressed by $h(\cdot, \cdot, \cdot)$: $m_1 = \sum_{n=1}^{\infty} \sum_s \sum_t h(n, s, t)$, $m_2 = \sum_{n=0}^{\infty} \sum_s h(n, s, 2) = \sum_{n=2}^{\infty} h(n)$, $z_1 = \sum_{n=0}^{\infty} \sum_t h(n, 1, t)$, and $z_2 = \sum_{n=0}^{\infty} \sum_t h(n, 2, t)$.

When all agents play the strategy (16)–(19), the time derivative of $h(n)$ is written as

$$(22) \quad \dot{h}(0) = \mu(z_1 h(1) + (1 - z_1)h(2)) - \mu m_1 h(0, 1, 0) - \mu m_2 h(0, 2, 0)$$

$$\dot{h}(1) = \mu(m_1 h(0, 1, 0) + z_1 h(2) + (1 - z_1)h(3))$$

$$(23) \quad -\mu(m_1 + z_1)h(1, 1, 1) - \mu(m_2 + z_1)h(1, 2, 1)$$

$$\dot{h}(n) = \mu(m_2 h(n - 2, 2, t) + m_1 h(n - 1, 1, t) + z_1 h(n + 1) + (1 - z_1)h(n + 2))$$

$$(24) \quad -\mu(m_1 + 1)h(n, 1, 2) - \mu(m_2 + 1)h(n, 2, 2) \quad \forall n \geq 2.$$

For example, in (24), $\dot{h}(n)$ is the difference between the measure of agents whose money holdings become np as results of trades and that of agents whose money holdings change from np to other ones as results of trades at any moment. There are four cases that agents' money holdings become np : an agent with $(n - 2)p$ sells her product at $2p$, an agent with $(n - 1)p$ sells at p , an agent with $(n + 1)p$ purchases her consumption good at p , and an agent with $(n + 2)p$ purchases at $2p$. The

probabilities of these cases, when she meets a potential trader, are m_2 , m_1 , z_1 , and $1 - z_1$, respectively. On the other hand, there are three cases that the money holdings change from np to other ones: an agent with offer strategy p sells her product at p , an agent with offer strategy $2p$ sells at $2p$, and she purchases her consumption good. The probability of these cases when she meets a potential trader are m_1 , m_2 , and 1 , respectively. Similar arguments apply to (22) and (23).

The stationarity of distribution H requires that h is invariant, i.e., (22)–(24) are equal to zero. Thus, for the strategy (16) – (19), H is stationary if and only if

$$(25) \quad z_1 h(1) + (1 - z_1) h(2) = m_1 h(0, 1, 0) + m_2 h(0, 2, 0),$$

$$(26) \quad m_1 h(0, 1, 0) + z_1 h(2) + (1 - z_1) h(3) = (m_1 + z_1) h(1, 1, 1) + (m_2 + z_1) h(1, 2, 1),$$

for $n \geq 2$,

$$(27) \quad \begin{aligned} m_2 h(n - 2, 2, t) + m_1 h(n - 1, 1, t) + z_1 h(n + 1) + (1 - z_1) h(n + 2) \\ = (m_1 + 1) h(n, 1, 2) + (m_2 + 1) h(n, 2, 2), \end{aligned}$$

and, for m_1 and m_2 ,

$$(28) \quad h(0) = 1 - m_1$$

$$(29) \quad h(1) = m_1 - m_2$$

$$(30) \quad \sum_{n=2}^{\infty} h(n) = m_2.$$

Note that among (28)–(30), two of them are independently determined.

The following lemma gives a stationary distribution.

LEMMA 1 *The distribution*

$$(31) \quad h(n) = m_1^n (1 - m_1)$$

$$(32) \quad h(n, 1, t) = z_1 m_1^n (1 - m_1)$$

$$(33) \quad h(n, 2, t) = (1 - z_1) m_1^n (1 - m_1),$$

for $n = 0, 1, 2, \dots$, and $m_2 = m_1^2$ satisfy (25) – (30).²

PROOF : Substituting (31) – (33) and $m_2 = m_1^2$ into (25) – (30), we can easily verify the stationarity of the distribution.

²There may exist another distribution satisfying (25) – (30).

Q.E.D.

Under this stationary distribution, the proportion of the agents who offer p is exactly the same at all money holdings.

In fact, the stationary distribution of money holdings, $h(n)$, is the same as that in a single-price equilibrium in GZ. Therefore,

$$(34) \quad M = p \sum_{n=1}^{\infty} nh(n) = \frac{m_1}{1 - m_1} p$$

holds as in GZ. Later, we will show that m_1 is uniquely determined in the equilibrium. Thus, for a given M , p is uniquely determined.

3.4 Equilibrium Value Function

First, for convenience, we denote by $V(n) = \mathcal{V}(np)$, the value function of an agent with money holding np . We restrict our attention to the case that money holdings are integer multiples of p . Thus the value function becomes a step function. Let $[x]$ denotes the integer part of a real number x ,³ the value function $\mathcal{V}(\eta)$ can be rewritten as

$$(35) \quad \mathcal{V}(\eta) = V([\eta/p]) = V(n), \quad n = 0, 1, 2, \dots$$

Next, we investigate a necessary condition for the existence of a p - $2p$ equilibrium. For the strategy (16)–(19), let the expected value of a money holding np , an offer price sp , and a reservation price tp be $W(n, s, t)$. On the equilibrium path, if it exists, $W(n, s, t)$ can be written as follows:

for $n = 0$,

$$(36\text{-a}) \quad W(0, 1, 0) = \frac{1}{\phi + 2} (V(0) + (1 - m_1)V(0) + m_1V(1))$$

$$(36\text{-b}) \quad W(0, 2, 0) = \frac{1}{\phi + 2} (V(0) + (1 - m_2)V(0) + m_2V(2)),$$

for $n = 1$,

$$(37\text{-a}) \quad W(1, 1, 1) = \frac{1}{\phi + 2} (z_1(u + V(0)) + ((1 - m_1) + z_2)V(1) + m_1V(2))$$

$$(37\text{-b}) \quad W(1, 2, 1) = \frac{1}{\phi + 2} (z_1(u + V(0)) + ((1 - m_2) + z_2)V(1) + m_2V(3)),$$

³Consequently, $x = [x] + \varepsilon$ for some $\varepsilon \in [0, 1)$

for $n \geq 2$ and $t \geq 2$,

$$(38\text{-}(a)) \quad \begin{aligned} W(n, 1, t) &= \frac{1}{\phi + 2} (z_2(u + V(n - 2)) + z_1(u + V(n - 1))) \\ &\quad + (1 - m_1)V(n) + m_1V(n + 1) \end{aligned}$$

$$(38\text{-}(b)) \quad \begin{aligned} W(n, 2, t) &= \frac{1}{\phi + 2} (z_2(u + V(n - 2)) + z_1(u + V(n - 1))) \\ &\quad + (1 - m_2)V(n) + m_2V(n + 2). \end{aligned}$$

In case of (38-(b)), the strategy of an agent is to offer $2p$ and to accept offer prices p and $2p$. Suppose she meets a partner. If the partner is a seller, he offers p with probability z_1 and $2p$ with probability z_2 . She accepts both offer and the transaction results in a purchase, and then obtains utility u . Her money holding becomes $(n - 1)p$ if the offer price is p and $(n - 2)p$ if it is $2p$. If the partner is a buyer, she offers $2p$ and so the transaction results in a sale with probability m_2 and in no trade with probability $1 - m_2$. Her money holding becomes $(n + 2)p$ in the former case and does not change in the latter case. Similar arguments apply to (36-(a))–(38-(a)).

On the equilibrium path, (38-(a)) and (38-(b)) should be equal,⁴ and the same arguments apply to (37-(a)) and (37-(b)), and to (36-(a)) and (36-(b)). From these equalities, we obtain the same relation

$$V(n + 2) = \frac{m_1}{m_2}V(n + 1) - \frac{m_1 - m_2}{m_2}V(n), \quad n = 0, 1, 2, \dots$$

This is equivalent to the following homogeneous second-order difference equation:

$$m_2V(n + 2) - m_1V(n + 1) + (m_1 - m_2)V(n) = 0.$$

The solution to the difference equation is

$$(39) \quad V(n) = \frac{m_2}{2m_2 - m_1} \left[\left(1 - \left(\frac{m_1 - m_2}{m_2} \right)^n \right) V(1) - \left(\frac{m_1 - m_2}{m_2} - \left(\frac{m_1 - m_2}{m_2} \right)^n \right) V(0) \right].$$

Nontrivial solutions $\{V(n) \mid n = 0, 1, \dots\}$ are bounded if and only if $m_1 < 2m_2$.

The following lemma summarizes the above results.

LEMMA 2 *If the value function of a p -2 p equilibrium with the strategy (16) – (19) exists, the following conditions hold.*

1. $m_1 < 2m_2$.

⁴Of course, they are equal to $V(n)$.

2. Given $V(0)$ and $V(1)$, $V(n)$ is given by (39).

3. For all n , $W(n, 1, t) = W(n, 2, t) = V(n)$.

The relation $m_1^2 = m_2$ in Lemma 1, together with the condition $m_1 < 2m_2$ above, implies $m_1 > 1/2$.

LEMMA 3 *If the value function of a p -2 p equilibrium with the strategy (16) – (19) exists, $m_1 > 1/2$ holds.*

3.5 Existence of Equilibrium Value Function

In the previous two subsections, we presented necessary conditions for the existence of a p -2 p equilibrium with the strategy (16)–(19). The conditions are expressed by $V(0)$, $V(1)$, m_1 , and z_1 . We will show, in this subsection, that there exist such $V(0)$, $V(1)$, m_1 , and z_1 , and, in the next subsection, that they indeed constitute a p -2 p equilibrium even if we take off-equilibrium-paths into consideration. For a while, we suppose that m_1 and m_2 satisfy $m_1 < 2m_2$, $m_1^2 = m_2$ and thus $m_1 > 1/2$, and later we will show these inequalities are indeed satisfied.

Lemma 2-2 says that $V(n)$, the value of state n , is given by (39). Substituting $m_2 = m_1^2$, (39) can be rewritten as

$$(40) \quad V(n) = \frac{m_1}{2m_1 - 1} \left[\left(V(1) - \left(\frac{1 - m_1}{m_1} \right) V(0) \right) - \left(\frac{1 - m_1}{m_1} \right)^n (V(1) - V(0)) \right].$$

Lemma 2-3 says that $W(n, 1, t)$ (= $W(n, 2, t)$) equals $V(n)$ for all n . Below, we rewrite them using (40).

We first consider $V(n) = W(n, 1, 2)$ for $n \geq 2$. Substituting (40) into the both sides of $V(n) = W(n, 1, 2)$, we obtain, for all $n \geq 2$,

$$(41) \quad \begin{aligned} & \frac{1}{\phi + 2} \left\{ u + \frac{m_1}{2m_1 - 1} \left[2 \left(V(1) - \left(\frac{1 - m_1}{m_1} \right) V(0) \right) \right. \right. \\ & \quad \left. \left. - \left(\left(\frac{m_1}{1 - m_1} \right)^2 (1 - z_1) + \left(\frac{m_1}{1 - m_1} \right) z_1 + 2 - 2m_1 \right) \left(\frac{1 - m_1}{m_1} \right)^n (V(1) - V(0)) \right] \right\} \\ & = \frac{m_1}{2m_1 - 1} \left[\left(V(1) - \left(\frac{1 - m_1}{m_1} \right) V(0) \right) - \left(\frac{1 - m_1}{m_1} \right)^n (V(1) - V(0)) \right]. \end{aligned}$$

In order to show that (41) holds for all $n \geq 2$, it suffices to show

$$(42) \quad \frac{1}{\phi + 2} \left\{ u + \frac{m_1}{2m_1 - 1} \left[2 \left(V(1) - \left(\frac{1 - m_1}{m_1} \right) V(0) \right) \right] \right\} = \frac{m_1}{2m_1 - 1} \left(V(1) - \left(\frac{1 - m_1}{m_1} \right) V(0) \right)$$

$$(43) \quad \frac{1}{\phi + 2} \left(\left(\frac{m_1}{1 - m_1} \right)^2 (1 - z_1) + \left(\frac{m_1}{1 - m_1} \right) z_1 + 2 - 2m_1 \right) = 1.$$

From (42),

$$(44) \quad V(0) = \frac{m_1}{1 - m_1} V(1) - \frac{2m_1 - 1}{\phi(1 - m_1)} u,$$

holds and, from (43)

$$(45) \quad z_1 = 2 - m_1 - \frac{\phi(1 - m_1)^2}{m_1(2m_1 - 1)}$$

holds.

We next consider $V(0) = W(0, 1, 0)$. This implies

$$(46) \quad V(0) = \frac{m_1}{\phi + m_1} V(1).$$

Then, by (44), $V(0)$ and $V(1)$ can be expressed as:

$$(47) \quad V(0) = \frac{1}{\phi} \frac{2m_1 - 1}{\phi + 2m_1 - 1} u$$

$$(48) \quad V(1) = \frac{\phi + m_1}{m_1} \frac{1}{\phi} \frac{2m_1 - 1}{\phi + 2m_1 - 1} u.$$

Finally, we consider $V(1) = W(1, 1, 1)$. Together with (40), (45), (46), and (47), this implies

$$\frac{1}{\phi + 2} \left(\frac{2m_1 - 1}{m_1} \left(\frac{1}{\phi + 2m_1 - 1} z_1 + 1 \right) \right) u = \frac{1}{\phi + 2} z_1 u,$$

and then

$$(49) \quad z_1 = \frac{(\phi + 2m_1 - 1)(2m_1 - 1)}{2m_1^2 + (\phi - 3)m_1 + 1}$$

holds.

So far, we have shown the relations between $V(0)$, $V(1)$, m_1 and parameters ϕ and u ((47) and (48)), and the relations between m_1 , z_1 and a parameter ϕ ((45) and (49)). The remaining conditions we should show are (a) there exist $V(0)$, $V(1)$, m_1 , and z_1 satisfying these relations, (b) $1/2 < m_1 < 1$, (c) $0 < z_1 < 1$, and (d) the

incentive to play the strategy. In this subsection, we are going to show that (a),(b), and (c) are satisfied for all $\phi > 0$. (d) will be shown in the next subsection.

First, eliminating z_1 from (45) and (49), we obtain the following fifth-degree polynomial of m_1 :

$$4m_1^5 + (4\phi - 8)m_1^4 + (\phi^2 - 8\phi + 9)m_1^3 + (-2\phi^2 + 7\phi - 5)m_1^2 + (\phi^2 - 4\phi + 1)m_1 + \phi = 0.$$

The left-hand side of this equation is expressed as the product of two polynomials of m_1 , i.e.,

$$(50) \quad 2m_1^2 + (\phi - 1)m_1 - \phi = 0$$

or

$$(51) \quad 2m_1^3 + (\phi - 3)m_1^2 + (3 - \phi)m_1 - 1 = 0.$$

holds.

However, by a simple calculation, the solution to (50) always gives $z_1 = 1$. Thus it does not satisfy condition (c).⁵ So we will focus only on the third-degree polynomial (51).

For all $\phi > 0$, (51) always has a unique solution $m_1^* \in (1/2, 1)$, i.e., conditions (a) and (b) hold. Indeed, denoting the left-hand side of (51) by $F(m_1, \phi)$ and substituting $m_1 = 1/2$ and $m_1 = 1$ into $F(m_1, \phi)$, we obtain $F(1/2, \phi) = -\phi/4 < 0$ and $F(1, \phi) = 1 > 0$. The uniqueness follows from $\frac{\partial F}{\partial m_1} > 0$ for $m_1 \in (1/2, 1)$.

Next, solving (51) with respect to ϕ , we obtain

$$(52) \quad \phi = \frac{2m_1^3 - 3m_1^2 + 3m_1 - 1}{m_1(1 - m_1)}.$$

Then, substituting this into (45) or (49), we obtain $z_1 = \frac{2m_1^* - 1}{m_1^*}$. Since $m_1^* \in (1/2, 1)$, $z_1^* = \frac{2m_1^* - 1}{m_1^*} \in (0, 1)$, condition (c), holds.

Figures 1 and 2 show the graphs of the solution to (51) and of (49). We can also see that they satisfy condition (b) and (c) for all $\phi > 0$.

So far, we have proved the following Lemma.

LEMMA 4 *For the stationary distribution H in Lemma 1, there exist $V(0)$, $V(1)$, m_1^* and z_1^* satisfying (6) for all $\phi > 0$.*

⁵It corresponds to one of single-price equilibria in GZ.

3.6 Incentive

First, as for the reservation price strategy, we define $\rho(np)$ by the perfectness condition (5) for all $n \geq 3$, i.e.,

$$(53) \quad \rho(\eta) = \max\{r \in [0, \eta] \mid u + \mathcal{V}(\eta - r) \geq \mathcal{V}(\eta)\},$$

where \mathcal{V} is the solution to the Bellman equation obtained in the previous subsections. Finally, we show that, for $V(n)$, m_1^* , z_1^* , and the stationary distribution obtained in the above subsections, each agent has incentive to play the strategy (16)–(19) and ρ specified by (53).

First, we investigate the reservation price strategy. In a p - $2p$ equilibrium with the above strategy, $V(n) \leq u + V(n - 2)$ must hold for $n \geq 2$, since an agent with $n \geq 2$ accepts offer $2p$.⁶⁷ This inequality holds if $m_1 < 2m_2$ and

$$(54) \quad V(2) - V(0) \leq u,$$

since, by using (39), inequality (54) can be written as

$$(55) \quad \frac{m_2}{2m_2 - m_1} \left(1 - \left(\frac{m_1 - m_2}{m_2} \right)^2 \right) (V(1) - V(0)) \leq u,$$

and so that

$$(56) \quad \begin{aligned} V(n) - V(n - 2) &= \frac{m_2}{2m_2 - m_1} \left(\left(\frac{m_1 - m_2}{m_2} \right)^{n-2} - \left(\frac{m_1 - m_2}{m_2} \right)^n \right) (V(1) - V(0)) \\ &= \frac{m_2}{2m_2 - m_1} \left(\frac{m_1 - m_2}{m_2} \right)^{n-2} \left(1 - \left(\frac{m_1 - m_2}{m_2} \right)^2 \right) (V(1) - V(0)) \\ &< \frac{m_2}{2m_2 - m_1} \left(1 - \left(\frac{m_1 - m_2}{m_2} \right)^2 \right) (V(1) - V(0)) \\ &< u \end{aligned}$$

holds if $m_1 < 2m_2$. Of course, $m_1^* < 2m_2^* = 2(m_1^*)^2$ holds.

By using (40) and (47), $V(2) - V(0) \leq u$ can be written as

$$\frac{1}{m_1^2} \frac{2m_1 - 1}{\phi + 2m_1 - 1} u \leq u,$$

⁶This condition is equivalent to the one that the expected value of $t = 2$ is not less than that of $t = 1$ for all $n \geq 2$. Indeed, $W(n, 1, 1) \leq W(n, 1, 2)$ is equivalent to $V(n) - V(n - 2) \leq u$. The same argument applies to $W(n, 2, 1) \leq W(n, 2, 2)$.

⁷An agent accepts an offer $2p$ also accepts an offer p . Clearly, $V(n) - V(n - 1) < V(n) - V(n - 2) \leq u$ holds if $V(n)$ is increasing. Indeed, $V(n)$ given by (39) is increasing if $m_1 < 2m_2$.

and it is equivalent to

$$2m_1^3 + (\phi - 1)m_1^2 - 2m_1 + 1 \geq 0,$$

since $u > 0$. Thus, by (51), we can rewrite the above inequality as

$$(57) \quad 2m_1^2 + (\phi - 5)m_1 + 2 \geq 0.$$

Substituting (52) into inequality (57), we obtain

$$(58) \quad \frac{(2m_1 - 1)^2}{1 - m_1} \geq 0.$$

This inequality always holds strictly if $1/2 < m_1 < 1$. Therefore, $V(2) - V(0) \leq u$ is satisfied for $m_1^* \in (1/2, 1)$.

We can easily show that an agent with $n = 1$ accept offer p . Indeed, since $V(n)$ is increasing in n , $V(1) - V(0) \leq V(2) - V(0) \leq u$ holds.

Next, we focus on the offer strategy. It suffices to show that the expected value of offering $3p$ is less than that of offering $2p$. Let $a(j)$ and $A(j)$ denote a minimal money holding necessary for an agent to accept an offer jp and the set of money holdings larger than or equal to $a(j)$, respectively. Note that $\{n | np \in A(j)\} = \{a(j), a(j) + 1, a(j) + 2, \dots\}$ holds and an agent with money holding $np \in A(j)$ always accepts the offer jp .

(21), (31), (32), and (33) imply $H\{(n, s, t) | np \in A(j)\} = m_1^{a(j)} \leq m_1^j$. Let $W(n, j)$ be the expected value of an agent with money holding np and with offer price jp . $W(n, j)$ can be written as

$$(59) \quad \begin{aligned} W(n, j) &= H\{(n, s, t) | np \in A(j)\}V(n + j) + H\{(n, s, t) | np \notin A(j)\}V(n) \\ &\leq m_1^j V(n + j) + (1 - m_1^j)V(n). \end{aligned}$$

Thus, we can see that the expected value of offer price $3p$ is lower than that of offer price $2p$ if

$$(60) \quad \begin{aligned} m_1^3 V(n + 3) + (1 - m_1^3)V(n) &< m_2 V(n + 2) + (1 - m_2)V(n) \\ &= m_1^2 V(n + 2) + (1 - m_1^2)V(n). \end{aligned}$$

Substituting (40) into this inequality (60) yields $(2m_1 - 1)(m_1 - 1) < 0$. For $m_1^* \in (1/2, 1)$, this clearly holds. That is the expected value of offer price $3p$ is less than that of offer price $2p$.

For $j \geq 3$, $W(n, j + 1) \leq W(n, 2)$ follows from (60) if

$$m_1^{j+1} V(n + j + 1) + (1 - m_1^{j+1})V(n) < m_1^j V(n + j) + (1 - m_1^j)V(n)$$

holds. Substituting (40) into the above inequality, we obtain $m_1^{j+1} - (1 - m_1)^{j+1} < m_1^j - (1 - m_1)^j$ and thus $(1 - m_1)^{j-1} < m_1^{j-1}$. For $m_1^* \in (1/2, 1)$, this clearly holds.

Note that the probability of acceptance of an offer larger than $2p$ is not zero, but too small for the value to be larger than that of offer price $2p$. Thus no seller makes an offer larger than $2p$.

So far, we have proved the following Lemma.

LEMMA 5 *For $V(n)$, m_1^* , z_1^* , the stationary distribution H in Lemma 4, and the strategy specified above, it is optimal for an agent to play the strategy.*

By the lemmas, all the conditions in Definition 2 are clearly satisfied. This completes the proof of Theorem 1.

4 Uniqueness

In this section, confining our attention to geometric distributions of money holdings, we show that the equilibrium obtained in the previous section is the unique p - $2p$ equilibrium.

For $m_1 \in (0, 1)$ and $p > 0$, let $H_{m_1, p}$ be the distribution of money holdings such that the proportion of agents with money holding np , $n = 0, 1, 2, \dots$, is $m_1^n(1 - m_1)$.

Since marginal utilities of money of rich sellers are lower than those of poor sellers, it seems at first glance that there exists a p - $2p$ equilibrium in which the former only offer p and the latter only offer $2p$. However, there does not exist such an equilibrium; such a strategy is not consistent with the stationarity of $H_{m_1, p}$. As for reservation price strategies, a strategy satisfying $\rho(2p) = p$ and $\rho(\hat{n}p) \geq 2p$ for some $\hat{n} > 2$ is a candidate for an equilibrium strategy. However, there does not exist such an equilibrium; there do not exist $m_1, z_1, V(0)$, and $V(1)$ consistent with the Bellman equation.

THEOREM 2 *If $\langle H_{m_1, p}, R, \Omega, \omega, \rho, \mathcal{V} \rangle$ is a p - $2p$ equilibrium, then it must be the one given in the previous section, i.e., (i) $H_{m_1, p}, R$, and Ω must be the ones determined by $h(n, s, t)$ in Lemma 1, where m_1^* and z_1^* are given in Subsection 3.5 and $p^* = \frac{1-m_1^*}{m_1^*}M$, (ii) $\mathcal{V}(n), n = 0, 1, \dots$, must be the ones given by (40), (47), and (48) for m_1^* , and (iii) ω and ρ must be the ones given by (16) – (19) and (53).*

PROOF : See Appendix.

5 Welfare

Next, we investigate welfare. We obtained the stationary geometric distribution $h(n) = m_1^n(1-m_1)$ as in GZ and therefore our p - $2p$ equilibrium and one of GZ's single-price equilibria coexist under this money holding distribution. A natural question is in which equilibrium welfare is higher. For a given $\phi > 0$, m_1^* is uniquely determined by (51). We use the standard welfare measure $U(m_1, \phi, u) = \sum_{n=0}^{\infty} h(n)V(n) = (1-m_1) \sum_{n=0}^{\infty} m_1^n V(n)$. Substituting the value of $V(n)$ yields the welfare of p - $2p$ equilibrium $U_2(m_1^*, \phi, u) = V(1) = \frac{\phi+m_1^*}{m_1^*} \frac{1}{\phi} \frac{2m_1^*-1}{\phi+2m_1^*-1} u$. On the other hand, the welfare of the single-price equilibrium under the same distribution is $U_1(m_1^*, \phi, u) = \frac{m_1^* u}{\phi}$ as obtained in GZ. By tedious calculation, we can show that $U_1(m_1^*, \phi, u) > U_2(m_1^*, \phi, u)$ for any $\phi > 0$. Figure 3 shows this for $\phi \in (0, 0.001)$ when $u = 0.01$. This is due to the fact that, in the p - $2p$ equilibrium, there are more matches result in no trade than in the single-price equilibrium.

There exists a continuum of single-price equilibria in GZ. The second question is whether the welfare of any single-price equilibrium is higher than that of p - $2p$ equilibrium. The proportion of the agents with positive money holdings, m , can be any value in $(0, \hat{m})$ for some $\hat{m} > 0$.⁸ Since the welfare is in proportion to m in single-price equilibria, the welfare can be any small positive value. However, in p - $2p$ equilibrium, the welfare is of some positive value for any given ϕ . Thus $U_2(m_1^*, \phi, u) > U_1(m, \phi, u)$ holds for sufficiently small m .

Appendix: Proof of Theorem 2

Let $\alpha_n \in [0, 1]$, $n = 0, 1, \dots$, denote the proportion of the agents with offer price p among the agents with money holding np . Suppose $\rho(p) = p$ and $\rho(2p) = 2p$, the case in Section 3. Given $H_{m_1, p}$ and $m_2 = m_1^2$, (25)–(27) can be rewritten as

$$(A1) \quad z_1 m_1 + (1 - z_1) m_1^2 = m_1 \alpha_0 + m_1^2 (1 - \alpha_0),$$

$$(A2) \quad m_1 \alpha_0 + z_1 m_1^2 + (1 - z_1) m_1^3 = (m_1 + z_1) \alpha_1 m_1 + (m_1^2 + z_1) (1 - \alpha_1) m_1,$$

for $n \geq 2$,

$$(A3) \quad m_1^n (1 - \alpha_{n-2}) + m_1^n \alpha_{n-1} + z_1 m_1^{n+1} + (1 - z_1) m_1^{n+2} = (m_1 + 1) \alpha_n m_1^n + (m_1^2 + 1) (1 - \alpha_n) m_1^n.$$

By (A1), $\alpha_0 = z_1$ holds. Then, by induction, $\alpha_n = z_1$ holds for all n .

⁸ \hat{m} must be the solution of (50) since all sellers post p and the offer prices p and $2p$ are indifferent. If the proportion of money holders is larger than this value, the proportion of the buyers who accept $2p$ is large enough for the sellers to have an incentive to offer $2p$. Figure 4 shows \hat{m} and m_1^* .

Suppose $\rho(p) = 0$ and $\rho(2p) = 2p$. Then the stationarity of $H_{m_1,p}$ for $n = 0$ becomes $(1 - z_1)m_1^2 = m_1\alpha_0 + m_1^2(1 - \alpha_0)$, so that $-z_1m_1^2 = (m_1 - m_1^2)\alpha_0$ holds. Since z_1 and m_1 are in $(0,1)$, α_0 must be negative. This contradicts $\alpha_0 \in [0, 1]$.

Therefore, the distribution given in Lemma 1 is the unique distribution satisfying $\rho(2p) = 2p$ and the stationarity of $H_{m_1,p}$. Thus the p - $2p$ equilibrium in Section 3 is the unique one in case of $\rho(2p) = 2p$.

Next, suppose there exists an integer $\hat{n} \geq 3$ such that $\rho(np) \geq 2p$ for all $n \geq \hat{n}$ and $\rho(np) < 2p$ for all $n < \hat{n}$. For the proof of Theorem 2, it suffices to show that there is no p - $2p$ equilibrium strategy consistent with the stationarity of $H_{m_1,p}$. Below, we will prove this.

First, we will show that α_n must be in $(0,1)$ for all $n = 0, 1, \dots$. Note that, in this case, $m_2 = m_1^{\hat{n}}$ holds by (30). Suppose $\rho(p) = 0$. Then the stationarity of $H_{m_1,p}$ for $n = 0$ implies $0 = m_1\alpha_0 + m_1^{\hat{n}}(1 - \alpha_0)$. This holds only if $m_1 = 0$. Therefore, $\rho(p) = p$ holds and thus $\rho(np) = p$ holds for $n = 2, 3, \dots, \hat{n} - 1$, since ρ is an increasing function. Thus, the stationarity of $H_{m_1,p}$ implies the following equations:⁹

$$(A4) \quad z_1m_1 = m_1\alpha_0 + m_1^{\hat{n}}(1 - \alpha_0),$$

$$(A5) \quad m_1\alpha_0 + z_1m_1^2 = (m_1 + z_1)\alpha_1m_1 + (m_1^{\hat{n}} + z_1)(1 - \alpha_1)m_1,$$

for $n = 2, \dots, \hat{n} - 3$,

$$(A6) \quad m_1^{\hat{n}+n-2}(1 - \alpha_{n-2}) + m_1^n\alpha_{n-1} + z_1m_1^{n+1} = (m_1 + z_1)\alpha_n m_1^n + (m_1^{\hat{n}} + z_1)(1 - \alpha_n)m_1^n,$$

for $n = \hat{n} - 2$ and $n = \hat{n} - 1$,

$$(A7) \quad m_1^{\hat{n}+n-2}(1 - \alpha_{n-2}) + m_1^n\alpha_{n-1} + z_1m_1^{n+1} + (1 - z_1)m_1^{n+2} = (m_1 + z_1)\alpha_n m_1^n + (m_1^{\hat{n}} + z_1)(1 - \alpha_n)m_1^n,$$

for $n \geq \hat{n}$,

$$(A8) \quad m_1^{\hat{n}+n-2}(1 - \alpha_{n-2}) + m_1^n\alpha_{n-1} + z_1m_1^{n+1} + (1 - z_1)m_1^{n+2} = (m_1 + 1)\alpha_n m_1^n + (m_1^{\hat{n}} + 1)(1 - \alpha_n)m_1^n.$$

By (A4),

$$(A9) \quad \alpha_0 = \frac{z_1 - m_1^{\hat{n}-1}}{1 - m_1^{\hat{n}-1}}.$$

By (A4), $m_1\alpha_0 = z_1m_1 - m_1^{\hat{n}}(1 - \alpha_0)$ holds. Substituting this into the first term of the left-hand side of (A5) and solving it with respect to α_1 yield a relationship between α_0 and α_1 : $\alpha_1 = \frac{z_1m_1^2 - m_1^{\hat{n}}(1+m_1)}{m_1^2(1-m_1^{\hat{n}-1})} + \frac{m_1^{\hat{n}}}{m_1^2(1-m_1^{\hat{n}-1})}\alpha_0$. Solving this equation for α_0 and substituting it into (A6) for $n = 2$, yield a relationship between α_1 and α_2 : $\alpha_2 = \frac{z_1m_1^2 - m_1^{\hat{n}}(1+m_1)}{m_1^2(1-m_1^{\hat{n}-1})} + \frac{m_1^{\hat{n}}}{m_1^2(1-m_1^{\hat{n}-1})}\alpha_1$. Similarly, repeating the operation in turn, we obtain the following first-order difference equations of α_n :

for $n = 0, 1, \dots, \hat{n} - 4$,

$$(A10) \quad \alpha_{n+1} = \frac{z_1m_1^2 - m_1^{\hat{n}}(1+m_1)}{m_1^2(1-m_1^{\hat{n}-1})} + \frac{m_1^{\hat{n}}}{m_1^2(1-m_1^{\hat{n}-1})}\alpha_n,$$

⁹Precisely, these representations are applied only to $\hat{n} \geq 5$. If $\hat{n} = 4$, (A7) holds for $n = 2$ and 3, and if $\hat{n} = 3$, (A7) holds for $n = 2$ and $m_1\alpha_0 + z_1m_1^2 + (1 - z_1)m_1^3 = (m_1 + z_1)\alpha_1m_1 + (m_1^{\hat{n}} + z_1)(1 - \alpha_1)m_1$ holds for $n = 1$. However, since these modifications do not affect the following argument, we will restrict our attention to the general case (A4)–(A8).

for $n = \hat{n} - 3$,

$$(A11) \quad \alpha_{\hat{n}-2} = \frac{(1-z_1)m_1^3 + z_1m_1^2 - m_1^{\hat{n}}(1+m_1)}{m_1^2(1-m_1^{\hat{n}-1})} + \frac{m_1^{\hat{n}}}{m_1^2(1-m_1^{\hat{n}-1})}\alpha_{\hat{n}-3},$$

for $n = \hat{n} - 2, \hat{n} - 1, \dots$,

$$(A12) \quad \alpha_{n+1} = \frac{(1-z_1)m_1^3 + m_1^2 - m_1^{\hat{n}}(1+m_1)}{m_1^2(1-m_1^{\hat{n}-1})} + \frac{m_1^{\hat{n}}}{m_1^2(1-m_1^{\hat{n}-1})}\alpha_n,$$

and α_0 is given by (A9).

By using these relations, below we will show that $\alpha_{n'} = 0$ or $\alpha_{n'} = 1$ for some n' contradicts the fact that $m_1, z_1 \in (0, 1)$, and $\alpha_n \in [0, 1]$ should hold for all n in p -2 p equilibria. Thus $\alpha_n \in (0, 1)$ should hold for all $n = 0, 1, \dots$, if a p -2 p equilibrium exists.

The case of α_0 : First, by (A9), $\alpha_0 = 1$ only if $z_1 = 1$, and $\alpha_0 = 0$ only if $z_1 = m_1^{\hat{n}-1}$. However, if $\alpha_0 = 0$ and $z_1 = m_1^{\hat{n}-1}$ hold, then α_1 must be negative by (A10). Therefore, $\alpha_0 \in (0, 1)$ holds.

The case of $\alpha_n = 0$ for $n \in \{1, 2, \dots, \hat{n} - 4\}$: Suppose $\alpha_n = 0$ for some $n \in \{1, 2, \dots, \hat{n} - 4\}$. Then, by (A10),

$$(A13) \quad \alpha_{n+1} = \frac{z_1m_1^2 - m_1^{\hat{n}}(1+m_1)}{m_1^2(1-m_1^{\hat{n}-1})}$$

$$(A14) \quad 0 = \frac{z_1m_1^2 - m_1^{\hat{n}}(1+m_1)}{m_1^2(1-m_1^{\hat{n}-1})} + \frac{m_1^{\hat{n}}}{m_1^2(1-m_1^{\hat{n}-1})}\alpha_{n-1}.$$

Suppose $\alpha_{n+1} \in (0, 1]$, then by (A14), $\alpha_{n-1} < 0$ holds. Suppose $\alpha_{n+1} = 0$, then $\alpha_{n-1} = 0$ holds also by (A14). Applying the same operation backward, we finally obtain $\alpha_0 = 0$, but α_0 cannot be zero as we have already shown. Therefore, α_n cannot be zero.

The case of $\alpha_n = 1$ for $n \in \{1, 2, \dots, \hat{n} - 4\}$: Suppose $\alpha_n = 1$ for some $n \in \{1, 2, \dots, \hat{n} - 4\}$. Then, by (A10), $1 = \frac{z_1m_1^2 - m_1^{\hat{n}}(1+m_1)}{m_1^2(1-m_1^{\hat{n}-1})} + \frac{m_1^{\hat{n}}}{m_1^2(1-m_1^{\hat{n}-1})}\alpha_{n-1}$ holds and arranging this yields $(1-z_1)m_1^2 = m_1^{\hat{n}}(\alpha_{n-1} - 1)$. This equation holds only if $m_1 = 0$ or both $z_1 = 1$ and $\alpha_{n-1} = 1$ hold. Therefore, α_n cannot be one.

The case of $\alpha_n = 0$ for $n \in \{\hat{n} - 3, \hat{n} - 2\}$: Suppose $\alpha_{\hat{n}-3} = 0$, then $\alpha_{\hat{n}-2}, \alpha_{\hat{n}-1}, \alpha_{\hat{n}}, \dots$ can be written by (A11) and (A12) as follows.

$$(A15) \quad \alpha_{\hat{n}-2} = \frac{A}{B}$$

$$(A16) \quad \alpha_{\hat{n}-1} = \frac{1}{B}C + \frac{m_1^{\hat{n}}}{B^2}A$$

$$(A17) \quad \alpha_{\hat{n}} = \frac{1}{B}C + \frac{m_1^{\hat{n}}}{B^2}C + \frac{(m_1^{\hat{n}})^2}{B^3}A$$

$$(A18) \quad \alpha_{\hat{n}-2+\ell} = C \sum_{x=1}^{\ell} \frac{(m_1^{\hat{n}})^{x-1}}{B^x} + \frac{(m_1^{\hat{n}})^{\ell}}{B^{\ell+1}}A,$$

where, $A \equiv (1 - z_1)m_1^3 + z_1m_1^2 - m_1^{\hat{n}}(1 + m_1)$, $B \equiv m_1^2(1 - m_1^{\hat{n}-1})$, and $C \equiv (1 - z_1)m_1^3 + m_1^2 - m_1^{\hat{n}}(1 + m_1)$. Note that, since $\alpha_{\hat{n}-2} \geq 0$ should hold in equilibria, $A \geq 0$ holds. Thus $C > 0$ holds. Since $\{\alpha_{\hat{n}-2+\ell}\}_{\ell=1}^{\infty}$ should converge to a real number less than or equal to one, then $\frac{(m_1^{\hat{n}})}{B} < 1$ holds. Thus $m_1^2 - m_1^{\hat{n}} - m_1^{\hat{n}+1} > 0$ holds. Thus

$$\begin{aligned}
\lim_{\ell \rightarrow \infty} \alpha_{\hat{n}-2+\ell} &= C \sum_{x=1}^{\infty} \frac{(m_1^{\hat{n}})^{x-1}}{B^x} + \lim_{\ell \rightarrow \infty} \frac{(m_1^{\hat{n}})^{\ell}}{B^{\ell+1}} A \\
&= C \frac{1}{m_1^2(1 - m_1^{\hat{n}-1})} \\
&\quad \frac{1}{1 - \frac{m_1^{\hat{n}}}{m_1^2(1 - m_1^{\hat{n}-1})}} \\
&= \frac{(1 - z_1)m_1^3 + m_1^2 - m_1^{\hat{n}}(1 + m_1)}{m_1^2 - m_1^{\hat{n}} - m_1^{\hat{n}+1}} \\
(A19) \quad &= 1 + \frac{(1 - z_1)m_1^3}{m_1^2(1 - m_1^{\hat{n}-1})}
\end{aligned}$$

holds. The right-hand side of (A19) is larger than one. That is, if $\alpha_{\hat{n}-3} = 0$, then there always exists some n such that $\alpha_n > 1$. This is a contradiction.

As for the case of $\alpha_{\hat{n}-2}$, besides a slight modification, i.e., $\alpha_{\hat{n}-2+\ell} = C \sum_{x=1}^{\ell} \frac{(m_1^{\hat{n}})^{x-1}}{B^x}$, the same argument applies. Therefore, neither $\alpha_{\hat{n}-3} = 0$ nor $\alpha_{\hat{n}-2} = 0$ hold.

The case of $\alpha_n = 1$ for $n \in \{\hat{n} - 3, \hat{n} - 2\}$: As for the case of $\alpha_{\hat{n}-3} = 1$, the same argument as the case of $\alpha_n = 1$ for $n = 1, 2, \dots, \hat{n} - 4$ applies.

Next, suppose $\alpha_{\hat{n}-2} = 1$, then by (A11) and (A12),

$$(A20) \quad 1 = \frac{(1 - z_1)m_1^3 + z_1m_1^2 - m_1^{\hat{n}}(1 + m_1)}{m_1^2(1 - m_1^{\hat{n}-1})} + \frac{m_1^{\hat{n}}}{m_1^2(1 - m_1^{\hat{n}-1})} \alpha_{\hat{n}-3}$$

$$(A21) \quad \alpha_{\hat{n}-1} = \frac{(1 - z_1)m_1^3 + m_1^2 - m_1^{\hat{n}+1}}{m_1^2(1 - m_1^{\hat{n}-1})}$$

hold. (A21), together with $(1 - z_1)m_1^3 - (1 - z_1)m_1^2 - m_1^{\hat{n}}(1 - \alpha_{\hat{n}-3}) = 0$, which is derived from (A20), implies

$$\begin{aligned}
\alpha_{\hat{n}-1} &= \frac{(2 - z_1)m_1^2 - m_1^{\hat{n}+1} + m_1^{\hat{n}}(1 - \alpha_{\hat{n}-3})}{m_1^2(1 - m_1^{\hat{n}-1})} \\
(A22) \quad &= 1 + \frac{(1 - z_1)m_1^2 + m_1^{\hat{n}}(1 - \alpha_{\hat{n}-3})}{m_1^2(1 - m_1^{\hat{n}-1})}.
\end{aligned}$$

The right-hand side of (A22) is always larger than one. Therefore, $\alpha_{\hat{n}-2}$ cannot be one.

The case of $\alpha_n = 0$ for $n \in \{\hat{n} - 1, \hat{n}, \dots\}$: Suppose $\alpha_n = 0$ for some $n \in \{\hat{n} - 1, \hat{n}, \dots\}$. The same argument as the case of $\alpha_{\hat{n}-2} = 0$ applies. Thus α_n cannot be zero in this case.

The case of $\alpha_n = 1$ for $n \in \{\hat{n} - 1, \hat{n}, \dots\}$: Finally, suppose $\alpha_n = 1$ for some $n \in \{\hat{n} - 1, \hat{n}, \dots\}$. Then by

$$(A23) \quad 1 = \frac{(1 - z_1)m_1^3 + m_1^2 - m_1^{\hat{n}}(1 + m_1)}{m_1^2(1 - m_1^{\hat{n}-1})} + \frac{m_1^{\hat{n}}}{m_1^2(1 - m_1^{\hat{n}-1})} \alpha_{n-1}$$

$$(A24) \quad \alpha_{n+1} = \frac{(1 - z_1)m_1^3 + m_1^2 - m_1^{\hat{n}+1}}{m_1^2(1 - m_1^{\hat{n}-1})},$$

we obtain

$$(A25) \quad \alpha_{n+1} = 1 + \frac{m_1^{\hat{n}}(1 - \alpha_{n-1})}{m_1^2(1 - m_1^{\hat{n}-1})}.$$

The right-hand side of (A25) is larger than one if $\alpha_{n-1} \in [0, 1)$. If $\alpha_{n-1} = 1$, then, applying the above argument for $\alpha_{n-1} = 1$, $\alpha_{n-2} = 1$ holds. Applying this argument backward, $\alpha_n = 1$ holds for $n = \hat{n} - 2, \hat{n} - 1, \dots, n - 3$. However, $\alpha_{\hat{n}-2} = 1$ cannot hold as we have shown. Therefore, α_n cannot be one in this case.

So far, we have shown that $\alpha_n \in (0, 1)$ holds for all $n = 0, 1, \dots$. In other words, $\omega(np) = \{p, 2p\}$ must hold for all n . Thus $V(n), n = 0, 1, \dots$, is given by (39) even in the case that, for some $\hat{n} \geq 3$, $\rho(np) \geq 2p$ for all $n \geq \hat{n}$ and $\rho(np) < 2p$ for all $n < \hat{n}$. What remains to show is there *do not* exist $V(0), V(1), m_1$ and z_1 satisfying necessary conditions for the existence of a p - $2p$ equilibrium in this case.

Substituting $m_2 = m_1^{\hat{n}}$, (39) can be rewritten as

$$(A26) \quad V(n) = \frac{m_1^{\hat{n}}}{2m_1^{\hat{n}} - m_1} \left[\left(V(1) - \left(\frac{m_1 - m_1^{\hat{n}}}{m_1^{\hat{n}}} \right) V(0) \right) - \left(\frac{m_1 - m_1^{\hat{n}}}{m_1^{\hat{n}}} \right)^n (V(1) - V(0)) \right].$$

If $V(n)$ is given by (A26) for all n , $W(n, 1, t) = V(n) (= W(n, 2, t))$ must hold for all n on the equilibrium path as in Section 3.5. $W(n, 1, t)$ and $W(n, 2, t)$ can be written as

for $n = 0$,

$$(A27-(a)) \quad W(0, 1, 0) = \frac{1}{\phi + 2} (V(0) + (1 - m_1)V(0) + m_1V(1))$$

$$(A27-(b)) \quad W(0, 2, 0) = \frac{1}{\phi + 2} (V(0) + (1 - m_2)V(0) + m_2V(2)),$$

for $n = 1, \dots, \hat{n} - 1$,

$$(A28-(a)) \quad W(n, 1, 1) = \frac{1}{\phi + 2} (z_1(u + V(n - 1)) + ((1 - m_1) + z_2)V(n) + m_1V(n + 1))$$

$$(A28-(b)) \quad W(n, 2, 1) = \frac{1}{\phi + 2} (z_1(u + V(n - 1)) + ((1 - m_2) + z_2)V(n) + m_2V(n + 2)),$$

for $n \geq \hat{n}$ and $t \geq 2$,

(A29-(a))

$$W(n, 1, t) = \frac{1}{\phi + 2} (z_2(u + V(n - 2)) + z_1(u + V(n - 1)) + (1 - m_1)V(n) + m_1V(n + 1))$$

(A29-(b))

$$W(n, 2, t) = \frac{1}{\phi + 2} (z_2(u + V(n - 2)) + z_1(u + V(n - 1)) + (1 - m_2)V(n) + m_2V(n + 2)).$$

By $W(0, 1, 0) = V(0)$,

$$(A30) \quad V(0) = \frac{m_1}{\phi + m_1} V(1)$$

holds, by $W(n, 1, 1) = V(n)$ for $n = 1, \dots, \hat{n} - 1$,

(A31)

$$\frac{1}{\phi + 2} \left\{ z_1 u + \frac{m_1^{\hat{n}}}{2m_1^{\hat{n}} - m_1} \left[2 \left(V(1) - \left(\frac{m_1 - m_1^{\hat{n}}}{m_1^{\hat{n}}} \right) V(0) \right) \right] \right\} = \frac{m_1^{\hat{n}}}{2m_1^{\hat{n}} - m_1} \left(V(1) - \left(\frac{m_1 - m_1^{\hat{n}}}{m_1^{\hat{n}}} \right) V(0) \right)$$

(A32)

$$\frac{1}{\phi + 2} \left(\left(\frac{m_1^{\hat{n}}}{m_1 - m_1^{\hat{n}}} \right) z_1 + 2 - m_1 - z_1 + \left(\frac{m_1 - m_1^{\hat{n}}}{m_1^{\hat{n}-1}} \right) \right) = 1$$

holds, and by $W(n, 1, t) = V(n)$ for $n \geq \hat{n}$,

(A33)

$$\frac{1}{\phi + 2} \left\{ u + \frac{m_1^{\hat{n}}}{2m_1^{\hat{n}} - m_1} \left[2 \left(V(1) - \left(\frac{m_1 - m_1^{\hat{n}}}{m_1^{\hat{n}}} \right) V(0) \right) \right] \right\} = \frac{m_1^{\hat{n}}}{2m_1^{\hat{n}} - m_1} \left(V(1) - \left(\frac{m_1 - m_1^{\hat{n}}}{m_1^{\hat{n}}} \right) V(0) \right)$$

(A34)

$$\frac{1}{\phi + 2} \left(\left(\frac{m_1^{\hat{n}}}{m_1 - m_1^{\hat{n}}} \right)^2 (1 - z_1) + \left(\frac{m_1^{\hat{n}}}{m_1 - m_1^{\hat{n}}} \right) z_1 + 1 - m_1 + \left(\frac{m_1 - m_1^{\hat{n}}}{m_1^{\hat{n}-1}} \right) \right) = 1$$

holds. However, clearly, (A31) cannot be consistent with (A33) unless $z_1 = 1$.

Thus, given $H_{m_1, p}$, any strategy such that $\rho(np) \geq 2p$ for all $n \geq \hat{n}$ and $\rho(np) < 2p$ for all $n < \hat{n}$ does not constitute a p -2 p equilibrium. This completes the proof of Theorem 2.

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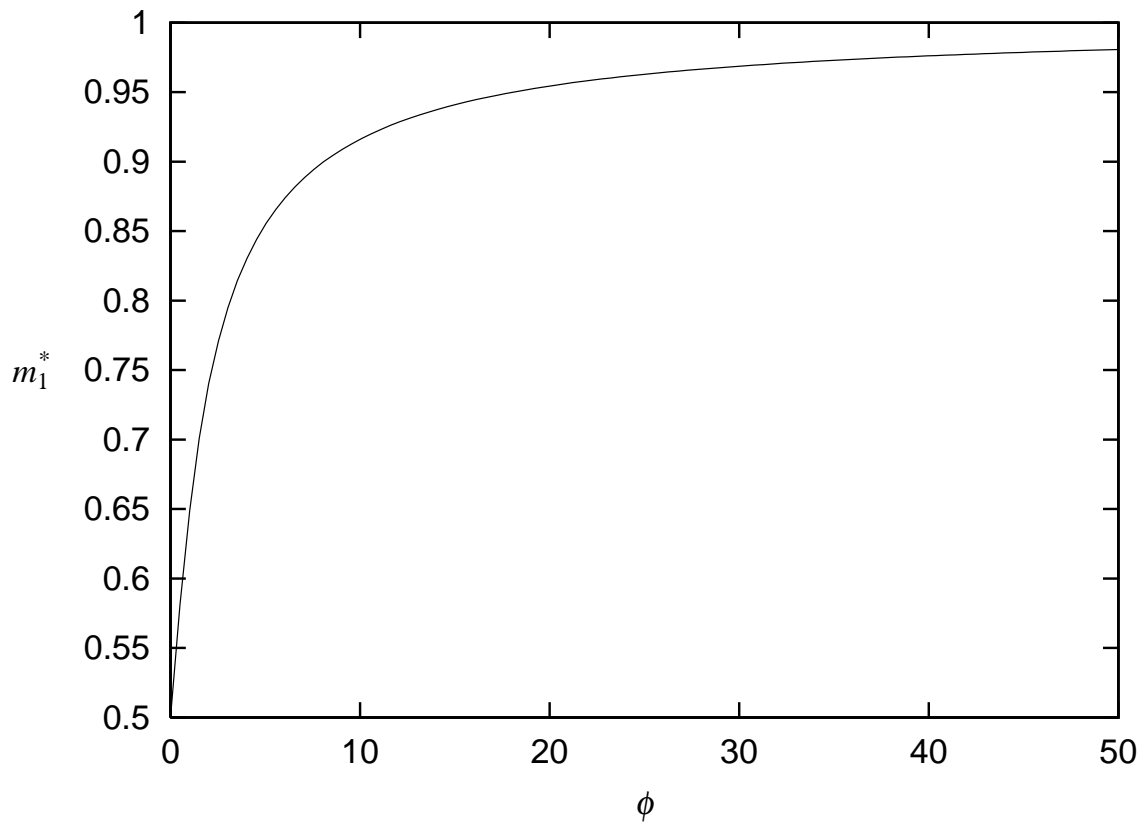


Figure 1

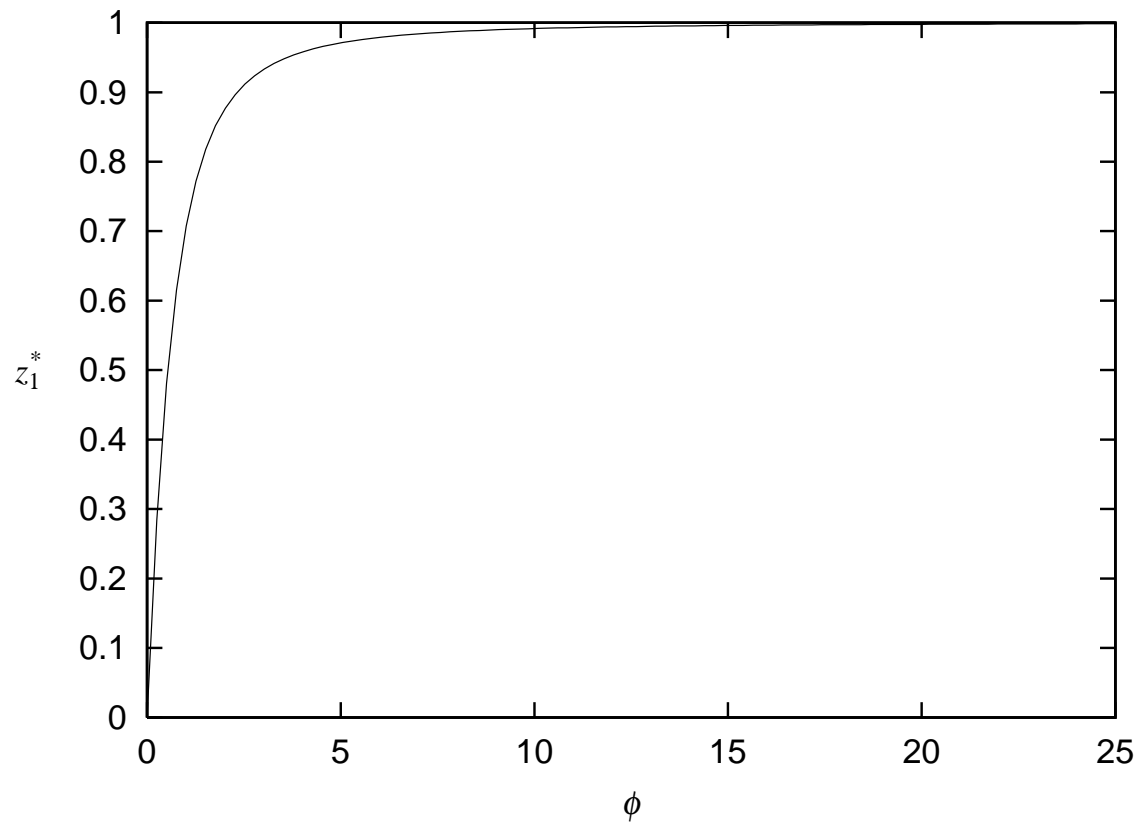


Figure 2

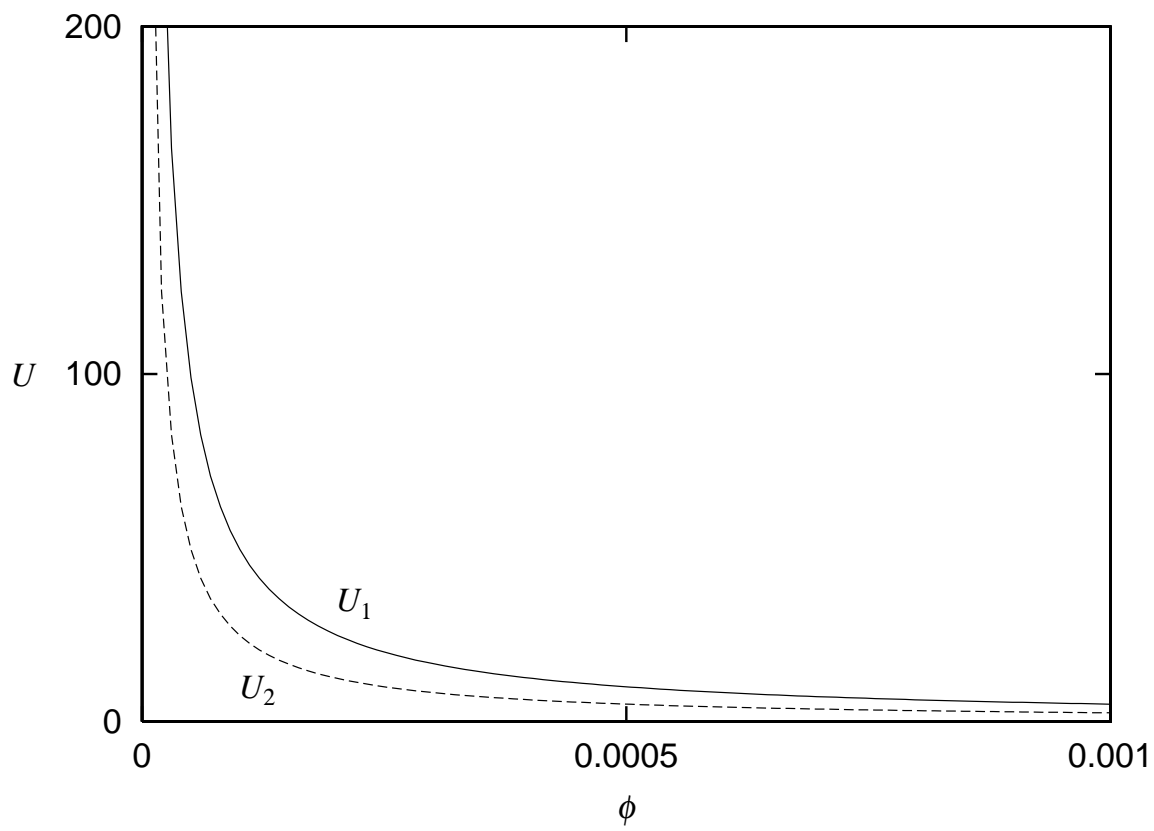


Figure 3

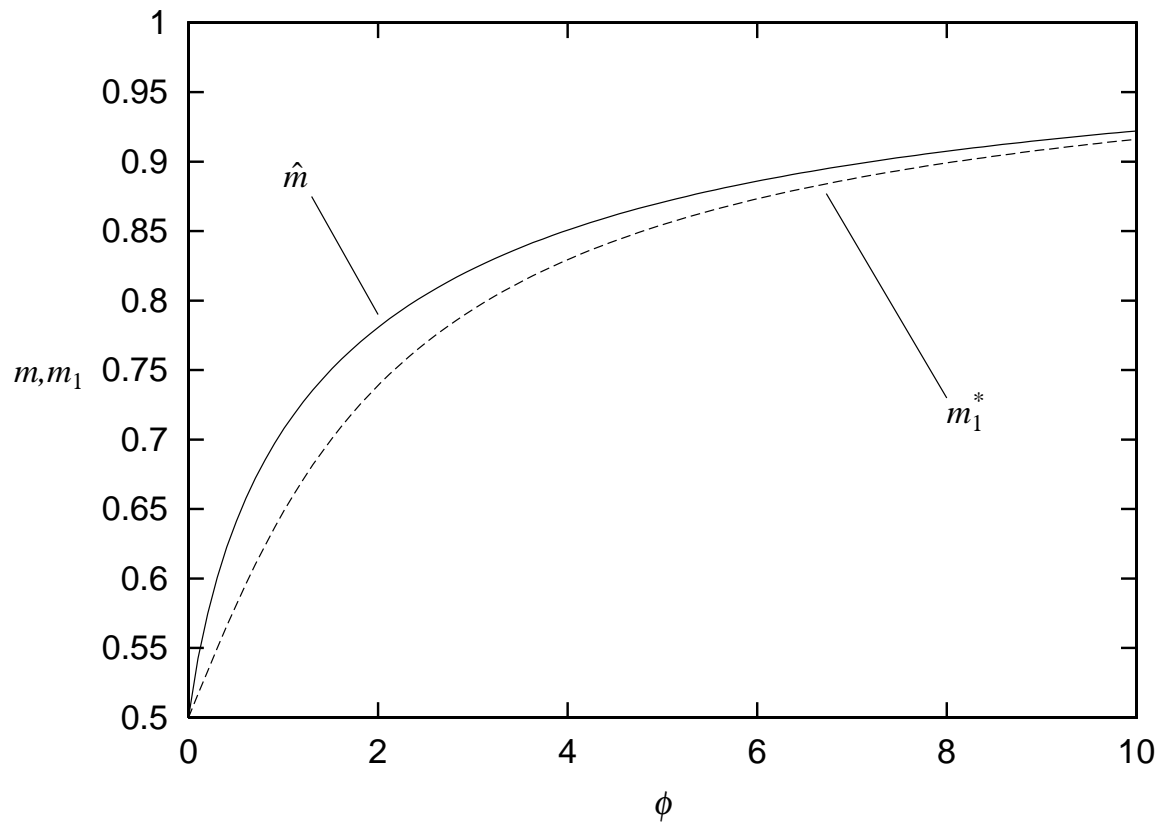


Figure 4