CentER

Discussion paper

No. 2002-114

ROBUST ONE PERIOD OPTION MODELLING

By Frank Lutgens, Jos Sturm

December 2002

ISSN 0924-7815

ROBUST ONE PERIOD OPTION MODELLING

FRANK LUTGENS, JOS STURM

Abstract. This paper considers robust optimization to cope with uncertainty about the stock return process in one period portfolio selection problems involving options. The robust approach relates portfolio choice to uncertainty, making more cautious portfolios when uncertainty is high. We represent uncertainty by a set of plausible expected returns of the underlyings and show that for this set the robust problem is a second order cone program that can be solved efficiently. We illustrate the approach for a benchmark tracking problem and discuss the added value of adopting the robust approach in a stochastic programming framework.

1. Introduction

The portfolio selection problem concerns the allocation of wealth to assets such that return is maximized and uncertainty (risk) is minimized. The best known mathematical model for portfolio selection is the Markowitz portfolio selection model [Mar52]. The Markowitz model measures return by the expected value of random portfolio return and uncertainty by the variance of the portfolio return. The mathematical model is a quadratic programming model. A good reference on portfolio optimization is Zenios [Zen93].

Critics have shown that the Markowitz model is very sensitive to the parameters of the model, in particular to the expected return. The numerical values for the parameters are output from financial (econometric) models, possibly combined with subjective beliefs. These models are estimated from noisy data and as such subject to statistical and judgemental error. However in classical portfolio optimization the parameters, once passed on to the optimization tool, are treated as being oracle prophecies; the reliability of the parameters is not questioned anymore. This also happens in the Markowitz model that does consider uncertainty, but only to the extent modelled *by* the parameters, ignoring uncertainty *in* the parameters.

Financial literature has proposed a number of solutions to deal with the parameter sensitivity, see Jagannathan and Ma [JM02], Black and Litterman [BL90], Ter Horst et al. [tHdRW02].

Date: December 5, 2002.

1991 *Mathematics Subject Classification.* 90C15, 90C20, 90C90, 49M29. JEL codes: C61, G11.

Key words and phrases. Robust Optimization, Stochastic Programming, Portfolio Optimization, Nonnegative Cones.

`f.lutgens@ke.unimaas.nl`, Maastricht University, The Netherlands

`j.sturm@uvt.nl`, Tilburg University, The Netherlands; research of this author supported by the Netherlands Organisation for Scientific Research (NWO), file 016.025.026.

These approaches adapt the parameters to reduce the exposure of the optimization to uncertain values but do not improve the modelling of uncertainty.

Much of the recent research is directed to modelling the parameter uncertainty more explicitly. A max-min variant of the Markowitz porfolio selection model was developed by [RBM00]. Robust versions of the Markowitz model fit also nicely in the robust optimization methodology developed by El Ghaoui et al. [EGOL98] and Ben-Tal and Nemirovsky [BTN98]. This methodology, which builds on semidefinite programming, is exploited in [BTMN00, CP02, Lob00, GI02], among others.

Most robust portfolio selection models that we found in the literature are two-stage (i.e. one period) models. A multi-stage model was developed in [BTMN00]. The robust approach in continuous time is studied in [AHS00] and [Mae99], among others.

In the robust optimization framework [BTN98, EGOL98], a model is formulated with similar structure to the original model, but now the constraints are not only imposed over the one (most likely) instance of parameter values but over a set $\mathcal{U}$ of (empirically) plausible parameter values. Consequently, the problem is solved assuming worst case behavior of parameter values within this set of plausible parameters.

We use econometric methods to quantify uncertainty in, and empirical plausibility of the parameters. For financial problems these parameters concern future asset returns. To describe the random asset returns, we use a simple econometric model:

$$(1) \qquad \ln r = \ln \mu + \varepsilon, \qquad \mathbb{E}(\varepsilon\varepsilon^T) = \Sigma$$

where μ is a vector of mean returns which may have multiple constituents (e.g. a factor model). The natural logarithm of a vector is interpreted in a component-wise fashion. The residual returns are multivariate normally distributed, with covariances given by Σ. We consider the uncertainty in μ, the expected future return. In practice we merely have an estimate $\hat{\mu}$ at our disposal. Nevertheless, we can be confident that the true vector μ is contained in a confidence ellipsoid $\mathcal{U}$ around $\hat{\mu}$ as follows:

$$(2) \qquad \mathcal{U} = \{\mu | \|C(\mu - \hat{\mu})\| \leq \theta\}$$

where θ denotes the degree of robustness that is required (typically around 2). Indeed, if the estimator $\hat{\mu}$ is a sample mean, than $\hat{\mu}$ is approximately normally distributed with mean μ and a covariance matrix Ω. Letting C be such that $C^T C = \Omega^{-1}$ yields approximately a 95% confidence ellipsoid when $\theta = 2$. The eigenvalues of CC^T will then be of the same order as the sample size T underlying the computation of $\hat{\mu}$.

In robust optimization models such as considered in [BTMN00, EGOL98], constraints are imposed for any $\mu \in \mathcal{U}$, thus achieving models that are robust to parameter uncertainty. The resulting models are second order cone programming models, which can be solved efficiently using standard software, such as [Stu99].

The robust optimization approach can also be used to model uncertainty in the actual returns r rather than uncertainty in the parameter μ; see Section 2. As such, robust optimization is often seen as an alternative to stochastic programming. However, even if we use a stochastic programming approach, as we do for the target tracking problem as considered in Section 4, we need robust optimization to insure against uncertainty in the parameters, in our case uncertainty in μ. The stochastic and robust optimization approaches thus complement each other.

The main contribution of this paper is that we generalize this approach by adding options to the investment opportunity set. An option is the right but not the obligation to buy or sell a particular asset for a predetermined price, called the exercise price. It is necessary to treat options separately as these are derivative assets. The option return is an affine function of its underlying stock return if in the money and zero otherwise. This has two consequences. The 'break' in the option return as a function of the underlying asset changes the form of the uncertainty set. This demands new theory for dealing with more complicated uncertainty sets. Secondly, we may not look at the stocks and options return separately as there is a relation. For example, a long position in both a stock and a put on this stock have opposite dependence on the stock price; higher stock returns are profitable for the stock holding but have a negative effect on the option value and vice versa. Ignoring that relation causes unnecessary conservatism, which we must surely avoid.

As a first step we only consider one period options. In this way we avoid the difficulties associated with pricing options at intermediate time periods. The outline of the paper is as follows. In Section 2 we introduce the problem associated with assimilating options in a financial optimization problem. We formulate the robust version of the portfolio return relation. In Section 3 we develop the tools to transform the class of robust relations into second order cone constraints constraints, which can be handled by standard optimization software. We illustrate the approach by performing an empirical study on a benchmark tracking problem in Section 4. We try to track the American Dow Jones index by the European EUREX Stoxx 50 and the options on this index. Section 5 presents some preliminary results.

2. Problem description: one period options

In this text we study the modelling of one period options: we can buy the option, and if we do, keep it until expiration, which happens in the next time period. We adopt the usual notation in the financial literature that X denotes the exercise price, S denotes the price of the underlying when the option matures, and $S_0 > 0$ denotes the current price of the underlying. The return of the underlying is denoted $r_s := S/S_0$. Thus, X and S_0 are known quantities in $\Re_+$, whereas S and r_s are quantities in $\Re_+$; their value will be revealed only at the next time epoch.

The payoff of a call option (the right to buy) with exercise price X is $\max\{0, S - X\}$. If the call option costs $c_0 > 0$, then the return r'_c is

$$(3) \qquad r'_c = \max\left\{0, \frac{S - X}{c_0}\right\}.$$

Since $S = r_s S_0$, we may rewrite r'_c as a piece-wise linear function of r_s as follows:

$$r'_c = \max\{0, a_c r_s + b_c\} \text{ with } a_c := \frac{S_0}{c_0} \text{ and } b_c := -\frac{X}{c_0}. \tag{4}$$

Hence, r'_c is a piece-wise linear function of r_s with known coefficients $a_c > 0$ and $b_c \leq 0$.

Similarly, the payoff of a put option (the right to sell) with exercise price X is $\max\{0, X - S\}$. If the put option costs $p_0 > 0$, then the return r'_p is

$$r'_p = \max\{0, a_p r_s + b_p\} \text{ with } a_p := -\frac{S_0}{p_0} \text{ and } b_p := \frac{X}{p_0}. \tag{5}$$

Consider now the case that there are n underlying assets (stocks and bonds) with unknown returns $r_1, r_2, \ldots, r_n$. Suppose that there are m derivatives (options) with return r'_i, $i = 1, 2, \ldots, m$, where

$$r'_i = \max\left\{0, b_i + \sum_{j=1}^{n} a_{ij} r_j\right\}, \tag{6}$$

for some given b_i and a_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Call and put options on a single underlying asset k correspond to the special case where $a_{ij} = 0$ for $j \neq k$ and $a_{ik} > 0$ or $a_{ik} < 0$ respectively; cf. (6) with (4)–(5).

We will use vector notation, i.e. $r \in \Re^n$ is the vector with r_j as its jth component, and similarly $r' \in \Re^m$ is the vector with r'_i as its ith component. It is important to observe that (6) defines r'_i as an explicit function of r. One could write $r'_i(r)$ to make this functional relationship explicit; however, we omit the argument r in our notation for brevity.

We say that the ith derivative is *in-the-money* if $r'_i > 0$ and *out-of-the-money* if $r'_i = 0$. The moneyness of the derivatives is determined by the realizations of $r_1, r_2, \ldots, r_n$ through relation (6).

For any given realization $r \in \Re^n_+$, the derivatives $\{1, 2, \ldots, m\}$ can be partitioned into the set $M \subseteq \{1, 2, \ldots, m\}$ of derivatives that are in-the-money, and the set $N := \{1, 2, \ldots, m\}$ of derivatives that are out-of-the-money. Conversely, given a partition (M, N), $M \cup N = \{1, 2, \ldots, m\}$ and $M \cap N = \emptyset$, we let

$$P(M, N) := \{r \in \Re^n_+ | r'_i > 0 \text{ for } i \in M,\ r'_j = 0 \text{ for } j \in N\}. \tag{7}$$

As a matter of notation, we let $A \in \Re^{m \times n}$ denote the matrix with entries a_{ij} on the ith row and the jth column. Let A_M denote the $|M| \times n$ submatrix of A consisting of the rows $i \in M$, where $|M|$ denotes the cardinality of M. Similarly, we let $b_M \in \Re^{|M|}$ denote the subvector of b with entries b_i, $i \in M$. Thus, after a suitable row permutation we have

$$A = \begin{bmatrix} A_M \\ A_N \end{bmatrix},\ b = \begin{bmatrix} b_M \\ b_N \end{bmatrix}.$$

Using (6) and (7), we obtain that

$$P(M, N) = \{r \in \Re^n_+ | b_M + A_M r > 0,\ b_N + A_N r \leq 0\}. \tag{8}$$

Observe that $P(M, N)$ is a polyhedral set. Furthermore, we have that

(9) $r'_M = b_M + A_M r$ for $r \in \text{cl } P(M, N)$

and

(10) $r'_N = 0$ for $r \in \text{cl } P(M, N)$.

We have shown that r' is a linear function of the uncertain parameter r on $P(M, N)$, where (M, N) is an arbitrary partition of $\{1, 2, \ldots, m\}$. Strictly speaking, the set $\{1, 2, \ldots, m\}$ can be partitioned in 2^m ways. Fortunately most of these moneyness configurations have an empty set of supporting returns $P(M, N)$. In view of 9 and 10 we are interested in moneyness configurations that have a non-empty set of supporting returns $P(M, N)$. These configurations are characterized by grouping derivatives on the same underlying according to the exercise price. For one single underlying asset, the moneyness configurations follow from each return interval $\left[\frac{X_i}{S_0}, \frac{X_{i+1}}{S_0}\right]$ defined by two subsequent exercise prices X_i and X_{i+1}. In-the-money options are call options with $X \le X_i$ and put options with $X \ge X_{i+1}$. If we let m_j denote the number of derivatives on the underlying j, then there are at most $m_j + 1$ of these subdomains. For n underlying assets, each asset return r_j is cut in at most $m_j + 1$ subdomains. The total number of configurations is therefore limited to $\prod_{j=1}^{n}(m_j + 1)$ where $\sum_{j=1}^{m} m_j = m$. This number will be reduced further by the following point.

In practice, not all nonnegative return vectors $r \in \Re^n_+$ are conceivable. The subset $\mathcal{U} \subseteq \Re^n_+$ of conceivable (or realistic) return vectors of the n underlying assets is called the *uncertainty set*. In this section, we assume that $\mathcal{U}$ is the intersection of $\Re^n_+$ with an n-dimensional ellipsoid, i.e.

(11) $\mathcal{U} = \{r \in \Re^n_+ \mid \|C(r - \tilde{r})\| \le \theta\}$

where C is a given $k \times n$ matrix (typically $k = n$), and θ is a given positive scalar constant. The quantity $\tilde{r}$ can be $\hat{\mu}$, an estimator of the mean return, as stipulated in (2). Since here we model uncertainty in r, eigenvalues of $C^T C$ will be much smaller than if $\mathcal{U}$ were to model uncertainty in the parameter μ. However, the theory developed to deal with uncertainty in r can also be used to deal with uncertainty in μ, see Section 4 later in this paper. We remark that the C-matrix allows us to model both volatility of individual assets and correlation between the various assets.

In the sequel, we will formulate our financial models as second order cone optimization models, which can be efficiently solved. A second order cone (or Lorentz cone) is defined as

(12) $$\text{S}OC = \left\{ x \in \Re^n \;\middle|\; x_1 \ge \sqrt{x_2^2 + x_3^2 + \cdots x_n^2} \right\},$$

where n is the dimension of the second order cone. The interior of the second order cone is denoted $\text{int}(\text{S}OC)$, i.e.

(13) $$\text{int}(\text{S}OC) = \left\{ x \in \Re^n \;\middle|\; x_1 > \sqrt{x_2^2 + x_3^2 + \cdots x_n^2} \right\}.$$

Observe that the ellipsoid $\mathcal{U}$ in (11) can be modelled as a conic section, viz.

$$\mathcal{U} = \left\{ r \in \Re^n_+ \,\middle|\, \left[\begin{array}{c} \theta \\ C(r-\tilde{r}) \end{array} \right] \in \mathrm{SOC}(k+1). \right\} \tag{14}$$

We let $\mathcal{F}$ be the family of conceivable moneyness configurations of the m derivative assets, i.e.

$$\mathcal{F} := \{(M,N) \mid M \cup N = \{1,2,\ldots,m\},\ M \cap N = \emptyset,\ P(M,N) \cap \mathcal{U} \neq \emptyset\}. \tag{15}$$

The moneyness configurations partition the uncertainty set $\mathcal{U}$ into at most $|\mathcal{F}|$ *ellipsoidal cuts* of the form

$$\mathcal{U}(M,N) := \mathcal{U} \cap P(M,N) \text{ for } (M,N) \in \mathcal{F}. \tag{16}$$

A *portfolio* is a pair $(x,x') \in \Re^n \times \Re^m$, where x_j denotes the number amount invested in the jth underlying asset, $j = 1,2,\ldots,n$, and x'_i is the amount invested in the ith derivative asset, $i = 1,2,\ldots,m$. Positive and negative values of x_j correspond to long and short positions respectively.

The task of a portfolio manager is to design a portfolio (x,x') such that budget restrictions and other portfolio constraints hold. If the the restriction is linear in the portfolio, we may depict it as a function

$$f(r;x_0,x,x') := x_0 + x^{\mathrm{T}} r + (x')^{\mathrm{T}} r' \tag{17}$$

such that the following restriction holds:

$$f(r;x_0,x,x') \geq 0 \text{ for all } r \in \mathcal{U}. \tag{18}$$

The design parameters (decision variables) are the quantity x_0 and the portfolio (x,x'); r is the vector of uncertain parameters. For example, if the value of the portfolio in the next period must be at least \$100, then one should add the constraint '$x_0 = -100$'. If one also likes to minimize the amount invested, then the objective function becomes 'min $\sum_{j=1}^n x_j + \sum_{i=1}^m x'_j$'. The success of the portfolio manager in solving the problem depends on her ability to transform the (possibly) infinite number of restrictions in 18 to a finite number of manageable restrictions.

Ensuing from (9) and (16), $f(r;x_0,x,x')$ is a linear function of r on each ellipsoidal cut $\mathcal{U}(M,N)$, i.e.

$$f(r;x_0,x,x') = f_0^{(M)}(x_0,x,x') + \sum_{j=1}^n f_j^{(M)}(x_0,x,x') r_j \text{ for all } r \in \mathcal{U}(M,N). \tag{19}$$

Since the coefficients of this function are different for each ellipsoidal cut $\mathcal{U}(M,N)$, we have added a superscript $^{(M)}$. In particular, we have for $r \in \mathcal{U}(M,N)$ that

$$f(r) = x_0 + x^{\mathrm{T}} r + (x')^{\mathrm{T}} r' \tag{20}$$

$$= x_0 + x^{\mathrm{T}} r + (x'_M)^{\mathrm{T}} (b_M + A_M r) \tag{21}$$

$$= x_0 + b_M^{\mathrm{T}} x'_M + (x + A_M^{\mathrm{T}} x'_M)^{\mathrm{T}} r. \tag{22}$$

It follows that

(23) $$f_0^{(M)}(x_0, x, x') = x_0 + b_M^{\mathrm{T}} x'_M$$

and

(24) $$f_j^{(M)}(x_0, x, x') = x_j + \sum_{i \in M} x'_i a_{ij} \text{ for } j = 1, 2, \ldots, n.$$

Since the uncertainty set $\mathcal{U}$ is not finite and in fact not countable, (18) represents an infinite number of constraints on the design parameters. However, we will show that it can be modelled by a finite number of constraints in a second order cone programming problem.

3. Duality to achieve standard-form expressions

Given a nonempty set $D \subseteq \Re^n$, its homogenized cone in $\Re^{n+1}$ is defined as

$$\mathcal{H}(D) := \text{cl } \{(s, y) | s > 0,\ y/s \in D\}.$$

A set $\mathcal{K} \subseteq \Re^n$ is a cone if and only if $\mathcal{K} \neq \emptyset$ and

$$x \in \mathcal{K},\ t \geq 0 \implies tx \in \mathcal{K}.$$

If in addition,

$$x, y \in \mathcal{K} \implies x + y \in \mathcal{K}$$

then $\mathcal{K}$ is a convex cone. It is easily verified that $\mathcal{H}(D)$ is a cone; if D is convex then $\mathcal{H}(D)$ is a convex cone. The dual of a cone $\mathcal{K} \subseteq \Re^n$ is defined as

$$\mathcal{K}^* := \{s \in \Re^n | x^{\mathrm{T}} s \geq 0 \text{ for all } x \in \mathcal{K}\}.$$

A dual cone is always closed and convex. If $\mathcal{K}$ is convex, then the bi-polar relation holds:

(25) $$(\mathcal{K}^*)^* = \text{cl } \mathcal{K}.$$

In the proof of Theorem 1 below, we need the following technical lemmas.

Lemma 1. *Let $D \neq \emptyset$. It holds that*

$$\mathcal{H}(D)^* = \{(f_0, f) \mid f_0 + f^{\mathrm{T}} r \geq 0 \text{ for all } r \in D\}.$$

The above lemma is a special case of Corollary 1 in [SZ01].

Lemma 2. *Let $\mathcal{K} \subseteq \Re^n$ be a cone and B an $m \times n$ matrix. Then*

$$\{x \mid Bx \in \mathcal{K}^*\} = \{B^{\mathrm{T}} y \mid y \in \mathcal{K}\}^*.$$

For a proof, see relation (17) in [SZ01]. An important special case is that for two cones $\mathcal{K}_1$ and $\mathcal{K}_2$ one has

(26) $$\mathcal{K}_1^* \cap \mathcal{K}_2^* = (\mathcal{K}_1 + \mathcal{K}_2)^*,$$

as obtained by setting $B := \begin{bmatrix} I, & I \end{bmatrix}^{\mathrm{T}}$ and $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2$.

Lemma 3. *Let*

$$D = \{r \mid Pr + q \in \mathrm{S}OC,\ \tilde{A}r + \tilde{b} \geq 0\}.$$

If $D \neq \emptyset$ *then*

$$\mathcal{H}(D) = \{(s, y) | Py + sq \in \mathrm{S}OC,\ s \geq 0,\ \tilde{A}y + s\tilde{b} \geq 0\}.$$

Proof. From the definition of $\mathcal{H}(D)$, it is clear that if $(s, y) \in \mathcal{H}(D)$ then

$$Py + sq \in \mathrm{S}OC,\ s \geq 0,\ \tilde{A}y + s\tilde{b} \geq 0 \tag{27}$$

Conversely, suppose that (s, y) satisfies (27). If $s > 0$ then $y/s \in D$ and hence $(s, y) \in \mathcal{H}(D)$. Suppose now that $s = 0$. Since $D \neq \emptyset$, there exists $r \in D$. Let $\sigma > 0$ be arbitrary. We have from the definition of D and (27) that

$$P(r + \frac{1}{\sigma}y) + q \in \mathrm{S}OC,\ \tilde{A}(r + \frac{1}{\sigma}y) + \tilde{b} \geq 0.$$

Hence $(\sigma r + y)/\sigma \in D$ and $(\sigma, \sigma r + y) \in \mathcal{H}(D)$ for all $\sigma > 0$. Letting $\sigma \downarrow 0$ it follows that $(0, y) \in \mathcal{H}(D)$. □

Theorem 1. *Let*

$$D = \{r \mid Pr + q \in \mathrm{S}OC,\ \tilde{A}r + \tilde{b} \geq 0\}$$

and consider the cone of linear functions that are nonnegative on D*, i.e.*

$$\mathcal{K} = \{(f_0, f) \mid f_0 + f^{\mathrm{T}}r \geq 0 \ \textit{for all}\ r \in D\}.$$

If $D \neq \emptyset$ *then*

$$\mathcal{K} = \mathrm{cl}\ \left\{ \begin{bmatrix} q^{\mathrm{T}}u + \tilde{b}^{\mathrm{T}}v + v_0 \\ P^{\mathrm{T}}u + \tilde{A}^{\mathrm{T}}v \end{bmatrix} \middle| u \in \mathrm{S}OC,\ v \geq 0,\ v_0 \geq 0 \right\}.$$

Proof. We have from Lemma 1 that

$$\mathcal{K} = \mathcal{H}(D)^*.$$

Applying Lemmas 3 and 2 respectively, we have

$$\begin{aligned} \mathcal{H}(D) &= \{(s, y) | Py + sq \in \mathrm{S}OC\} \cap \left\{ (s, y) \middle| \begin{bmatrix} 1 & 0^{\mathrm{T}} \\ \tilde{b} & \tilde{A} \end{bmatrix} \begin{bmatrix} s \\ y \end{bmatrix} \geq 0 \right\} \\ &= \left\{ \begin{bmatrix} q^{\mathrm{T}}u \\ P^{\mathrm{T}}u \end{bmatrix} \middle| u \in \mathrm{SOC} \right\}^* \cap \left\{ \begin{bmatrix} v_0 + \tilde{b}^{\mathrm{T}}v \\ \tilde{A}^{\mathrm{T}}v \end{bmatrix} \middle| v_0 \geq 0,\ v \geq 0 \right\}^*. \end{aligned}$$

(It is well known that $\mathrm{S}OC$ and $\Re^n_+$ are self-dual cones.) Further using (26) and (25), we have

$$\mathcal{H}(D)^* = \mathrm{cl}\ \left\{ \begin{bmatrix} q^{\mathrm{T}}u + \tilde{b}^{\mathrm{T}}v + v_0 \\ P^{\mathrm{T}}u + \tilde{A}^{\mathrm{T}}v \end{bmatrix} \middle| u \in \mathrm{S}OC,\ v \geq 0,\ v_0 \geq 0 \right\}.$$

□

The following theorem states that the closure operator in the above characterization of $\mathcal{K}$ is redundant if a Slater condition holds.

Theorem 2. *Let*

$$D^o := \{r \mid Pr + q \in \mathrm{int}(SOC),\ \tilde{A}r + \tilde{b} > 0\}$$

and let $\mathcal{K}$ be defined as in Theorem 1. If $D^o \neq \emptyset$ then

$$\mathcal{K} = \left\{ \begin{bmatrix} q^{\mathrm{T}}u + \tilde{b}^{\mathrm{T}}v + v_0 \\ P^{\mathrm{T}}u + \tilde{A}^{\mathrm{T}}v \end{bmatrix} \middle| u \in \mathrm{SOC},\ v \geq 0,\ v_0 \geq 0 \right\}.$$

Proof. Let

$$\Gamma = \left\{ \begin{bmatrix} q^{\mathrm{T}}u + \tilde{b}^{\mathrm{T}}v + v_0 \\ P^{\mathrm{T}}u + \tilde{A}^{\mathrm{T}}v \end{bmatrix} \middle| u \in \mathrm{SOC},\ v \geq 0,\ v_0 \geq 0 \right\}.$$

We know from Theorem 1 that $\mathcal{K} = \mathrm{cl}\ \Gamma$. It remains to show that Γ is closed, i.e. $cl\ \Gamma = \Gamma$. Let $(t, x) \in \mathcal{K} = \mathrm{cl}\ \Gamma$, and let $(u^{(k)}, v^{(k)}, v_0^{(k)})$, $k = 1, 2, \ldots$ be a sequence such that

$$u^{(k)} \in SOC,\ v^{(k)} \geq 0,\ v_0^{(k)} \geq 0 \text{ for all } k = 1, 2, \ldots$$

and

$$\begin{bmatrix} t \\ x \end{bmatrix} = \lim_{k\to\infty} \begin{bmatrix} q^{\mathrm{T}}u^{(k)} + \tilde{b}^{\mathrm{T}}v^{(k)} + v_0^{(k)} \\ P^{\mathrm{T}}u^{(k)} + \tilde{A}^{\mathrm{T}}v^{(k)} \end{bmatrix}.$$

By definition of Γ, such a sequence must exist, because $(t, x) \in \mathrm{cl}\ \Gamma$. Let $r \in D^o$. We have

$$\begin{aligned} t + r^{\mathrm{T}}x &= \lim_{k\to\infty} q^{\mathrm{T}}u^{(k)} + \tilde{b}^{\mathrm{T}}v^{(k)} + v_0^{(k)} + r^{\mathrm{T}}(P^{\mathrm{T}}u^{(k)} + \tilde{A}^{\mathrm{T}}v^{(k)}) \\ &= \lim_{k\to\infty} (Pr + q)^{\mathrm{T}}u^{(k)} + (\tilde{A}r + \tilde{b})^{\mathrm{T}}v^{(k)} + v_0^{(k)} \\ &\geq \lim_{k\to\infty} (Pr + q)^{\mathrm{T}}u^{(k)} + (\tilde{A}r + \tilde{b})^{\mathrm{T}}v^{(k)}. \end{aligned} \tag{28}$$

Since $Pr + q \in \mathrm{int}(SOC)$ and $\tilde{A}r + \tilde{b} > 0$, we have

$$\begin{cases} (Pr + q)^{\mathrm{T}}u > 0 \text{ for all } u \in \mathrm{SOC}\backslash\{0\} \\ (\tilde{A}r + \tilde{b})v > 0 \text{ for all } v \in \Re^n_+\backslash\{0\}. \end{cases} \tag{29}$$

We claim that the sequence $\{u^{(k)}\}$ is bounded. Indeed, suppose to the contrary that $\limsup_{k\to\infty} \|u^{(k)}\| = \infty$. From (29), it follows that

$$\liminf_{k\to\infty} \frac{(Pr + q)^{\mathrm{T}}u^{(k)}}{\|u^{(k)}\|} > 0,$$

and hence, using also (28), we arrive at the impossible inequality

$$t + r^T x \geq \limsup_{k\to\infty} (Pr + q)^{\mathrm{T}}u^{(k)} = \infty.$$

Similarly, we can show by contradiction from (28) and (29) that $v^{(k)}$ must be bounded. Hence this sequence $\{u^{(k)}, v^{(k)}, v_0^{(k)}\}$ has a cluster point (u, v, v_0), $u \in \mathrm{SOC}$, $v \geq 0$, $v_0 \geq 0$, and

$$\begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} q^{\mathrm{T}}u + \tilde{b}^{\mathrm{T}}v + v_0 \\ P^{\mathrm{T}}u + \tilde{A}^{\mathrm{T}}v \end{bmatrix} \in \Gamma.$$

This concludes the proof. □

Define

$$\mathcal{K}(M,N) := \{(f_0, f) \mid f_0 + f^{\mathrm{T}} r \geq 0 \text{ for all } r \in \mathcal{U}(M,N)\},$$

where $\mathcal{U}(M,N) := \mathcal{U} \cap P(M,N)$, see (16). We find an explicit representation of $\mathcal{K}(M,N)$ by applying Theorem 1 with

$$\tilde{A} := \begin{bmatrix} A_M \\ -A_N \\ I_n \end{bmatrix}, \tilde{b} := \begin{bmatrix} b_M \\ -b_N \\ 0 \end{bmatrix},$$

see (8) and

$$P = \begin{bmatrix} 0^{\mathrm{T}} \\ C \end{bmatrix}, q = \begin{bmatrix} \theta \\ -C\tilde{r} \end{bmatrix},$$

see (14). We have deduced that (18) is equivalent with

$$f^{(M)}(x_0, x, x') \in \mathcal{K}(M,N) \text{ for all } (M,N) \in \mathcal{F}, \tag{30}$$

where $f^{(M)}$ is defined in (23)–(24) and $\mathcal{F}$ is defined in (15). We have reduced the infinite set of constraints in (18) to at most $|\mathcal{F}|$ conic constraints in (30).

4. Illustration: benchmark tracking

We continue by illustrating the method we developed above for a practical problem: benchmark tracking. A benchmark is a quantity that may vary over time, possibly caused by changing asset returns. The aim of benchmark tracking is to imitate the movements of a particular benchmark with a portfolio that consists merely of financial assets at one's disposal. The level of imitation is measured by the discrepancy between the portfolio and benchmark returns, also called the tracking-error. Consequently we express the benchmark tracking problem as a tracking error minimization problem, as follows:

$$\min_{x,x'} \left\{ \mathbb{E}[(g^T r - f(r,x,x'))^2] \,\middle|\, (\sum_{i=1}^{n} x_i) + \sum_{j=1}^{m} x'_j = 1,\ (x,x') \in \Xi \right\}. \tag{31}$$

Here, $f(r,x,x')$ and $g^T r$ denote respectively the portfolio and benchmark return. The set Ξ models restrictions faced by the portfolio manager. These restrictions make it in particular impossible to invest in the index $g^T r$ itself. Observe that the benchmark in our illustration is a stock index with weights g_i, $i = 1, \ldots, n$, making the return vector r the only determinant for the benchmark value. As in (17), the portfolio return is written explicitly as

$$f(r; x, x') = x^{\mathrm{T}} r + (x')^{\mathrm{T}} r',$$

see (17). Furthermore, for all (M,N) in $\mathcal{F}$ we have

$$f(r; x, x') = x^T r + (x'_M)^T (b_M + A_M r) \text{ for } r \in P(M,N), \tag{32}$$

see (9)–(10). The constraint $(\sum_{i=1}^n x_i) + \sum_{j=1}^m x'_j = 1$ expresses the budget restriction: we invest our capital, scaled to unity.

Recall from (1) that

$$\ln(r) = \ln(\mu) + \varepsilon,\ \mathbb{E}(\varepsilon) = 0,\ \mathbb{E}(\varepsilon\varepsilon^T) = \Sigma.$$

In order to make the problem in (31) precise, we further assume in this section that ε is multivariate normally distributed. However, our approach remains valid also if a different (but specific) distribution is assumed. The objective function in (31) thus involves an n-dimensional integral. Although this integral cannot be computed exactly, it can be computed with reasonably accuracy using Monte Carlo methods. In particular, we have

$$\begin{aligned} \mathbb{E}[(g^T r - f(r,x,x'))^2] &= \mathbb{E}[\big(g^T e^{\ln(\mu)+\varepsilon} - f(e^{\ln(\mu)+\varepsilon}, x, x')\big)^2] \\ &\approx \sum_{k=1}^{\kappa} \pi_k \left(g^T e^{\ln(\mu)+\varepsilon_k} - f(e^{\ln(\mu)+\varepsilon_k}, x, x')\right)^2, \end{aligned} \tag{33}$$

where ε_k, $k = 1, 2, \dots, \kappa$ is a sample and $\sum \pi_k = 1$. We will not get into the details of the sampling technique here. The reader may simply consider as an obvious possibility a sample from $N(0, \Sigma)$ of size κ with $\pi_1 = \dots = \pi_\kappa = 1/\kappa$. To simplify notations, we define a mapping $r_k : \Re^n \to \Re^n$ as follows:

$$r_k(\mu) = e^{\ln(\mu)+\varepsilon_k} \text{ for } k = 1, 2, \dots, \kappa. \tag{34}$$

We remark that $r_k(\mu)$ is a linear function of μ, namely

$$(r_k(\mu))_i = \mu_i e^{(\varepsilon_k)_i} \text{ for } i = 1, 2, \dots, n.$$

Replacing the objective function in (31) by the right hand side in (33), we arrive at the stochastic programming formulation of (31):

$$\min\{t_0 | (x, x') \in \Xi \text{ and (36)–(39)}\} \tag{35}$$

with constraints

$$t_0 \geq \sqrt{\sum_{k=1}^{\kappa} \pi_k t_k^2} \tag{36}$$

$$t_k \geq g^T r_k(\mu) - f(r_k(\mu), x, x') \text{ for all } k = 1, \dots, \kappa \tag{37}$$

$$t_k \geq f(r_k(\mu), x, x') - g^T r_k(\mu) \text{ for all } k = 1, \dots, \kappa \tag{38}$$

$$(\sum_{i=1}^{n} x_i) + \sum_{j=1}^{m} x'_j = 1. \tag{39}$$

Besides the genuine decision variables x and x', we have incorporated auxiliary variables t_k, $k = 0, 1, \dots, \kappa$. Observe that (37)–(38) holds if and only if

$$t_k \geq |g^T r_k(\mu) - f(r_k(\mu), x, x')|$$

and hence (36)–(38) holds if and only if $t_0 \geq 0$ and

$$t_0^2 \geq \sum_{k=1}^{\kappa} \pi_k (g^T r_k(\mu) - f(r_k(\mu), x, x'))^2 \approx \mathbb{E}[(g^T r - f(r, x, x'))^2].$$

Since (36) is a second order cone constraint and (37)–(39) are linear constraints, the problem (35) can be solved using standard second order cone programming software. This is the classical stochastic programming approach [BL97].

However, there are a number of problems with this approach. First, the value of μ is not known exactly; one can merely work with an estimate $\hat{\mu}$. Second, knowledge of the randomly selected ε_ks will be misused by the optimization routine that selects x and x'. This typically makes the error in (33) larger than if the ε_ks were selected *after* x and x' are determined, although various convergence results are known for this situation [BL97]. Third, even if we could solve (31) exactly, i.e. without approximating the integral, the underlying assumption that $\varepsilon \sim N(0, \Sigma)$ is highly debatable. In summary, the above developed target tracking model lacks robustness.

In the sequel of this section, we develop a modification of (35) which is robust against uncertainty in the parameter μ. As discussed in Section 1, we construct an estimate $\hat{\mu}$ for μ based on historical data. This yields a region $\mathcal{U}$ such that $\mu \in \mathcal{U}$ with a certain confidence, see (2). Next, we replace the constraints (37)–(38) by an infinite set of robust constraints:

$$t_k \geq g^T r_k(\mu) - f(r_k(\mu), x, x') \text{ for all } \mu \in \mathcal{U} \tag{40}$$

$$t_k \geq f(r_k(\mu), x, x') - g^T r_k(\mu) \text{ for all } \mu \in \mathcal{U}, \tag{41}$$

for $k = 1, 2, \ldots, \kappa$. As discussed in Section 2, the presence of options makes these constraints piece-wise linear in μ.

Analogous to (8), we define for $(M, N) \in \mathcal{F}$,

$$P_k(M, N) = \{\mu \in \Re^n_+ | b_M + A_M r_k(\mu) > 0,\ b_N + A_N r_k(\mu) \leq 0\} \tag{42}$$

as the polyhedral set associated with the (M, N) moneyness configuration. The expected portfolio value $f(r_k(\mu), x, x')$ is therefore linear in μ for $\mu \in P_k(M, N)$. Recall from (6) that $A \in \Re^{m \times n}$ and $b \in \Re^m$ define the payoff structure of the options under consideration.

Since we are confident that $\mu \in \mathcal{U}$, it suffices to consider only those moneyness configurations for which $\mathcal{U}_k(M, N) \neq \emptyset$, where $\mathcal{U}_k(M, N) := P_k(M, N) \cap \mathcal{U}$; cf. (16). Given k, there are only very few of such moneyness configurations, especially if the sample size underlying the computation of $\hat{\mu}$ is sufficiently large (and hence the uncertainty is low). For this reason it is important to draw scenarios such that most moneyness configurations are covered. Stratified sampling offers a solution here.

The moneyness configurations $(M, N) \in \mathcal{F}$ that produce relevant subsets $\mathcal{U}(M, N)$ are defined by subsequent exercise prices of the derivatives on each asset j. Between each two subsequent exercise prices, a constant moneyness configuration applies. Therefore in order to define the relevant moneyness configurations, we sort the derivatives on each underlying (including boundaries 0, ∞) according to the derivatives exercise price in ascending order. Subsequent exercise prices X_i and X_{i+1} define a return interval for the underlying j: $\left[\frac{X_i}{S_{0,j}}, \frac{X_{i+1}}{S_{0,j}}\right]$. Only calls with $X \leq X_i$ and puts $X \geq X_{i+1}$ are in the money for this interval and are included in M. We combine the intervals of different underlying assets to form a non-empty $P_k(M, N)$. Consequently $P_k(M, N)$ as defined in (42) can be written more specifically as

$$P_k(M,N) = \left\{ \mu \in \Re^n_+ \,\middle|\, \text{for all } j = 1,\ldots,n : \frac{X^l_{M,j}}{S_{0,j}} \le r_k(\mu)_j \le \frac{X^u_{M,j}}{S_{0,j}} \right\},$$

where $X^l_{M,j}$ and $X^u_{M,j}$ denote the exercise prices of options on underlying j that specify the boundaries on r_j for a certain configuration (M, N).

For each scenario $k = 1, 2, \ldots, \kappa$, we replace (40)–(41) by

$$t_k \ \ge \ g(r_k(\mu)) - f(r_k(\mu), x, x') \text{ for all } \mu \in \mathcal{U}_k(M,N) \tag{43}$$

$$t_k \ \ge \ f(r_k(\mu), x, x') - g(r_k(\mu)) \text{ for all } \mu \in \mathcal{U}_k(M,N), \tag{44}$$

for all $(M,N) \in \mathcal{F}$ for which $\mathcal{U}_k(M,N) \ne \emptyset$.

Due to Theorem 1, relations (43)–(44) are in fact second order cone constraints. In summary, our robust target tracking stochastic programming model is reduced to the following second order cone problem:

$$\min\{t_0 | (x, x') \in \Xi \text{ and } (36),(39),\ (43)\text{–}(44)\}. \tag{45}$$

So far we have focussed on uncertainty in the parameter μ. However, one may deal with uncertainty in the parameter Σ as well, see for example [GI02]. However for mean variance problems it is shown (Korkie et al [JK81] and Michaud [Mic98]) that the uncertainty in the expected return estimate is the driving force behind the misbehavior of the Markowitz model.

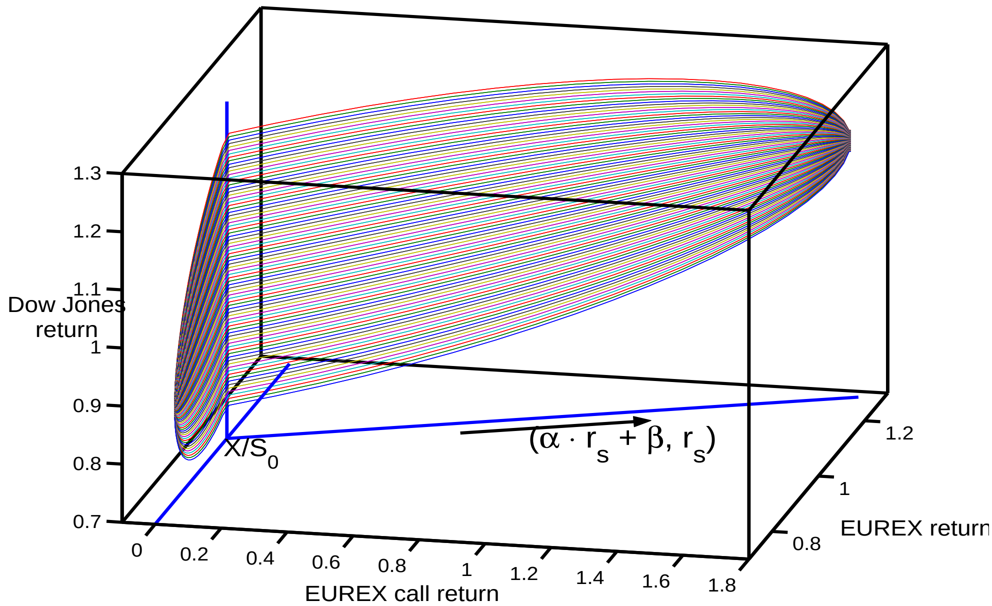

Figure 1. Uncertainty set for a world with two stock indices, Dow Jones and EUREX, and a call option on the EUREX

5. Computational study

In this section we compare the performance of the robust approach with the classical approach on real market data. The specific problem we consider is to track the Dow-Jones index with the EUREX stoxx 50 index and all options on this index. The test on real market data naturally introduces uncertainty: uncertainty of future returns. To provide sensible estimates for these future returns, we use a model to describe the return process. As we calibrate (estimate) this return process on a limited set of historical data, the parameters of the return process suffer from uncertainty. This will introduce uncertainty in the estimates of the future returns, which the robust approach will deal with.

Strictly speaking, the uncertainty is not confined to the parameters *in* the return model, but also concerns the selection of the particular return model itself. From this perspective, the results of our test will depend on the adequacy of the model we use to describe the returns. We may expect that the use of a poor return model, will affect the classical approach more than the robust approach: The parameter estimates in a poor model display large uncertainty; uncertain estimates lead to conservative strategies in the robust approach, while the classical approach does not compensate for this uncertainty. Hence using a return model inferior to the best known model may color the results somewhat in favor of the robust approach. Nevertheless, the test remains appropriate as the true return process is not known in reality and we must rely on a reasonable guess for the return process.

As in the previous section, we use a simple but common model to describe the return process with time index t:

$$\ln r_t = \ln \mu + \varepsilon_t, \qquad \varepsilon_t \sim N(0, \Sigma), \tag{46}$$

i.e. μ and Σ are assumed to be time invariant. The parameters of the model are estimated from historical data according to the maximum likelihood principle.

Our (limited) data set consists of monthly returns from March 1997 to March 2002. We start at January 2000 and use the following procedure for the test. First we estimate the return model (46) based on the last $T = 3, 6, 20$ observations. Next we formulate the portfolio optimization model. Hereto we need future return scenarios $i = 1, .., N$ and for the robust version the uncertainty in μ. We ignore the uncertainty in the covariance matrix and estimate it using the full sample. The scenarios are generated from return model (46) by a combination of random and stratified sampling such that all 'exercise intervals', i.e. the interval between subsequent exercise prices, are covered. Depending on the number of options maturing at the subsequent time period (20-43 options), this produces between 80 and 250 scenarios. The uncertainty set $\mathcal{U}$ is formed by letting $CC^T = T\Sigma^{-1}$ and the degree of robustness $\theta = 1.6$ or 2.5. This approximately corresponds with a 75% resp. 95% confidence level of the solution (assuming (46) is an appropriate description of the return process). The classical stochastic programming problem (35) has dimensions of order $\kappa \times (m + n)$, where κ is the number of scenarios, m is the number of options and n is the number of underlying stocks. In our numerical study, $n = 2$, viz. the Dow-Jones index and the EUREX stoxx 50 index. The set Ξ in (35) is defined as $\Xi = \{0\} \times \Re^{m+1}$, i.e. it is only allowed to invest in the EUREX stoxx 50 and options on it in order to track the Dow-Jones index. The robust stochastic optimization problem (45) is much larger than the classical model (dimensions up to: 15.000×10.000); it is also degenerate and sparse. We use SeDuMi 1.05 [Stu99] which exploits this sparsity to solve the problems. The final step is to evaluate the solutions of the classical and robust approach using the next periods return. By repeating this procedure for each month between January 2000 and March 2002, we get an idea of the performance of the classical and robust approach.

The historical means and variances of the indices are given in Table 1. The correlation between the indices is low, 25%, making the benchmark tracking a real challenge.

Table 1. Monthly return statistics for period March 1997 to March 2002

	Mean	Std.dev
Dow Jones return r_b	0.31%	7.47 %
EUREX return r_u	2.21%	6.06 %

As a check on the relevance of the approaches, we compare the results of the classic and robust approach to a portfolio where everything is invested in the EUREX, i.e. we try to track the Dow Jones with the EUREX index. For the period January 2000 - March 2002 this results in an average tracking error of 9.43%.

5.1. **Results.** Table 2 summarizes the test results. The first panel depicts the results for a robustness level of $\theta = 1.6$. The columns present the different portfolio, the robust portfolio, classic portfolio and the EUREX stoxx 50 only portfolio as a check. For convenience, we use the acronym TE to denote the tracking error:

$$\text{(47)} \qquad \texttt{TE} = \text{tracking error} = |g^T r - f(r, x, x')|;$$

observe that although $g^T r - f(r, x, x')$ can take positive and negative values, TE is always nonnegative. The rows provide the expected and actual results, $E(\texttt{TE})$ gives the average expected tracking error (TE) under the estimated parameters, $R(\texttt{TE})$ is the average realized (actual) tracking error and min(TE) resp. max(TE) give the smallest and largest TE in the simulation. If we use 20 observations to estimate our return process, the expected tracking errors for robust, classic and the EUREX only portfolio are resp. 8.4%, 7.4% and 9.4%. Obviously the expected TE of the classic portfolio is smaller than the restricted EUREX only portfolio and the robust portfolio that distorts the objective of minimizing the expected TE by using robust relations. The relevant question is: What happens ex post where the combination of selection of the return model, parameter uncertainty and the approach to portfolio composition play a role.Also for a comparison on real returns, the classical approach appears to be best, although the differences become smaller: tracking errors become 7.6%, 7.2% and 7.7% respectively. Somewhat surprisingly, these similar results are achieved by strikingly different portfolios. The classical portfolio invests fanatically in options (portfolio norm equals 10-20 times the budget), with figures up to 10 times the budget into options with the smallest and largest exercise prices. The reason becomes clear if we look at Figure 2 which presents the return of a typical classic portfolio. The solid line presents the portfolio return and the dotted horizontal line the expected benchmark return. The two vertical dot-dash lines on the left and the right present the extreme options exercise prices and the dotted line on the bottom depicts the distribution of the underlying EUREX stoxx 50 asset. As the objective is to minimize the expected tracking error, we want to stay close to the (expected) benchmark return for those returns that have reasonable probability, given by the distribution on the bottom. Between the two extreme options exercise limits this can be achieved by taking positions in the options such that the course of the portfolio return has a horizontal sawtooth pattern. Outside that interval, as far as there is still probability mass, the extreme options try to flatten the underlying return somewhat to overcome a too extreme course as the correlation between the underlying and the benchmark is only 25%.

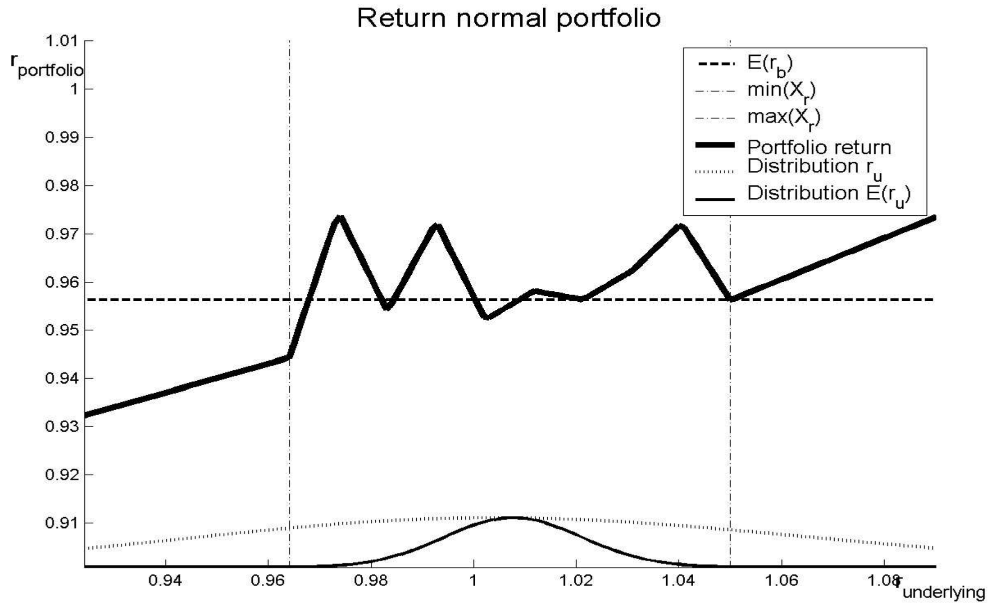

FIGURE 2. Portfolio returns

The robust portfolio invests more moderate (a portfolio norm around 4); ±90% is invested in the underlying asset and seldom more than 5% invested in each individual option. We can further stylize the portfolio if we look at Figure 3. The return profile for the robust portfolio, as given in the lower left panel, is more fluent than the one of the classical approach (lower right panel). This is due to a decrease in perseverance of the robust approach for exploiting every possibility to decrease expected tracking error, i.e. fit as good as possible at every particular point: The robust approach is interested in a good worst case performance in the small neighborhood of each scenario and indifferent about the tracking error within this neighborhood.

On some instances the return profile of the robust portfolio is roughly downward sloping. This occurs if the relatively small ($< 25\%$) correlation is dominated by the uncertainty in the mean and the relevant set of returns (returns with significant probabilities as given by the distribution on the bottom) is concentrated between the option exercise limits. This never

Test, 20 options, Jan.1991-Sept.2000	Robust Approach	Classic Approach	EUREX only
3 obs., 21 sim., theta: 1.60,			
E(TE) (std.err.) in %	9.57 (2.26)	7.43 (0.51)	10.61 (1.77)
R(TE) (std.err.) in %	8.09 (6.60)	8.58 (6.67)	7.64 (5.79)
min(TE)-max(TE) in %	0.46-26.64	0.01-22.58	0.72-22.70
Portfolio norm	36.40	28.45	1.00
Eurex investment	1.21	2.51	1.00
6 obs., 21 sim., theta: 1.60,			
E(TE) (std.err.) in %	8.58 (0.97)	7.39 (0.37)	9.44 (0.74)
R(TE) (std.err.) in %	7.35 (5.87)	7.63 (5.45)	7.64 (5.79)
min(TE)-max(TE) in %	0.67-21.45	0.37-18.97	0.72-22.70
Portfolio norm	6.37	20.32	1.00
Eurex investment	0.94	2.16	1.00
20 obs., 21 sim., theta: 1.60,			
E(TE) (std.err.) in %	8.48 (0.79)	7.42 (0.35)	9.39 (0.82)
R(TE) (std.err.) in %	7.25 (5.88)	7.12 (5.56)	7.64 (5.79)
min(TE)-max(TE) in %	0.32-21.58	0.10-19.02	0.72-22.70
Portfolio norm	11.74	19.44	1.00
Eurex investment	0.91	1.95	1.00
6 obs., 14 sim., theta: 2.50,			
E(TE) (std.err.) in %	8.48 (0.89)	7.33 (0.43)	9.47 (0.90)
R(TE) (std.err.) in %	7.22 (5.06)	6.95 (4.68)	6.88 (5.80)
min(TE)-max(TE) in %	1.80-17.44	1.30-15.59	0.72-22.70
Portfolio norm	3.99	20.18	1.00
Eurex investment	0.83	1.82	1.00
6 obs., 16 sim., theta: 2.50,			
E(TE) (std.err.) in %	8.47 (0.86)	7.47 (0.49)	9.66 (0.85)
R(TE) (std.err.) in %	6.79 (5.37)	6.83 (4.79)	6.28 (4.33)
min(TE)-max(TE) in %	0.39-15.66	0.75-16.71	0.72-16.55
Portfolio norm	10.88	23.74	1.00
Eurex investment	0.82	2.08	1.00
20 obs., 11 sim., theta: 2.50,			
E(TE) (std.err.) in %	8.13 (0.70)	7.28 (0.30)	8.57 (0.31)
R(TE) (std.err.) in %	8.39 (7.53)	7.70 (6.37)	7.56 (6.89)
min(TE)-max(TE) in %	0.17-22.46	1.20-20.47	0.72-22.70
Portfolio norm	3.34	11.18	1.00
Eurex investment	0.84	0.87	1.00

TABLE 2. Test results for robustness level $\theta = 1.6, 2.5$ and using an estimation window of length $T = 3, 6, 20$. The investment set is limited to EUREX stoxx 50 index and 20 options with exercise prices closest to expected return. Some simulations are excluded due to numerical problems during robust portfolio construction.

happens for the classical approach as the small correlation is never doubted and reflected in a somewhat upward sloping portfolio return profile.

The upper two panels of Figure 3 depict the tracking discrepancy ($\mathtt{TE} = r_b - r_P$) for various returns of the underlying and benchmark asset. Naturally the figure is upward sloping in the benchmark return r_b as large benchmark returns make TE larger. Ideally we strive to a flat surface at $\mathtt{TE} = 0$, making the tracking error zero everywhere. Unfortunately this is not achievable with the EUREX and its options. So we aim at finding a surface that is as close as possible to the flat surface at $\mathtt{TE} = 0$. Typically we want it to be close for returns with large probabilities. Of course the true probabilities are unknown and we use the classic and robust approach to deal with this.

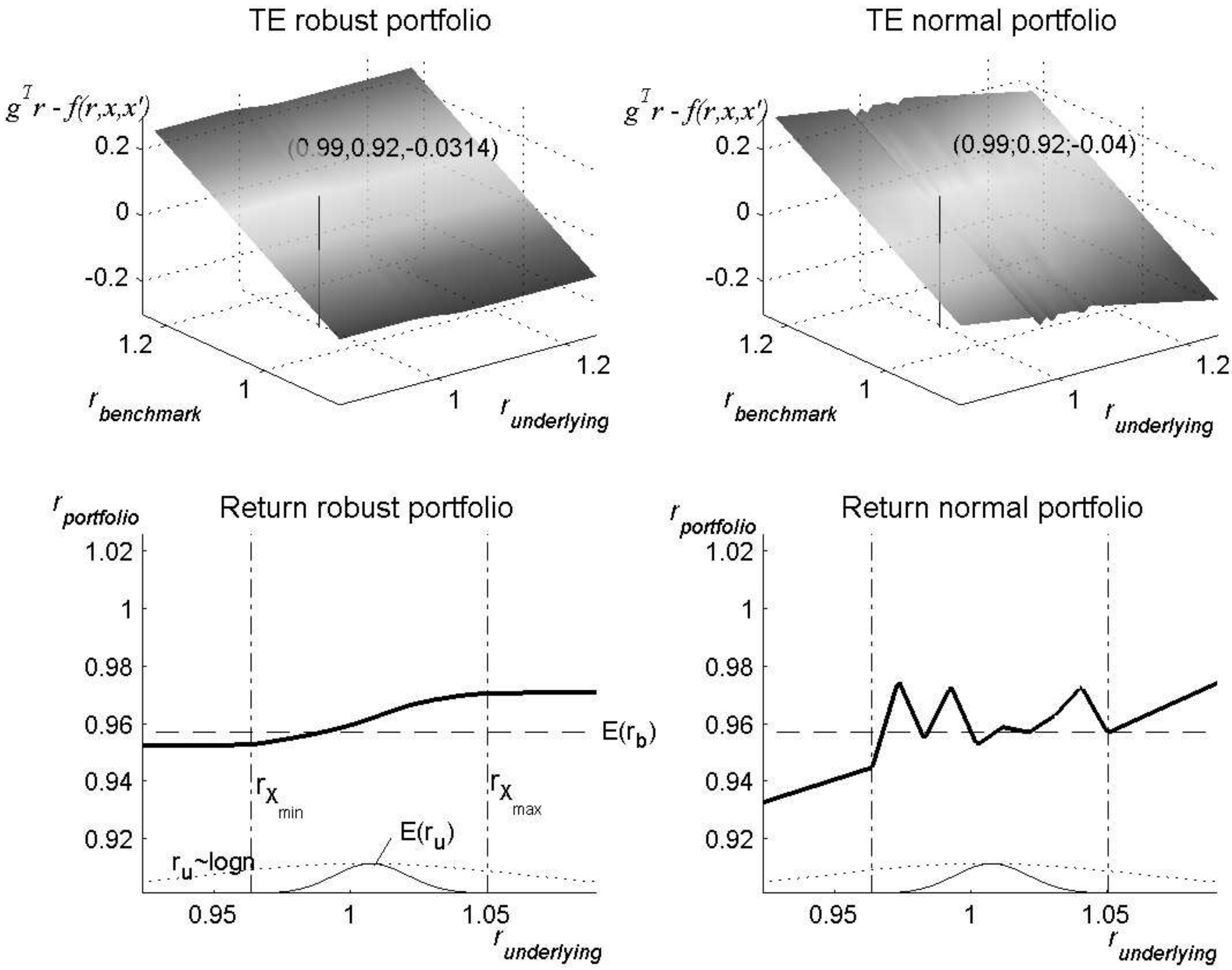

FIGURE 3. Typical TE and portfolio return distribution

The surfaces of the robust and classic approach have some minor differences. As we have seen in the portfolio profiles, the surface of the robust approach is smoother due to the worst case property. Another recurring property is that the robust approach sacrifices the situation where both returns are large. This is somewhat strange as there is a small positive correlation between the returns. However the phenomenon occurs at the boundary of the area and has small probability. So the cost of large TE's, or large local worst case TE's is small in terms of increasing *expected* TE. Moreover the robust approach artificially decreases the correlation between the benchmark and underlying's return, by using the uncertainty sets for the mean. Therefore the robust approach is more willing to sacrifice tracking precision in that corner than the classic approach. The classic approach generally sacrifices larger tracking precision in more remote areas in terms of the estimated distribution. In Figure 3 this happens for small r_b combined with large r_u and large r_b combined with small r_u. Which of the two sacrifices is better, once again depends on the correctness of the econometric model and the estimated parameters.

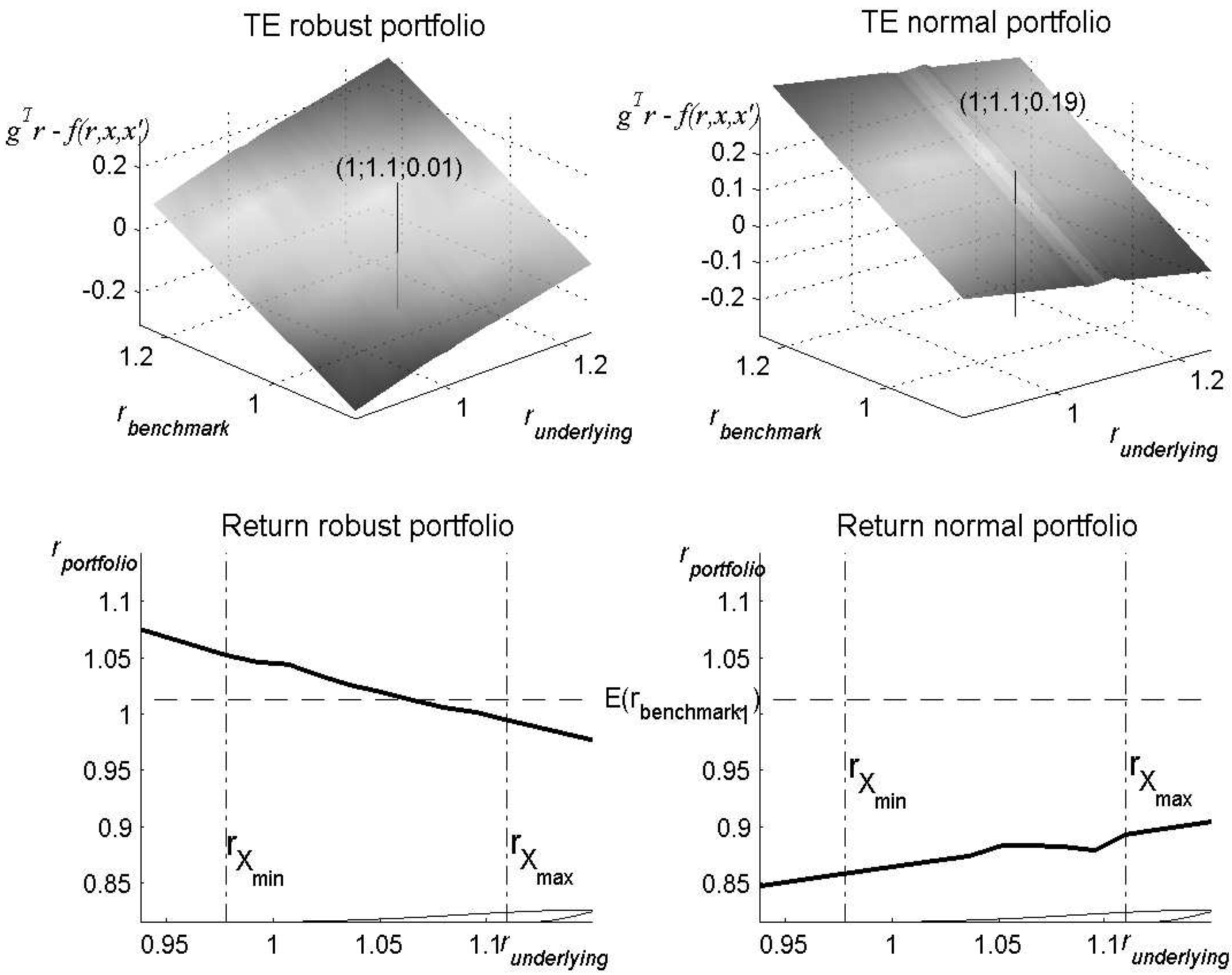

FIGURE 4. TE and portfolio return when expected return forecast is wrong

Figure 4 plots a similar graph for a hypothetical situation where uncertainty is large and the estimators were far off the true estimators. The lower panels show the errors in the estimates: the probabilities estimates of the relevant returns are almost zero. Clearly the robust approach performs much better as its portfolio is more prudent due to the large uncertainty.

We note that a somewhat similar behavior of the classical and robust portfolio follows naturally from the employed definition of uncertainty. We deduced uncertainty from the return variance. This implies that large uncertainty and large variance go hand in hand. A large variance also produces dispersed scenarios, assigning larger probabilities to outlying events. Just like the conservative robust approach, the classic approach will not risk large tracking errors for probable outlying events as this drives up the expected tracking error. Thus in cases of large uncertainty caused by large variance, the classic approach is also more prudent. The merit of the robust approach remains to label the sources of uncertainty: modelled uncertainty (given by the variance of ε) and unmodelled uncertainty, characterized by Ω.

The second panel of Table 2 depicts the results for a test with a small number of observations and thus more parameter uncertainty; we use resp. 3 or 6 observations to estimate the return process. We consider this situation for two reasons. First, if returns do indeed follow (46), this is a situation where the few number of observations makes parameter uncertainty an important issue. This means that if the robust approach contributes, this should be visible in this situation. On the other hand, we may motivate the small window estimations by the empirical phenomenon of momentum. Momentum is the persistence of returns: high returns are more likely to be followed by high return as low returns are more likely to be followed by low returns. An estimation window of 3 to 6 months, catches this sort of dynamic effect although we did not account for this in our simple return model.

The robust approach performs relatively well if only the 3 most recent observations are used for estimating the return process. However the other approaches perform worse here. One reason could be that the return process is not entirely correct and misses some dynamic effects. By using the short estimation window, we introduce this dynamic effect but also make the estimators imprecise. The robust approach can deal with this imprecision, the classic approach cannot.

However these conclusions remain premature as the simulation size is small (due to a limited dataset) and the conclusions are based on a particular return model. Further computational experiments are needed to provide reliable conclusions.

6. Discussion

The main mathematical result of this paper is a description of the cone dual to the intersection of a second order cone and linear half spaces. This description enables us to develop a formulation for the robust portfolio optimization problem with options that is efficiently solvable. In particular, for a fixed number of options the robust portfolio return relation is shown to be equivalent to a second order cone relation.

We employed the former for developing a robust version of a benchmark tracking problem including options. An empirical test for this problem shows promising results for the robust approach in situations of considerable uncertainty.

Further and current research treats the robust formulation for the multi period portfolio model with options. This demands that we can price options at intermediate time periods. Current models for option pricing (e.g. [BS73]) lack precision to blindly adopt these, causing uncertainty in the options prices. We can handle this imprecision in the (parameters of the) option pricing model in a similar way as we treat uncertainty in the return process here.

References

[AHS00] E.W. Anderson, L.P. Hansen, and T.J. Sargent. Robustness, detection and the price of risk. *Working Paper*, 2000.

[BL90] F. Black and R. Litterman. Asset allocation: Combining investor views with market equilibrium. Technical report, Goldman, Sachs & Co., Fixed Income Research, 7 1990.

[BL97] J. R. Birge and F. Louveaux. *Introduction to stochastic programming.* Springer, 1997.

[BS73] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–59, 1973.

[BTMN00] A. Ben-Tal, T. Margalit, and A. Nemirovski. Robust modeling of multi-stage portfolio problems. In *High Performance Optimization*, chapter 12. Kluwer Academic Publishers, 2000.

[BTN98] A. Ben-Tal and A. Nemirovski. Robust convex optimization. *Mathematics of Operations Research*, 23:769–805, 4 1998.

[CP02] O. Costa and A. Paiva. Robust portfolio selection using linear matrix inequalities. *Journal of Economic Dynamics and Control*, 26:889–909, 2002.

[EGOL98] L. El Ghaoui, H. Oustry, and H. Lebret. Robust solutions to uncertain semidefinite programs. *SIAM J. Optimization*, 9, 1 1998.

[GI02] D. Goldfarb and G. Iyengar. Robust portfolio selection problems. *To Appear*, 2002.

[JK81] J.D. Jobson and B. Korkie. Putting markowitz theory to work. *The Journal of Portfolio Management*, 7:70–74, 1981.

[JM02] R. Jagannathan and T. Ma. Risk reduction in large portfolios: A role for portfolio weight constraints. *SSRN*, 2002.

[Lob00] M.S. Lobo. *Robust and convex optimization with applications in finance.* PhD thesis, Stanford University, 2000.

[Mae99] P. Maenhout. *Robust portfolio rules and asset pricing.* Ph.d. thesis, Harvard University, Cambridge, MA, 1999.

[Mar52] H.M. Markowitz. Portfolio selection. *Journal of Finance*, 7(1):77–91, 1952.

[Mic98] R.O. Michaud. *Efficient Asset Management: A Pratical Guide to Stock Portfolio Optimization and Asset Allocation.* Harvard Business School Press, 1998.

[RBM00] B. Rustem, R. Becker, and W. Marty. Robust min-max portfolio strategies for rival forecast and risk scenarios. *Journal of Economic Dynamics and Control*, 24:1591–1621, 2000.

[Stu99] J.F. Sturm. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11–12:625–653, 1999. Version 1.05 available from `http://fewcal.kub.nl/sturm`.

[SZ01] J.F. Sturm and S. Zhang. On cones of nonnegative quadratic functions. *To Appear in* Mathematics of Operations Research, 2001.

[tHdRW02] J.R. ter Horst, F.A. de Roon, and B.J.M. Werker. Incorporating estimation risk in portfolio choice, 2002.

[Zen93] S.A. Zenios. *Financial Optimization.* Cambridge university press, 1993.