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By M.H. ten Raa, R.P. Gilles

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TILBURG UNIVERSITY

Lindahl Equilibrium and Schweizer's Open Club Model with Semi-Public Goods*

Thijs ten Raa† Robert P. Gilles‡

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Abstract

Limit core allocations are the ones that remain in the core of a replicated economy. An equivalent notion for economies with public goods is Schweizer's concept of club efficiency under a variable number of economic agents. We extend this notion to economies with goods that have a semi-public nature. We show that given certain conditions the equivalence of club efficient allocations and Lindahl equilibria holds for a wide range of economies with semi-public club goods. We also show that extension to a more general class of economies seems implausible.

Key words: clubs; club efficiency; Lindahl equilibrium; limit cores.

JEL codes: H41, R51, D71

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†Tilburg University, Box 90153, 5000 LE Tilburg, the Netherlands, E-mail: tenRaa@UvT.nl

‡Department of Economics (0316), Virginia Tech, Blacksburg VA 24061, USA, E-mail: rgilles@vt.edu. Part of this research was done during this author's visit to the CentER for Economic Research at Tilburg University, Tilburg, the Netherlands. Financial support from the Netherlands Organization for Scientific Research (NWO), grant B46-390, is gratefully acknowledged.

1 Club efficiency and Lindahl pricing

Lindahl equilibrium is a well-known solution concept in the general equilibrium theory of public goods, but its competitive basis has been questioned. In this paper we show that if the public goods are not pure but feature some form of rivalry in terms of opportunity costs, Lindahl pricing within a club with a variable membership base has a firm competitive basis.

We will do so in the context of Schweizer's (1983) model of club efficiency. This model assumes that the club is "open" and has a variable membership base, drawn from an unlimited pool of potential members in the background. The issue of how to partition a given population of agents in a number of clubs is not addressed. The possible variation of the numbers of consumers amounts to replication of the economy and the club efficiency of an allocation indicates that the latter remains in the core. To explain the concept, a membership profile with private and club good consumption plans (for each type of agents) is feasible if the consumption plans are producible with the initial wealth of the club members and it is club efficient if no other feasible membership profile yields higher utility to all members. Schweizer (1983) shows that a club efficient allocation must be a Walrasian equilibrium for an economy with private goods and that agents whose numbers are variable do not and should not pay for any public good. These results consolidate the limit core theorem and the Henry George theorem, respectively.

One of the problems of the original formulation of Schweizer (1983) is that the use of a pure public good is unrealistic due to the non-crowding hypothesis. In this paper we try to remedy this particular problem and introduce intermediate types of goods. We investigate when a club efficient equilibrium is a Lindahl equilibrium.

Crowding does not enter the utility functions directly. The utility of an agent depends exclusively on his or her own consumption of private goods and club goods. The degree of "publicness" of club goods is determined by the costs of production. A cost function expresses the input requirements of a membership profile (the composition of a club by type of agents) for each level of club goods consumption (possibly varying by type of agents). In the polar cases of private and public goods, the cost function would be linear or constant, respectively. The main contribution of this paper is the delineation of cost functions of club goods such that a club efficient allocation is a Lindahl equilibrium. One may expect to encounter the membership profile of such a club efficient allocation in an economy with a sea of agents, not plagued by integer problems. More interesting, the prices of a club efficient allocation are Lindahl prices. It can be shown that the Debreu-Scarf limit core property, the Walrasian equilibrium concept, and club efficiency are equivalent for economies with purely private goods.[1]

The public goods literature knows a tricky division as regards the exogeneity or endogeneity of the number of consumers and the level of the public goods. In the older literature, going back to Foley (1970), the number of consumers is fixed and the level of the public goods is variable. He defined a Lindahl equilibrium as a set of prices, economy-wide for private goods and individualized for public goods, such that markets clear. Foley proceeded to demonstrate that Lindahl equilibria are in the core. Ellickson (1973) showed that a Lindahl equilibrium need no longer be in the core when public goods are not pure, but have opportunity costs that increase with the number of consumers; he also showed that the core may even be empty. His negative results stem directly from the fact that the aggregate technology set may not be convex. We follow Ellickson in admitting non-pure public goods, but the aggregate technology will be assumed convex, keeping scope for positive results. Milleron (1972) replicated the number of consumers, which remains exogenous. The trouble with pure public goods is that they are not replicated and their per capita opportunity costs vanish. To keep the Lindahl equilibria in the core, Milleron changed the preferences or endowments of the consumers as the economy becomes large. Even then the core did not shrink to the set of Lindahl equilibria in his paper. Vasil'ev, Weber, and Wiesmeth (1995) were able to let the core shrink to the set of Lindahl equilibria, but also had to change consumers (in fact, their utility functions) as they were replicated. Conley (1994) obtained this result assuming that consumers are either asymptotically satiated or strictly nonsatiated in public goods; these are extreme polar cases of consumer utility functions. We need no such assumptions in the context of semi-public goods.

The roles of consumer numbers and public good levels were reversed in Schweizer (1983). He solved for club allocations, including a membership profile. On the other hand, he fixed the level of the public goods, not necessarily at a desired level. He provided "Lindahlian" price support of club efficient allocations, but had to assume that some types of agents are given in fixed numbers. The other types escape taxation as they can bring in more members of their types and, thus, may spread the burden of a contribution to the pure public good. We follow Schweizer in letting the numbers of consumers be free, but the public goods will be neither pure nor fixed, and be determined by preferences.

We look at the provision of *club goods* with a *semi-public* nature. Such goods are

[1]An indirect proof can be based on noting that Schweizer (1983) showed equivalence of the Walrasian equilibrium concept and club efficiency. Debreu and Scarf (1963) showed equivalence of Walrasian equilibria and the limit core. Hence, club efficiency, the Walrasian equilibrium concept, and the limit core property are the same.

commodities of an intermediate nature. It is assumed that these commodities are provided through a club, and therefore are principally locally collective. But their rivalry properties might be different from that of a purely local public good. We model this by means of a cost function that associates input requirements with members' demands for these club goods. Our main theorem states that for certain club goods with a semi-public nature the notions of club efficiency and Lindahl equilibrium remain equivalent. For this we extend Schweizer's (1983) equivalence theorem (of Walrasian equilibrium and club efficiency) to a model in which the aggregation function for the club goods has a certain specification and certain properties. We also show that it cannot be expected that our Lindahl equivalence result can be extended further to more general specifications of the aggregation function.

The second section develops the model, Section 3 states and proves our equivalence result, and Section 4 concludes the paper with a discussion of the result, its relationship to the literature, and its implications.

2 Clubs and semi-public club goods

In this section we introduce a model of a club consisting of a variable membership base, an allocation of private goods consumed, and an allocation of so-called club goods, which are provided collectively. In our theory we use a club as a replication device discussed in the previous section.

We consider an economy with a finite set of consumer types denoted by $t = 1, \ldots, T$. A vector $n \in \mathbb{R}^T_+$ represents a *profile* of a coalition of economic agents, comprising n^t members of type t. A profile $n \in \mathbb{R}^T_+$ forms the membership base of a club. Throughout we assume that agents of the same type are treated equally, i.e., agents of the same type consume the same quantities of private as well as club goods. This assumption enables us to discuss replication properly. In the standard model of a replicated pure exchange economy, the equal treatment property can be shown as a proposition (Debreu and Scarf, 1963).

We consider a situation with $\ell \in \mathbb{N}$ private goods. Agents of type t are endowed with a commodity bundle $w^t \in \mathbb{R}^\ell_+$. It is assumed that $w^t > 0$ for all t.[2] Private consumption of an agent of type t is now given by $x^t + w^t \in \mathbb{R}^\ell$ where x^t denotes the net consumption of type t. A net consumption plan is given by a vector of net consumption bundles $x = \left(x^1, \ldots, x^T\right) \in \mathbb{R}^{\ell T}$. Total consumption of private goods in a club with membership base $n \in \mathbb{R}^T_+$ is represented by $\overline{x} = \left(n^1 x^1, \ldots, n^T x^T\right) \in \mathbb{R}^{\ell T}_+$.

[2]This means $w^t \geqq 0$, but $w^t \neq 0$.

There are $m \in \mathbb{N}$ club goods. Each club good is provided collectively by the club to its members. Again assuming equal treatment, an agent of type t now consumes the club goods at levels given by the vector $y^t \in \mathbb{R}^m_+$. The consumption plan for club goods is represented by the vector $y = \left(y^1, \ldots, y^T\right) \in \mathbb{R}^{mT}_+$. Total consumption of club goods in a club with membership base $n \in \mathbb{R}^T_+$ is now represented by $\overline{y} = \left(n^1 y^1, \ldots, n^T y^T\right) \in \mathbb{R}^{mT}_+$.

The nature of the club goods is introduced through the production technology used for their creation. The production technology is represented by the induced cost function $C : \mathbb{R}^{mT}_+ \to \mathbb{R}^\ell_+$ which for every membership base $n \in \mathbb{R}^T_+$ and consumption plan $y \in \mathbb{R}^{mT}_+$ assigns to the total consumption bundle $\overline{y} = \left(n^1 y^1, \ldots, n^T y^T\right) \in \mathbb{R}^{mT}_+$ a bundle of private goods $C\left(\overline{y}\right) \in \mathbb{R}^\ell_+$ that is used to create the club goods at these levels.[3] Notice that the membership base and the consumption plan are combined into the total consumption bundle. Only when costs follow this reduced form, our analysis applies.

Still, we encompass a number of interesting cases. The club goods have a *purely private* nature if $C\left(\overline{y}\right) = \widetilde{C}\left(\sum_{t=1}^T n^t y^t\right)$, where the cost function $\widetilde{C} : \mathbb{R}^m_+ \to \mathbb{R}^\ell_+$ represents a standard private goods production technology converting the ℓ private good inputs into m private good outputs. (This reduces the model to the standard setting of a pure exchange economy.)

Second, the club goods have a *purely public* nature in the sense of Schweizer (1983) if $C\left(\overline{y}\right) = Z \in \mathbb{R}^\ell_+$, where Z is some fixed input vector.

Finally, there are many intermediate possibilities, giving the club goods a *semi-public* nature. For example, if $C(\overline{y}) = \widetilde{C}(\max_{t=1,\ldots,T} n^t y^t)$, where the max operator on $\mathbb{R}^m$ is defined by $\max_i(y^1, y^2) = \max(y^1_i, y^2_i)$, $i = 1, ..., m$ and, as before, $\widetilde{C} : \mathbb{R}^m_+ \to \mathbb{R}^\ell_+$ represents a standard private goods production technology, we can interpret the club goods to be based on a fixed infrastructure such as a network. The capacity of the network has to handle the peak demands, which in turn determines the construction costs. A contemporary example of such a situation is that of the provision of access to Internet through a so-called "Internet Service Provider" (ISP). One can interpret an ISP as a club that provides access to Internet services to their members. The cost function $\widetilde{C}$ introduced here exactly represents the cost structure for such an ISP. Capacity of the ISP's server needs to be based on peak demands for Internet access at the different time moments during a standard period of time. These time moments can be represented by the discrete parameter t.

These examples feature an important commonality, namely convexity. In the

[3] We may allow substitution of inputs by generalizing C to a correspondence.

purely public case in the sense of Schweizer, the induced cost function C is constant, which is obviously convex. In the purely private and semi-public cases, C is induced by a private goods cost function $\widetilde{C}$. If $\widetilde{C}$ is convex, as is standard in neoclassical production theory (excluding increasing returns to scale in production), then so is C in either case, as the latter is the composition of $\widetilde{C}$ and either summation (of private goods) or maximization (of semi-public goods). The latter two operations are convex and the composition of convex operations is convex.

A *club* is formally introduced as a tuple $(n^t, x^t, y^t)_{t=1,\ldots,T}$, where $n = \left(n^1, \ldots, n^T\right) \in \mathbb{R}^T_+$ is a profile, $x = \left(x^1, \ldots, x^T\right) \in \mathbb{R}^{\ell T}$ a net private consumption plan, and $y = \left(y^1, \ldots, y^T\right) \in \mathbb{R}^{mT}_+$ is a club good consumption plan. A club $(n^t, x^t, y^t)_{t=1,\ldots,T}$ is *feasible* if

$$\sum_{t=1}^{T} n^t x^t + C\left(n^1 y^1, \ldots, n^T y^T\right) \leqq 0. \tag{1}$$

Net demands for the private goods and the costs for the provision of the club goods sum to zero at most.[4] For simplicity, there is no production of private goods. Its inclusion would be a straightforward extension of the model.

A consumer of type t has an extended utility function $U^t : \mathbb{R}^\ell \times \mathbb{R}^m \to \mathbb{R}$ over his total private and club good consumption. However, since his initial endowment w^t is fixed, we may simply write $U^t(x^t, y^t)$. In principle we allow an agent to have short positions in all commodities.

Next we introduce our main efficiency concept. Consider two feasible clubs given by $(n^t, x^t, y^t)_{t=1,\ldots,T}$ and $(n^t_0, x^t_0, y^t_0)_{t=1,\ldots,T}$. The club $(n^t, x^t, y^t)_{t=1,\ldots,T}$ is an *improvement* over the club $(n^t_0, x^t_0, y^t_0)_{t=1,\ldots,T}$ if

$$U^t(x^t, y^t) > U^t(x^t_0, y^t_0)$$

whenever $n^t > 0$. If no such improvement exists for a club $(n^t_0, x^t_0, y^t_0)_{t=1,\ldots,T}$, then $(n^t_0, x^t_0, y^t_0)_{t=1,\ldots,T}$ is called *club efficient*, following Schweizer (1983).

A feasible club $(n^t_0, x^t_0, y^t_0)_{t=,\ldots,T}$ is a *Lindahl equilibrium* if there exist a private goods price vector $p \in \mathbb{R}^\ell_+$ and personalized admission price vectors $p^1, \ldots, p^T \in \mathbb{R}^m_+$ such that the following conditions are satisfied:

1. For every $t \in \{1, \ldots, T\}$ with $n^t_0 > 0$ the allocation satisfies the consumer utility maximization condition

$$(x^t_0, y^t_0) = \text{argmax } U^t(x^t, y^t) \text{ subject to } px^t + p^t y^t \leqq 0.$$

[4] We remark that Schweizer (1983) introduces a given endowment for the club, denoted by $F \geqq 0$, that covers the provision costs of the public goods and the net demands for private goods. In that case, in equation (1) the zero is replaced by F. Here we limit our discussion to the case without such an endowment.

2. The club $(n_0^t, x_0^t, y_0^t)_{t=,\dots,T}$ satisfies a budget balance condition, i.e.,

$$\sum_{t=1}^{T} n_0^t\, p^t y_0^t = p\, C\left(n_0^1 y_0^1, \dots, n_0^T y_0^T\right).$$

3. The club $(n_0^t, x_0^t, y_0^t)_{t=,\dots,T}$ is optimal in the sense that for every alternative club $(n^t, x^t, y^t)_{t=1,\dots,T}$

$$\sum_{t=1}^{T} n^t\, p^t y^t \leqq p\, C\left(n^1 y^1, \dots, n^T y^T\right).$$

By the first condition, consumers maximize their utility given the market prices for the private goods and the personal admission prices for the semi-public club goods. The fees collected cover the costs of the provision of the club goods by the second condition. The third condition stipulates that a public administration is in charge of the provision of the club goods and admission prices, and as such has the objective to maximize its "profits." (This maximal profit is zero by the second condition.) This condition is not included by the authors who consider the number of consumers exogenous. (See Foley, 1970, and others referenced in section 1.) However, since our theorem will entail that club efficiency implies Lindahl pricing, the result is only strengthened by the inclusion of the third condition in the definition of Lindahl equilibrium.

3 A decentralization result

Relatively little is assumed to arrive at complete decentralization of efficient clubs through appropriate price systems. Following Foley (1970) and Schweizer (1983), positivity of prices is ensured to render a complete decentralization through Lindahl pricing.

Axiom *There are two properties that have to be satisfied.*

(a) *For every type $t = 1, ..., T$ the utility function U^t is assumed to be continuous, quasi-concave, and strongly monotonic.*

(b) *The club good production technology has to be convex in the sense that the cost function $C : R_+^{mT} \to R_+^{\ell}$ is convex.*

In the context of this assumption we have the following result.

Theorem *Under the properties stated in the Axiom, any efficient club* $(n_0^t, x_0^t, y_0^t)_{t=1,..,T}$ *with a strictly positive endowment,* $\sum_{t=1} n_0^t w^t \gg 0$*, can be supported as a Lindahl equilibrium with strictly positive prices.*

Proof. Let the club $(n_0^t, x_0^t, y_0^t)_{t=1,\dots,T}$ be efficient.
We construct the following sets. First, for every $t \in T$ we define the preferred set,

$$B^t = \left\{ \left(x^t, 0, \dots, 0, y^t, 0, \dots, 0\right) \, \middle| \, U^t\left(x^t, y^t\right) > U^t\left(x_0^t, y_0^t\right) \right\} \subset \mathbb{R}^{\ell+mT}$$

In this definition we let y^t be at location $1+t$.
Now for any profile $n \in \mathbb{R}_+^T$ we define the preferred set,

$$B_n = \sum_{t=1}^{T} n^t B^t = \left\{ \left(\sum_{t=1}^{T} n^t x^t, n^1 y^1, \dots, n^T y^T \right) \middle| \, U^t\left(x^t, y^t\right) > U^t\left(x_0^t, y_0^t\right) \text{ for all } t \right\}.$$

Finally, we let

$$B = \cup \left\{ B_n \, \middle| \, n \in \mathbb{R}_+^T \text{ such that } n > 0 \right\} \subset \mathbb{R}^{\ell+mT}.$$

Second, we introduce the feasible set,

$$D = \left\{ \left(-C\left(n^1 y^1, \dots, n^T y^T\right) - z, n^1 y^1, \dots, n^T y^T\right) \middle| \begin{array}{c} n > 0, \, z \in \mathbb{R}_+^\ell, \\ y^1, \dots, y^T \in \mathbb{R}_+^m \end{array} \right\}.$$

We remark that also $D \subset \mathbb{R}^{\ell+mT}$.

B^t is convex by quasi-concavity of U^t for every type t. Consequently, the set B is convex. Furthermore, from continuity of U^t for every type t the set B is open as well.

We show that D is convex. Let $(y^1, \dots, y^T, z, n)$ and $(\widehat{y}^1, \dots, \widehat{y}^T, \widehat{z}, \widehat{n})$ constitute (but not be) members of D. Define $v = \left(n^1 y^1, \dots, n^T y^T\right)$ and $\widehat{v} = \left(\widehat{n}^1 \widehat{y}^1, \dots, \widehat{n}^T \widehat{y}^T\right)$. Then $(-C(v) - z, v) \in D$ as well as $(-C(\widehat{v}) - \widehat{z}, \widehat{v}) \in D$.

Now consider $\lambda \in [0,1]$. We have to show that there exists a tuple $(\tilde{y}^1, \dots, \tilde{y}^T, \tilde{z}, \tilde{n})$ such that $(-C(\tilde{v}) - \tilde{z}, \tilde{v}) \in D$ where $\tilde{v} = \left(\tilde{n}^1 \tilde{y}^1, \dots, \tilde{n}^T \tilde{y}^T\right)$, $\tilde{v} = \lambda v + (1-\lambda)\widehat{v}$, and $C(\tilde{v}) + \tilde{z} = \lambda (C(v) + z) + (1-\lambda)(C(\widehat{v}) + \widehat{z})$. This can be accomplished by selecting $\tilde{y}^t = \lambda n^t y^t + (1-\lambda)\widehat{n}^t \widehat{y}^t$ for every $t, \widetilde{n}^t = 1$, and

$$\tilde{z} = \lambda C(v) + (1-\lambda) C(\widehat{v}) - C(\tilde{v}) + \lambda z + (1-\lambda)\widehat{z}.$$

Now $\tilde{v} = \lambda v + (1-\lambda)\widehat{v}$ and by convexity of the cost function C it follows that

$$\begin{aligned} \tilde{z} &= \lambda C(v) + (1-\lambda) C(\widehat{v}) - C(\tilde{v}) + \lambda z + (1-\lambda)\widehat{z} \\ &\geqq C(\lambda v + (1-\lambda)\widehat{v}) - C(\tilde{v}) + \lambda z + (1-\lambda)\widehat{z} \\ &= \lambda z + (1-\lambda)\widehat{z}. \end{aligned}$$

Hence, $\tilde{z} \geqq 0$ and thus indeed $(-C(\tilde{v}) - \tilde{z}, \tilde{v}) \in D$, finishing the proof that D is convex.

We define the cone generated by the feasible set D by

$$\overline{D} = \{\lambda d \mid d \in D \text{ and } \lambda \geqq 0\}.$$

By convexity of D it follows that $\overline{D}$ is a convex cone.

We claim that $B \cap \overline{D} = \emptyset$.

Suppose to the contrary that $(n^t, x^t, y^t)_{t=1,\ldots,T}$ constitutes a member of B, $(\widehat{n}^t, \widehat{y}^t)_{t=1,\ldots,T}$ and $\widehat{z} \in \mathbb{R}^\ell_+$ constitute a member of D, and $\lambda \geqq 0$ such that

$$\left(\sum_{t=1}^{T} n^t x^t, n^1 y^1, \ldots, n^T y^T\right) = \left(-\lambda C\left(\widehat{n}^1 \widehat{y}^1, \ldots, \widehat{n}^T \widehat{y}^T\right) - \lambda \widehat{z}, \lambda \widehat{n}^1 \widehat{y}^1, \ldots, \lambda \widehat{n}^T \widehat{y}^T\right).$$

This implies that $\lambda \neq 0$. From the equation it now follows that $\widehat{n}^t \widehat{y}^t = \frac{n^t}{\lambda} y^t$ and that

$$\sum_{t=1}^{T} \frac{n^t}{\lambda} x^t = -C\left(\frac{n^1}{\lambda} y^1, \ldots, \frac{n^T}{\lambda} y^T\right) - \widehat{z} \leqq -C\left(\frac{n^1}{\lambda} y^1, \ldots, \frac{n^T}{\lambda} y^T\right).$$

This implies that the club $\left(\frac{n^t}{\lambda}, x^t, y^t\right)_{t=1,\ldots,T}$ is feasible and improves upon the club $(n_0^t, x_0^t, y_0^t)_{t=1,\ldots,T}$. This is a contradiction to the efficiency hypothesis.

By the separating hyperplane theorem and the fact that $\overline{D}$ is a cone and B is open, there exist $p \in \mathbb{R}^\ell_+$ and $p^1, \ldots, p^T \in \mathbb{R}^m_+$ not all equal to zero such that

$$(p, p^1, \ldots, p^T) B \geqq 0 \geqq (p, p^1, \ldots, p^T) \overline{D}. \tag{2}$$

By strong monotonicity of U^t it can be concluded that B is comprehensive, and therefore $p, p^t > 0$.[5] Also, by assumption that the aggregated total endowment is strictly positive, we may conclude that $\sum n_0^t p w^t > 0$. Thus, there is a type t with $n_0^t > 0$ and $p w^t > 0$. For this type t an interior consumption plan is feasible with respect to $p x^t + p^t y^t \leqq 0$. Hence, by strong monotonicity and continuity of U^t, using a standard argument, $p \gg 0$ as well as $p^t \gg 0$. Hence, by nonzero endowment assumption, $p w^t > 0$ for all t. By the same argument, all $p^t \gg 0$. We will now prove that these prices constitute a Lindahl equilibrium.

First, we show the consumer's utility maximization condition. Suppose that the tuple given by $(x^t, 0, \ldots, 0, y^t, 0, \ldots, 0)$ — with y^t at location $1 + t$ — satisfies $U^t(x^t, y^t) > U^t(x_0^t, y_0^t)$. In fact, since $p \gg 0$, $p w^t > 0$, and the utility function is strongly monotonic and continuous, the same holds for a pair of slightly smaller

[5] Here we define $p > 0$ if $p \geqq 0$ and $p \neq 0$.

vectors. Now from the separation property (2) and the strict positivity of all prices it is concluded that $px^t + p^t y^t > 0$.

It remains to show that (x_0^t, y_0^t) satisfies the budget condition $px_0^t + p^t y_0^t = 0$ if $n_0^t > 0$. Indeed from the feasibility condition for $(n_0^t, x_0^t, y_0^t)_{t=1,\dots,T}$ it follows that there is some $z \in \mathbb{R}_+^\ell$ such that

$$\left(\sum_{t=1}^{T} n_0^t x_0^t, n_0^1 y_0^1, \dots, n_0^T y_0^T\right) = \left(-C\left(n_0^1 y_0^1, \dots, n_0^T y_0^T\right) - z, n_0^1 y_0^1, \dots, n_0^T y_0^T\right) \in D.$$

From the separation property (2) it then follows that

$$\sum_{t=1}^{T} n_0^t p x_0^t + \sum_{t=1}^{T} n_0^t p^t y_0^t = \sum_{t=1}^{T} n_0^t \left(px_0^t + p^t y_0^t\right) \leqq 0. \tag{3}$$

By strong monotonicity $(x_0^t, 0, \dots, 0, y_0^t, 0, \dots, 0)$ belongs to the boundary of $B^t \subset B$. From (2) it immediately follows that $px_0^t + p^t y_0^t \geqq 0$. Hence, each term in (3) must be zero. Since $n_0^t \geqq 0$ for all types t it now immediately can be concluded that $px_0^t + p^t y_0^t = 0$ if $n_0^t > 0$.

Together with previously shown statement, this proves that (x_0^t, y_0^t) indeed solves the consumer's problem if $n_0^t > 0$.

Second, we consider the financial balance condition. Since, as shown above, each term in (3) must be zero, it follows immediately that

$$\sum_{t=1}^{T} n_0^t p^t y_0^t = -\sum_{t=1}^{T} n_0^t p x_0^t = pC\left(n_0^1 y_0^1, \dots, n_0^T y_0^T\right), \tag{4}$$

where the last equality reflects the fact that the feasibility constraint is binding, using strong monotonicity.

Lastly, we consider the problem of the public administration. As pointed out above, by strong monotonicity and continuity of the U^t's, $(\sum n_0^t x_0^t,\ n_0^1 y_0^1, \dots, n_0^T y_0^T)$ belongs to the boundary of B. As the feasibility constraint is binding, this implies that $(-C\left(n_0^1 y_0^1, \dots, n_0^T y_0^T\right), n_0^1 y_0^1, \dots, n_0^T y_0^T)$ belongs to the boundary of B as well. Hence from the separation property it must be priced higher than regular D-member $(-C\left(n^1 y^1, \dots, n^T y^T\right), n^1 y^1, \dots, n^T y^T)$. The value of the former is zero by the just established balance (4). Hence the value of the latter is nonpositive, or

$$\sum_{t=1}^{T} n^t p^t y^t - p\ C\left(n^1 y^1, \dots, n^T y^T\right) \leqq 0.$$

This proves that $(y_0^1, \dots, y_0^T, n_0)$ indeed solves the public administration's problem.

This completes the proof of the theorem. ■

With regard to this equivalence theorem we have the following remarks. If the population is not replicated, i.e., $n_0 = (1, \ldots, 1)$, the financial balance condition of Lindahl equilibrium can be simplified further to $\sum_{t=1}^{T} p^t y_0 = p y_0$. Hence, if only one club good is supplied and it is designated the numerair, then the admission prices or fees sum to unity.

Also we emphasize that the converse of the theorem is true, implying that it is a true equivalence result. A Lindahl equilibrium is always efficient. The proof is an easy adaptation of Schweizer's (1983) proof of his second theorem.

Finally, we remark that the implementation of more general club good cost functions is probably very hard. In the next example we consider a cost function that is more general, but fails to lead to equivalence of efficient clubs and the Lindahl equilibria. Semi-public goods, as we defined them, have a distinct structure in that only total consumption by type, $\overline{y} = (n^1 y^1, ..., n^T y^T) \in \mathbb{R}_+^{mT}$, affects their provision. In general, a club with profile n and club goods demands y may impose resource requirements in a way that is not separable by type.

Counterexample Consider an economy setting with one private and one club good, i.e., $\ell = m = 1$, and two types of consumers, i.e., $T = 2$, with the following utility functions:

$$\begin{aligned} U^1(x, y) &= \min(2x + 4, y); \\ U^2(x, y) &= \min(2x + 3, 2y). \end{aligned}$$

Now consider a production structure for the club good that does not satisfy the convexity requirement considered in our model. The cost function is given by

$$C\left(n^1, n^2, y^1, y^2\right) = \max_{t \in \{1,2\}} n^t \cdot \max_{t \in \{1,2\}} y^t.$$

This cost function can be interpreted as representing a semi-public good of which the provision is based on the maximal consumption capacity requested, where the maximal capacity is $\max n^t$.

Consider the club given by $n_0 = (1, 1)$, $x_0^1 = x_0^2 = -1$, and $y_0^1 = y_0^2 = 2$. This club is efficient, as we demonstrate now.

We show that U^2 cannot be lifted over its club level, 1, whenever $n^2 > 0, U^1 \geqq 2$ (its club level), and feasibility is fulfilled. Invoking linear homogeneity with respect to n, feasibility can be written as

$$n^1 x^1 + x^2 + \max(n^1, 1) \cdot \max(y^1, y^2) \leqq 0.$$

Hence

$$x^2 \leqq -n^1 x^1 - \max(n^1, 1) \cdot y^1.$$

Substitute $x^1 \geqq -1$ and $y^1 \geqq 2$ (both from $U^1 \geqq 2$):

$$x^2 \leqq n^1 - 2\max(n^1, 1) \leqq -1.$$

Hence $U^2(x^2, y^2) \leqq 1$ indeed, proving club efficiency, and this level is obtained only if $n^1 = 1$ and the feasibility constraint is binding:

$$x_0^1 + x_0^2 + \max(y_0^1, y_0^2) = 0.$$

Lindahl pricing by p and substituting the Lindahl break-even constraint for the semi-public goods, the sum of the consumers' budgets is zero. Since each of them is nonpositive, they are all zero. Better clubs must be priced higher, hence positively. But this is not so. Consider any club with n arbitrary, $(x^1, y^1) = (-1, 2)$ again, but $(x^2, y^2) = (-1/2, 1)$. A consumer of type 2 prefers it. This consumption bundle is half the club-efficient bundle, $(x_0^2, y_0^2) = (-1, 2)$, which has zero value, hence it is affordable. The efficient club cannot be supported as a Lindahl equilibrium. This completes the example.

4 Discussion

Our theorem provides price support to allocations that cannot be improved upon by clubs. The prices are linear, unlike Mas-Colell's (1980) personalized price schedules (also used by Gilles and Scotchmer, 1997) or Barham and Wooders' (1998) admission fees or "wages." The theorem and its proof are adaptations of Schweizer's (1983) theorem on club efficient allocations. He obtains the Henry George Theorem for economies with fixed public goods and associated inputs and, if the latter are zero, the welfare and core limit theorems. In the present paper, club goods are not exogenous, but endogenous, namely the outcome of competition among utility maximizers. Moreover, these club goods are not purely public, but semi-public.

After all, it is well known that there is no competitive basis for Lindahl equilibria in pure public goods economies (Milleron, 1972, and Bewley, 1981). Wooders (1978) has conjectured that the core shrinks when there is crowding, but Conley and Wooders (1997) show that the second welfare theorem is generally false. Barham and Wooders (1998) provides useful relationships between optima and competitive equilibria, but all these papers concern economies with only one private and one public good. More generally, Conley (1994) conjectures that the core of a public goods economy converges only in the knife-edge case in which the increasing returns to coalitional size are precisely offset by crowding, diminishing marginal returns in production, or something similar. In a sense, we have articulated this intuition. For example, if the public goods function is $C(ny) = F + (ny)^2$ (everything scalar), then club efficiency brings about the efficient scale of production, $n_0 y_0 = \sqrt{F}$. It is interesting to note, however, that our model is quite general.

An alternative modelling of an economy with multiple public goods such that the Lindahl equilibrium emerges, has been undertaken by Vasil'ev, Weber, and Wiesmeth (1995). That paper uses an alternative core definition, with utility levels of members of blocking coalitions depending on the replica size and the coalition structure. Although our approach to club goods may seem cleaner, the two approaches are closely related, in the sense that the opportunity cost of individual public — or club — goods consumption is not reduced with the size of the economy in either paper. From this perspective the contribution of our paper is the demonstration that Schweizer's theorem encompasses the core limit theorem of Vasil'ev, Weber, and Wiesmeth (1995).

The just mentioned replication literature has attempted to provide a competitive basis for Lindahl equilibria by modelling congestion on the demand side, while we have kicked the problem to the supply side. In a way this is a return to the intuition of Ellickson (1973): all that matters is the convexity of the aggregate technology set. In fact, the convexity not only ensures the belonging of the Lindahl equilibrium to the core, but also the coincidence of the two solution concepts as the number of consumers varies freely. Lindahl equilibria have a competitive basis in economies with semi-public goods.

References

[1] Barham, V., and M.H. Wooders (1998), "First and Second Welfare Theorems for Economies with Collective Goods," in D. Pines, E. Sadka and I. Zilcha (editors), *Topics in Public Economics: Theoretical and Applied Analysis*, Cambridge UP, Cambridge.

[2] Bewley, T. (1981), "A Critique of Tiebout's Theory of Local Public Expenditures," *Econometrica*, 49, 713–740.

[3] Conley, J.P. (1994), "Convergence Theorems on the Core of a Public Goods Economy: Sufficient Conditions," *Journal of Economic Theory*, 62, 161-185.

[4] Conley, J.P., and M. Wooders (1997), "Equivalence of the Tiebout Equilibrium and the Core in a Model with Crowding Types," *Journal of Urban Economics*, 41, 421–440.

[5] Debreu, G., and H. Scarf (1963), "A limit Theorem on the Core of an Economy," *International Economic Review* 4, 235–246.

[6] Ellickson, B. (1973), "A Generalization of the Pure Theory of Public Goods," *American Economic Review* 63, 417-432.

[7] Foley, D.K. (1970), "Lindahl's Solution and the Core of an Economy with Public Goods," *Econometrica* 38, 66-72.

[8] Gilles, R.P., and S. Scotchmer (1997), "Decentralization in Replicated Club Economies with Multiple Private Goods," *Journal of Economic Theory* 72, 363–387.

[9] Mas-Colell, A. (1980), "Efficiency and Decentralization in the Pure Theory of Public Goods," *Quarterly Journal of Economics* XCIV, 625–641.

[10] Milleron, J.-C. (1972), "Theory of Value with Public Goods: A Survey Article," *Journal of Economic Theory* 5, 419–477.

[11] Schweizer, U. (1983), "Efficient Exchange with a Variable Number of Consumers," *Econometrica* 51, 575–584.

[12] Vasil'ev, V.A., S. Weber, and H. Wiesmeth (1995), "Core Equivalence with Congested Public Goods," *Economic Theory* 6, 373–387.

[13] Wooders, M.H. (1978), "Equilibria, the Core, and Jurisdiction Structures in Economies with a Local Public Good," *Journal of Economic Theory* 18, 328–348.