CentER

Discussion Paper

No. 2006–28

CONVERGENCE OF ARCHIMEDEAN COPULAS

By Arthur Charpentier, Johan Segers

April 2006

ISSN 0924-7815

Tilburg University

Convergence of Archimedean Copulas

Arthur Charpentier [a]

[a] *ENSAE-CREST, Timbre J120, 3 avenue Pierre Larousse, 92240 Malakoff, France*

Johan Segers [b,1]

[b] *Tilburg University, P.O. Box 90153, NL-5000 LE Tilburg, the Netherlands*

Abstract

Convergence of a sequence of bivariate Archimedean copulas to another Archimedean copula or to the comonotone copula is shown to be equivalent with convergence of the corresponding sequence of Kendall distribution functions. No extra differentiability conditions on the generators are needed.

JEL: C14, C16

Key words: Archimedean copula, generator, Kendall distribution function

1 Introduction

Let C_n be a sequence of bivariate Archimedean copulas with generators ψ_n and Kendall distribution functions K_n. In this note, we establish necessary and sufficient conditions in terms of ψ_n or K_n for convergence of the sequence of copulas C_n to a limiting copula C which is either Archimedean or comonotone. In particular, we extend results in Genest and MacKay (1986, Proposition 4.2 and 4.3) and Nelsen (1999, Theorems 4.4.7 and 4.4.8) to generators that are possibly not everywhere differentiable. Moreover, we show that convergence of the sequence of copulas is equivalent to convergence of the corresponding sequence of Kendall distribution functions.

Email addresses: `arthur.charpentier@ensae.fr` (Arthur Charpentier), `jsegers@uvt.nl` (Johan Segers).

[1] Supported by a VENI grant of the Netherlands Organization for Scientific Research (NWO).

The link between Archimedean copulas and their Kendall distribution functions has already been exploited in the context of statistical inference on Archimedean copulas (Genest and Rivest, 1993; Barbe et al., 1996; Wang and Wells, 2000; Genest et al., 2006). Our findings add perspective to these papers by showing that closeness of Kendall distribution functions really implies closeness of the corresponding Archimedean copulas.

The structure of the paper is as follows. We start with some preliminaries in section 2. In section 3, we extend to the case of general Archimedean copulas C the expression in Genest and MacKay (1986, Proposition 3.3) for the joint distribution function of the pair of random variables $(X, C(X, Y))$, where (X, Y) is itself a random pair with distribution function C. The main results in this paper involve characterizations for the convergence of a sequence of Archimedean copulas to another Archimedean copula or to the comonotone copula (sections 4 and 5). Extensions to higher dimensions are treated in section 6. We conclude in section 7 with a counterexample showing that not every limit copula of a sequence of Archimedean copulas is necessarily Archimedean or comonotone.

2 Preliminaries

A function $C : [0, 1]^2 \to [0, 1]$ is called a *(bivariate) copula* if it is the restriction to the domain $[0, 1]^2$ of a bivariate distribution function with uniform margins on $[0, 1]$. A function $\psi : [0, 1] \to [0, \infty]$ is called a *generator* if it is convex, decreasing and $\psi(1) = 0$. The generalized inverse of ψ is denoted by

$$\psi^{\leftarrow}(t) = \inf\{u \in [0, 1] \mid \psi(u) \leq t\}, \qquad t \in [0, \infty].$$

A copula C is called *Archimedean* if there exists a generator ψ such that

$$C(u, v) = \psi^{\leftarrow}\{\psi(u) + \psi(v)\}, \qquad (u, v) \in [0, 1]^2$$

(Schweizer and Sklar, 1983; Genest and MacKay, 1986). The conditions imposed on ψ are necessary and sufficient for the expression in the previous display to define a copula. The copula C determines the generator ψ uniquely up to a multiplicative constant.

The class of Archimedean copulas encompasses many well known bivariate parametric distributions, such as the Frank, Clayton or Gumbel copulas (Nelsen, 1999, Table 4.1). Furthermore, if the inverse of the generator, $\psi^{\leftarrow}$, is the Laplace transform of a nonnegative random variable, then the corresponding Archimedean copula C reduces to the proportional frailty model in Marshall and Olkin (1988); Oakes (1989).

If the generator is a natural way to identify the Archimedean copula, other functions can be considered as well. The *Kendall distribution function* K of a copula C is defined as the distribution function of the random variable $C(X,Y)$, where (X,Y) is a random pair with distribution function C, so

$$K(t) = \Pr[C(X,Y) \le t], \qquad t \in [0,1].$$

If the copula C is Archimedean with generator ψ, then $K(t) = t - \lambda(t)$ with $\lambda(t) = \psi(t)/\psi'(t)$ and ψ' is the right-hand derivative of ψ on $[0,1)$ (Genest and Rivest, 1993, Proposition 1.1). Conversely, from K or λ it is possible to reconstruct ψ up to a multiplicative constant via

$$\psi(u) = \psi(u_0) \exp\left(\int_{u_0}^{u} \frac{1}{\lambda(t)} \mathrm{d}t \right)$$

for $0 < u_0 < 1$ and $0 \le u \le 1$.

3 Auxiliary result

The following result is a useful device to deduce properties of the generator ψ of an Archimedean copula C from the copula itself. For twice continuously differentiable generators, the result can already be found in Genest and MacKay (1986, Proposition 3.3).

Proposition 1 *Let (X,Y) be a random pair with joint distribution function C, a bivariate Archimedean copula with generator ψ. Let ψ' be the right-hand derivative of ψ on $[0,1)$. Put $Z = C(X,Y)$. For $(z,x) \in [0,1]^2$,*

$$\Pr[X \le x,\, Z \le z] = \begin{cases} x & \text{if } x \le z \le 1, \\ z + \dfrac{\psi(x)}{\psi'(z)} - \dfrac{\psi(z)}{\psi'(z)} & \text{if } 0 < z < x \le 1, \\ \dfrac{\psi(x) - \psi(0)}{\psi'(0)} & \text{if } z = 0 < x \text{ and } \psi'(0) > -\infty, \\ 0 & \text{if } z = 0 < x \text{ and } \psi'(0) = -\infty. \end{cases}$$

Proof. Since $Z = C(X,Y) \le X$, we have $\Pr[X \le x,\, Z \le z] = \Pr[X \le x] = x$ for $x \le z \le 1$. Hence we can restrict attention to $z < x$.

The case $z = 0 < x$ follows from the case $0 < z < x$ by the fact that $\psi'(0) = \lim_{z \downarrow 0} \psi'(z)$ and the fact that $\lim_{z \downarrow 0} \psi(z)/\psi'(z) = 0$ if $\psi'(z) = -\infty$, the latter property following from convexity.

Hence we can restrict attention to the case $0 < z < x$. Since both ψ' and the function $z \mapsto \Pr[X \le x,\, Z \le z]$ are right-continuous, it suffices to prove the stated equality for z such that ψ' is continuous in z.

We have

$$
\begin{aligned}
\Pr[X \le x,\, Z \le z] &= \Pr[X \le z] + \Pr[z < X \le x,\, Z \le z] \\
&= z + \Pr[z < X \le x,\, Z \le z]
\end{aligned}
$$

We can focus on the last term on the right-hand side. Let n be a positive integer, and let

$$z = u_0 < u_1 < \cdots < u_n = x$$

be such that

$$\psi(u_i) = \left(1 - \frac{i}{n}\right)\psi(z) + \frac{i}{n}\psi(x), \qquad i = 0, 1, \ldots, n.$$

We have

$$\Pr[z < X \le x,\, Z \le z] = \sum_{i=1}^{n} \Pr[u_{i-1} < X \le u_i,\, Z \le z].$$

If $u_{i-1} < X \le u_i$, then $C(u_{i-1}, Y) \le Z \le C(u_i, Y)$. Hence

$$
\begin{aligned}
&\sum_{i=1}^{n} \Pr[U_{i-1} < X \le u_i,\, C(u_i, Y) \le z] \\
&\le \Pr[z < X \le x,\, Z \le z] \le \sum_{i=1}^{n} \Pr[u_{i-1} < X \le u_i, C(u_{i-1}, Y) \le z].
\end{aligned}
$$

Further, for $z \le u \le 1$, since ψ and $\psi^{\leftarrow}$ are decreasing, $C(u, Y) \le z$ is equivalent to $Y \le \psi^{\leftarrow}\{\psi(z) - \psi(u)\}$. We find that

$$
\begin{aligned}
&\Pr[z < X \le x,\, Z \le z] \\
&\le \sum_{i=1}^{n} \Pr[u_{i-1} < X \le u_i,\, Y \le \psi^{\leftarrow}\{\psi(z) - \psi(u_{i-1})\}] \\
&= \sum_{i=1}^{n} \left(C(u_i, \psi^{\leftarrow}\{\psi(z) - \psi(u_{i-1})\}) - C(u_{i-1}, \psi^{\leftarrow}\{\psi(z) - \psi(u_{i-1})\})\right) \\
&= \sum_{i=1}^{n} \left(\psi^{\leftarrow}\{\psi(u_i) + \psi(z) - \psi(u_{i-1})\} - \psi^{\leftarrow}\{\psi(z)\}\right).
\end{aligned}
$$

Our choice of the grid $\{u_i\}$ is such that

$$\psi(u_i) - \psi(u_{i-1}) = -\{\psi(z) - \psi(x)\}/n, \qquad i = 1, \ldots, n.$$

Hence

$$\Pr[z < X \leq x,\, Z \leq z] \leq n\Big(\psi^{\leftarrow}[\psi(z) - \{\psi(z) - \psi(x)\}/n] - \psi^{\leftarrow}\{\psi(z)\}\Big)$$

Since $\psi^{\leftarrow}$ is convex with nondecreasing derivative $1/(\psi' \circ \psi^{\leftarrow})$,

$$\psi^{\leftarrow}(a) - \psi(b) \leq (a - b)\frac{1}{\psi'\{\psi^{\leftarrow}(a)\}}, \qquad 0 < a < b < \psi(0).$$

Combine the two previous displays to find

$$\Pr[z < X \leq x,\, Z \leq z] \leq -\frac{\psi(z) - \psi(x)}{\psi'(\psi^{\leftarrow}[\psi(z) - \{\psi(z) - \psi(x)\}/n])}$$

Let n tend to infinity and use the fact that z is a continuity point of ψ' to find

$$\Pr[z < X \leq x,\, Z \leq z] \leq -\frac{\psi(z) - \psi(x)}{\psi'(z)}.$$

The inequality in the other direction follows in a similar fashion. We give the steps here in full. By the same arguments as above,

$$\begin{aligned}
&\Pr[z < X \leq x,\, Z \leq z] \\
&\geq \sum_{i=1}^{n} \Pr[u_{i-1} < X \leq u_i,\, Y \leq \psi^{\leftarrow}\{\psi(z) - \psi(u_i)\}] \\
&= \sum_{i=1}^{n} (C(u_i, \psi^{\leftarrow}\{\psi(z) - \psi(u_i)\}) - C(u_{i-1}, \psi^{\leftarrow}\{\psi(z) - \psi(u_i)\})) \\
&= \sum_{i=1}^{n} (\psi^{\leftarrow}\{\psi(z)\} - \psi^{\leftarrow}\{\psi(u_{i-1}) + \psi(z) - \psi(u_i)\}) \\
&= n\Big(\psi^{\leftarrow}\{\psi(z)\} - \psi^{\leftarrow}[\psi(z) + \{\psi(z) - \psi(x)\}/n]\Big) \\
&\geq -\frac{\psi(z) - \psi(x)}{\psi'(\psi^{\leftarrow}[\psi(z) + \{\psi(z) - \psi(x)\}/n])}.
\end{aligned}$$

Let n tend to infinity and use the fact that z is a continuity point of ψ' to arrive at

$$\Pr[z < X \leq x,\, Z \leq z] \geq -\frac{\psi(z) - \psi(x)}{\psi'(z)},$$

as required. □

4 Convergence to Archimedean copula

In this section, we investigate necessary and sufficient properties of a sequence of Archimedean copulas C_n with generators ψ_n to converge to an Archimedean copula C with generator ψ. Let ψ_n' and ψ be the right-hand derivatives of ψ_n and ψ, respectively, and denote $\lambda_n = \psi_n/\psi_n'$, $\lambda = \psi/\psi'$, $K_n(t) = t - \lambda_n(t)$, and $K(t) = t - \lambda(t)$.

For twice continuously differentiable generators, the equivalence of (i) and (ii) in Proposition 2 below was already established in Genest and MacKay (1986, Proposition 4.2). The claim that characterizations (iii) and (v) are sufficient for copula convergence seems to be new.

Proposition 2 *The following five conditions are equivalent:*

(i) $\lim_{n\to\infty} C_n(x,y) = C(x,y)$ *for all* $(x,y) \in [0,1]^2$
(ii) $\lim_{n\to\infty} \psi_n(x)/\psi_n'(y) = \psi(x)/\psi'(y)$ *for every* $x \in (0,1]$ *and* $y \in (0,1)$ *such that* ψ' *is continuous in* y.
(iii) $\lim_{n\to\infty} \lambda_n(x) = \lambda(x)$ *for every* $x \in (0,1)$ *such that* λ *is continuous in* x.
(iv) There exist positive constants κ_n *such that* $\lim_{n\to\infty} \kappa_n\psi_n(x) = \psi(x)$ *for all* $x \in [0,1]$.
(v) $\lim_{n\to\infty} K_n(x) = K(x)$ *for every* $x \in (0,1)$ *such that* K *is continuous in* x.

Proof. *(i) implies (ii).* Let (X,Y) and (X_n,Y_n) be pairs of random variables with joint distribution functions C and C_n, respectively. Also, put $Z = C(X,Y)$ and $Z_n = C_n(X_n,Y_n)$. By (i), (X_n,Y_n) converges in distribution to (X,Y) as $n \to \infty$. Moreover, since C is a continuous distribution function, the convergence of C_n to C is necessarily uniform in $(x,y) \in [0,1]^2$. Hence (X_n,Z_n) converges in distribution to (X,Z) as $n \to \infty$. By Proposition 1, we have

$$\lim_{n\to\infty} \frac{\psi_n(x) - \psi_n(y)}{\psi_n'(y)} = \frac{\psi(x) - \psi(y)}{\psi'(y)}$$

for all $0 < y < x \le 1$ such that ψ' is continuous in y. Choose $x = 1$ to find

$$\lim_{n\to\infty} \frac{\psi_n(y)}{\psi_n'(y)} = \frac{\psi(y)}{\psi'(y)}.$$

Combine the two previous displays to get

$$\lim_{n\to\infty} \frac{\psi_n(x)}{\psi_n'(y)} = \frac{\psi(x)}{\psi'(y)}$$

for every $0 < y \le x \le 1$ such that $y < 1$ and ψ' is continuous in y. Let $0 < x_i < 1$ for $i = 1,2$ and apply the above display to (x_1,y) and (x_2,y) for some $0 < y < \min(x_1,x_2)$ in which ψ' is continuous to arrive at

$$\lim_{n\to\infty} \frac{\psi_n(x_1)}{\psi_n(x_2)} = \frac{\psi(x_1)}{\psi(x_2)}.$$

Combine the last two displays to arrive at (ii).

(ii) implies (iii). Trivial.

(iii) implies (iv). For $0 < x < y < 1$, we have

$$\log \psi_n(y) - \log \psi_n(x) = \int_x^y \frac{\psi_n'(z)}{\psi_n(z)} \mathrm{d}z.$$

Suppose that we can show that the limit of the integral of the right-hand side of the previous display is equal to the integral of the (almost everywhere) limit of the integrand. Then we have

$$\lim_{n\to\infty} \{\log \psi_n(y) - \log \psi_n(x)\} = \log \psi(y) - \log \psi(x).$$

This, in turn, obviously implies (iv).

In order to justify interchanging limit and integral in the previous paragraph, we will show that (iii) implies

$$\limsup_{n\to\infty} \sup_{z\in[x,y]} \left| \frac{\psi_n'(z)}{\psi_n(z)} \right| < \infty.$$

Let $0 < \varepsilon < x$ be such that

$$|\psi'(x-\varepsilon)| \le \psi(y)/(4\varepsilon).$$

By (iii), we have

$$\lim_{n\to\infty} \mathbf{1}\left(\frac{|\psi_n'(z)|}{\psi_n(z)} > 2\frac{|\psi'(z)|}{\psi(z)} \right) = 0$$

for almost every $z \in [x-\varepsilon, y]$. Since the above indicator variables are bounded and converge pointwise to zero, there exists a positive integer n_ε such that

$$\int_{x-\varepsilon}^y \mathbf{1}\left(\frac{|\psi_n'(z)|}{\psi_n(z)} > 2\frac{|\psi'(z)|}{\psi(z)} \right) \mathrm{d}z < \varepsilon$$

for all integer $n \ge n_\varepsilon$. Hence, for $z \in [x, y]$ and integer $n \ge n_\varepsilon$, there exist $z - \varepsilon < u < z$ such that

$$\frac{|\psi_n'(u)|}{\psi_n(u)} \le 2\frac{|\psi'(u)|}{\psi(u)} \le 2\frac{|\psi'(x-\varepsilon)|}{\psi(y)} \le \frac{1}{2\varepsilon}$$

But then, since ψ_n and $|\psi_n'|$ are both nonincreasing,

$$\frac{\psi_n(z)}{|\psi_n'(z)|} \ge \frac{\psi_n(u) - (z-u)|\psi_n'(u)|}{|\psi_n'(u)|} \ge 2\varepsilon - \varepsilon = \varepsilon,$$

as required.

(iv) implies (i). Let $\phi_n = \kappa_n \psi_n$. Then ϕ_n is a generator of C_n. Since each ϕ_n is monotone and since ψ is monotone and continuous, we have $\lim_{n\to\infty} \phi_n(x_n) =$

$\psi(x)$ whenever $\lim_{n\to\infty} x_n = x$ in $[0,1]$. Hence also $\lim_{n\to\infty} \phi_n^{\leftarrow}(t_n) = \psi^{\leftarrow}(t)$ whenever $\lim_{n\to\infty} t_n = t$ in $[0,\infty]$. Hence, for every $(x,y) \in [0,1]^2$,

$$C_n(x,y) = \phi_n^{\leftarrow}\{\phi_n(x) + \phi_n(y)\} \to \psi^{\leftarrow}\{\psi(x) + \psi(y)\} = C(x,y),$$

as $n \to \infty$.

(v) implies (iii) and conversely. Trivial. □

5 Convergence to comonotone copula

The comonotone copula is itself not an Archimedean copula, so that Proposition 2 is not suitable for deciding whether a sequence of copulas converges to the comonotone copula. The following resulting, extending Nelsen (1999, Theorem 4.4.8) to arbitrary generators, gives such a criterion. Let again C_n (positive integer n) be a sequence of bivariate Archimedean copulas with generators ψ_n. Let ψ_n' be the right-hand derivative of ψ_n, and denote $\lambda_n = \psi_n/\psi_n'$ and $K_n(t) = t - \lambda_n(t)$.

Proposition 3 *The following four conditions are equivalent:*

(i) $\lim_{n\to\infty} C_n(x,y) = \min(x,y)$ *for all* $(x,y) \in [0,1]^2$
(ii) $\lim_{n\to\infty} \lambda_n(x) = 0$ *for every* $x \in (0,1)$.
(iii) $\lim_{n\to\infty} \psi_n(y)/\psi_n(x) = 0$ *for every* $0 \le x < y \le 1$.
(iv) $\lim_{n\to\infty} K_n(x) = x$ *for every* $x \in (0,1)$.

Proof. *(i) implies (ii).* Let (X_n, Y_n) be a pair of random variables with distribution function C_n. Since the limit of C_n is the comonotone copula, (X_n, Y_n) converges in distribution to (X, X), where X is a uniform random variable on $(0,1)$. But since the convergence in (i) is necessarily uniform, we find that $Z_n = C_n(X_n, Y_n)$ converges in distribution to $\min(X,X) = X$, whence $\lim_{n\to\infty} \Pr[Z_n \le z] = z$ for all $z \in [0,1]$. But by Proposition 1,

$$\Pr[Z_n \le z] = z + \frac{\psi_n(z)}{\psi_n'(z)}, \qquad 0 < z < 1.$$

Hence we arrive at (ii).

(ii) implies (iii). Let $0 < x < y < 1$ (the cases $x = 0$ or $y = 1$ follow by monotonicity of ψ_n). We have

$$\frac{\psi_n(x)}{\psi_n(y)} - 1 = \frac{\psi_n(x) - \psi_n(y)}{\psi_n(y)} \ge \frac{(y-x)|\psi_n'(y)|}{\psi_n(y)}.$$

By (ii), the right-hand side diverges to infinity as $n \to \infty$.

(iii) implies (i). Since each C_n is a symmetric copula, it suffices to consider $0 < x \le y < 1$. Take $0 < w < x$. By (ii), we have $\psi_n(w) \ge 2\psi_n(x) \ge \psi_n(x) + \psi_n(y)$ for all sufficiently large integer n, whence

$$w \le \psi_n^{\leftarrow}\{\psi_n(x) + \psi_n(y)\} = C_n(x, y) \le x.$$

Let first $n \to \infty$ and then $w \uparrow x$ to find that $\lim_{n\to\infty} C_n(x, y) = x$. □

6 Extension to higher dimensions

Propositions 2 and 3 can be readily extended to the general multivariate case. Let d be an integer at least two. A d-variate copula C is the distribution function of a d-variate random vector $(X_1, \ldots, X_d)$,

$$C(x_1, \ldots, x_d) = \Pr[X_1 \le x_1, \ldots, X_d \le x_d]$$

the components of which are uniformly distributed on the interval $[0, 1]$, that is, $\Pr[X_j \le x] = x$ for $j = 1, \ldots, d$ and $x \in [0, 1]$. A d-variate copula C is called Archimedean if there exists a generator ψ such that

$$C(x_1, \ldots, x_d) = \psi^{\leftarrow}\{\psi(x_1) + \cdots + \psi(x_d)\}$$

for all $(x_1, \ldots, x_d) \in [0, 1]^d$. In general, extra conditions on the generator ψ are required to ensure that the expression in the above display defines a genuine copula. A sufficient condition is for instance that $\psi^{\leftarrow}$ is d-times differentiable and $(-D)^j\psi^{\leftarrow} \ge 0$ for every $j = 1, \ldots, d$; see for instance Kimberling (1974, Theorems 1 and 2), Schweizer and Sklar (1983), Barbe et al. (1996, Example 3), and Nelsen (1999, Section 4.6).

Obviously, if the distribution function of the random vector $(X_1, \ldots, X_d)$ is given by the d-variate Archimedean copula C with generator ψ, then the distribution function of every bivariate subvector (X_i, X_j), with $i \ne j$, is given by the bivariate Archimedean copula with the same generator. This property can be used to upgrade Propositions 2 and 3 to the general multivariate case.

Let C_n be a sequence of d-variate Archimedean copulas with generators ψ_n. On the one hand, if C_n converges to another d-variate Archimedean copula C with generator ψ or to the d-variate comonotone copula, then the sequence of bivariate Archimedean copulas with generators ψ_n must converge to the bivariate Archimedean copula with generator ψ or to the bivariate comonotone copula, respectively. Hence, the stated conditions on the sequence of generators are certainly necessary for convergence of the sequence of copulas. On the other hand, they are also sufficient, as the proofs of the implications "(iv) implies (i)" in Proposition 2 and "(iii) implies (i)" in Proposition 3 carry over to the d-variate case with only notational changes.

7 Counterexample

From Propositions 2 and 3, one might get the impression that every limit copula of a sequence of Archimedean copulas is necessarily Archimedean or comonotone. This is not true, as is demonstrated by the following example.

For integer $n \geq 2$, define a generator ψ_n by

$$\psi_n(x) = \begin{cases} n - 2(n-1)x & \text{if } 0 \leq x \leq 1/2, \\ 2(1-x) & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

That is, ψ_n is piecewise linear with knots $\psi_n(0) = n$, $\psi_n(1/2) = 1$, and $\psi_n(1) = 0$. Denoting the right-hand derivative of ψ_n with ψ_n', we have

$$\lambda_n(x) = \frac{\psi_n(x)}{\psi_n'(x)} = \begin{cases} x - n/\{2(n-1)\} & \text{if } 0 \leq x < 1/2, \\ x - 1 & \text{if } 1/2 \leq x \leq 1, \end{cases}$$

and therefore

$$K_n(x) = x - \lambda_n(x) = \begin{cases} n/\{2(n-1)\} & \text{if } 0 \leq x < 1/2, \\ 1 & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

Let C_n be the Archimedean copula with generator ψ_n. By direct computation, one arrives at

$$\lim_{n\to\infty} C_n(x,y) = C(x,y) = \begin{cases} (x+y-1/2)_+ & \text{if } (x,y) \in [0,1/2]^2, \\ x & \text{if } 0 \leq x < 1/2 < y \leq 1, \\ y & \text{if } 0 \leq y < 1/2 < x \leq 1, \\ 1/2 + (x+y-3/2)_+ & \text{if } (x,y) \in [1/2,1]^2. \end{cases}$$

The copula C corresponds to the uniform distribution, with respect to one-dimensional Lebesgue measure, on the union of the two line segments $\{(x,y) \in [0,1]^2 \mid x+y = 1/2\}$ and $\{(x,y) \in [0,1]^2 \mid x+y = 3/2\}$. The copula C is not Archimedean, because the function

$$\lim_{n\to\infty} \frac{|\psi_n'(x)|}{\psi_n(x)} = \begin{cases} 1/(1/2-x) & \text{if } 0 \leq x < 1/2, \\ 1/(1-x) & \text{if } 1/2 \leq x < 1 \end{cases}$$

is not integrable around $x = 1/2$. Note also that K_n converges towards K as

n goes to infinity, where

$$K(x) = \begin{cases} 1/2 & \text{if } 0 \leq x < 1/2, \\ 1 & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

Hence, $\lim_{x \uparrow 1/2} K(x) = 1/2$, and from Genest and Rivest (1993, Proposition 1.2), the associated copula cannot be Archimedean.

References

Barbe, Ph., Genest, C., Ghoudi, K., Rémillard, B., 1996. On Kendall's process. Journal of Multivariate Analysis 58, 197–229.

Genest, C., MacKay, R.J, 1986. Copules archimédiennes et familles de lois bidimensionnelles dont les marges sont données. La revue canadienne de statistique 14, 145–159.

Genest, C., Quessy, J.-F., Rémillard, B., 2006. Goodness-of-fit procedures for copula models based on the probability integral transformation. Scandinavian Journal of Statistics 33, 337–366.

Genest, C., Rivest, L.P., 1993. Statistical Inference Procedures for Bivariate Archimedean Copulas. Journal of the American Statistical Association, 88, 1034-1043.

Juri, A., Wüthrich, M. V., 2002. Copula convergence theorems for tail events. Insurance: Mathematics and Economics 30, 411–427.

Kimberling, C.H., 1974. A probabilistic interpretation of complete monotonicity. Aequationes Math. 10, 152–164.

Ling, C.H., 1965, Representation of associative functions. Publicationes Mathematicae Debrecen, 12, 189-212.

Marshall, A.W., Olkin, I, 1988. Families of multivariate distributions. Journal of the American Statistical Association, 83, 834-841.

Nelsen, R. B., 1999. An Introduction to Copulas. Lecture Notes in Statistics 139, Springer, New York.

Oakes, D., 1989. Bivariate survival models induced by frailties. Journal of the American Statistical Association, 84, 487-493.

Schweizer, B., Sklar, A., 1983. Probabilistic Metric Spaces. North-Holland, New York.

Wang, W., Wells, M.T., 2000. Model Selection and Semiparametric Inference for Bivariate Failure-Time Data. Journal of the American Statistical Association, 95, 62-72.