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TAKE-AND-GUESS GAMES

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Take-and-Guess Games*

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Abstract

This paper studies two classes of two-person zero-sum games in which the strategies of both players are of a special type. Each strategy can be split into two parts, a *taking* and a *guessing* part. In these games two types of asymmetry between the players can occur. In the first place, the number of objects available for taking does not need to be the same for both players. In the second place, the players can be guessing sequentially instead of simultaneously; the result is asymmetric information. The paper studies the value and equilibria of these games, for all possible numbers of objects available to the players, for the case with simultaneous guessing as well as for the variant with sequential guessing.

Keywords: zero-sum games, morra, coin-guessing, asymmetric information.
JEL code: C72.

1 Introduction

Object of study are two classes of take-and-guess games. In both classes of games, each of the two players (I and II) has to take a number of objects out of a given private finite set of objects. After that, they both have to guess the total amount of objects taken by both players. For the objects, one can think of fingers, coins or matches. Player 1 has $m \in \mathbb{N}$ objects available: he can take any number in $\{0, 1, \ldots, m\}$. His opponent has n objects available. The values of m and n are common knowledge.

In the first class, the morra games, the objects used in general are the fingers of one hand of the player. Both players have to announce their guesses simultaneously. A player wins a particular play of this game if he guesses the total number of fingers correctly, while his opponent guesses a wrong number. If both players guess correctly, the play is a draw. This is also the case if both players guess a wrong total.

In morra with an equal number of fingers for both players, the player roles are symmetric. As expected, these games turn out to be fair (i.e., their value is zero). We prove this in section 2 and we also show that if one player can use more fingers than his opponent ($m \neq n$), then this player has an advantage in the game.

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In the other class of take-and-guess games, the so-called (m, n)-coin games, the players announce their guesses sequentially. The second player is not allowed to guess the same total as the first player. In the naming of the games, we follow Schwartz (1959), who studied the games with $m = n$. He called these games n-coin games. If a player guesses right, he wins. If neither player guesses the total correctly, the play ends in a draw.

Since coin games are not symmetric for any m and n, it is not clear at first sight if any of these games is fair. However, Schwartz (1959) has shown that the games with $m = n$ are fair. We show in section 3 that a much larger class of coin games is fair: the game value is zero for any coin game in which the starting player has at least as many coins as the opponent ($m \geq n$). Furthermore, (m, n)-coin games with $m < n$ are not fair. We give an overview of the values for all these games and we describe optimal strategies for both players for all m and n.

The remainder of this paper is organized as follows. Morra is discussed in section 2. In section 3, we study coin games in detail. Section 4 presents some concluding remarks and comparisons of morra and coin games.

2 Games of morra

Morra is a game that has been played since ancient Egyptian times. It is still played throughout different parts of the world, especially in Europe and Northern Africa. For a more detailed historic description we refer to Ifrah (1985, p. 67–70) and Perdrizet (1898). The game is fairly simple and can be played by two or more players, but it is usually played by two. The players face each other, each holding up a closed fist. At a given signal, they both hold up zero to five fingers and at the same time announce a number from zero to ten. If both hands are used, the number can range from zero to twenty. A player wins if the number he calls out is the total number of fingers shown by both players. However, if the opponent guesses the same number, the play ends in a draw. Also if neither of the players guesses the correct number, then there is no winner. Winning will be formally represented by getting one unit from the opponent. Payoffs in this zero-sum game can therefore only be -1, 0 and 1.

Variants of morra are a popular subject in game theory lectures (see, for example, Rector (1987)). The proof of the result that we derive in this section (or parts or variants of it), appears as an exercise in various course notes concerning non-cooperative game theory. Proposition 2.1 is mainly included to be able to compare morra with the coin games that are studied in section 3.

In the general version of morra that we study in this paper, the first player is allowed to hold up a maximum of $m \in \mathbb{N}$ fingers, while his opponent can choose to hold up at most $n \in \mathbb{N}$ fingers. We will refer to this game as (m, n)-morra, or briefly $M_{m,n}$. In the analysis of these games and the coin games that are studied in section 3, we will often encounter sets of integers of the form $\{a, a + 1, \ldots, b - 1, b\}$. It is therefore convenient to introduce a shorthand notation for such a set: $[a, b]$.

A pure strategy for player I in $M_{m,n}$ will be denoted by (x_1, y_1), where x_1 is the number of fingers he decides to hold up and y_1 is the sum he guesses. Clearly, with a strategy for which $y_1 < x_1$, player I can never win. Neither can he win with a strategy for which $y_1 > x_1 + n$. Such a strategy is called *infeasible*. We will restrict attention to *feasible* strategies. That is, the pure strategy space for player I is $S_1 = \{(x_1, y_1) \mid (x_1 \in [0, m]) \wedge (y_1 \in [x_1, x_1 + n])\}$. Analogously, the pure strategy space for player II is given by $S_2 = \{(x_2, y_2) \mid (x_2 \in [0, n]) \wedge (y_2 \in [x_2, x_2 + m])\}$. The cardinalities of the strategy spaces are equal: $|S_1| = |S_2| = (m + 1)(n + 1)$.

The game (m, n)-morra can be modelled as a matrix game and is then completely defined by the matrix $A = [a_{(x_1,y_1),(x_2,y_2)}]$, where

$$a_{(x_1,y_1),(x_2,y_2)} = \begin{cases} 1 & \text{if } (y_1 = x_1 + x_2) \wedge (y_1 \neq y_2), \\ -1 & \text{if } (y_2 = x_1 + x_2) \wedge (y_1 \neq y_2), \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.1 *Let $m, n \in \mathbb{N}$. The value $v(M_{m,n})$ of (m, n)-morra is $\frac{m-n}{(m+1)(n+1)}$.*

Proof. Let $x_1 \in [0, m]$ and $y_1 \in [x_1, x_1 + n]$. The strategy (x_1, y_1) of player I will win against all strategies $(x_2, y_2) \in S_2$ of player II for which $x_2 = y_1 - x_1$ and $y_1 \neq y_2$. Player II has exactly m strategies that fulfil these conditions. On the other hand, (x_1, y_1) will cause a victory for player II if he uses a strategy $(x_2, y_2) \in S_2$ for which $y_2 = x_2 + x_1$ and $y_2 \neq y_1$. That is, player I will lose against any of the n elements of the set $\{(x_2, x_2 + x_1) \mid x_2 \in [0, n] \backslash \{y_1 - x_1\}\}$. Against any other strategy of player II, (x_1, y_1) will cause a tie. Therefore, the elements of each row of A sum to $m - n$. Consequently, by playing all $(x_2, y_2) \in S_2$ with equal probability, $\frac{1}{|S_2|}$, player II can guarantee that player I will not get more than $\frac{m-n}{(m+1)(n+1)}$.

In an analogous way, one can show that player I can guarantee himself $\frac{m-n}{(m+1)(n+1)}$ by playing each of his pure strategies with probability $\frac{1}{|S_1|}$. This completes the proof. □

From the proof of Proposition 2.1, we can see that optimal strategies in this game are rather simple. Both players just have to play all their pure strategies with equal probability. It is interesting to notice that $v(M_{m,n}) = -v(M_{n,m})$. Furthermore, one can easily derive the following results by studying the effect of varying m and n on the value $v(M_{m,n})$.

Corollary 2.2 *Only the (m, n)-morra games with $m = n$ are fair. For $m \neq n$, the advantage is for the player who can use more fingers.*

Corollary 2.3

$$\lim_{m\to\infty} v(M_{m,n}) = \lim_{m\to\infty} \frac{m - n}{(m + 1)(n + 1)} = \frac{1}{n + 1}.$$

The intuition behind the limit of Corollary 2.3 is that if one of the players has extremely many objects available (in terms of fingers it becomes difficult to imagine), then his opponent will not be able to guess the number of objects he takes. The value of the game is therefore completely determined by the probability that this player guesses correctly the number of objects chosen by the other player.

3 Coin games

In this section we study a second class of take-and-guess games, the (m, n)-coin games. In contrast to morra, the players have to announce their guesses sequentially in these games. Schwartz (1959) studied the games with $m = n$ and called these games n-coin games. In the naming of our generalization, we also generalize the name he suggested.

The taking part of the (m, n)-coin game (or briefly $C_{m,n}$) is the same as in (m, n)-morra. The first player is allowed to take a maximum of $m \in \mathbb{N}$ objects, while his opponent can pick at most $n \in \mathbb{N}$ objects. The numbers m and n are common knowledge. When played in practice, the objects are not fingers, but things that can be hidden in a hand. As the name of the game suggests, coins are suitable. In Dutch bars the game used to be played with matches.

The difference with morra lies in the guessing part. The players have to announce their guesses sequentially instead of simultaneously. Player II hears the guess of player I and is not allowed to guess the same total as his opponent. If a player guesses right, he wins (i.e., obtains one unit of his opponent). If neither player guesses the total correctly, the play ends in a draw.

Now we can formally write down the strategy spaces of the players. Since coin games are games of perfect recall, the result of Kuhn (1953) tells us that we can restrict our analysis to behavioural strategies. A pure behavioural strategy for player I in $C_{m,n}$ is a choice $(x_1, y_1(x_1)) \in S_1$, where $S_1 = [0, m] \times [0, m + n]$. As in morra, x_1 represents the number of coins he takes in hand, while y_1 is his guess of the total number of coins taken by him and his opponent. Note that y_1 may depend on x_1. Player II picks a combination $(x_2, y_2(x_2, y_1)) \in S_2$, where $S_2 = [0, n] \times [0, m + n]$, such that $y_2(x_2, y_1) \neq y_1$ for all $x_2 \in [0, n]$. Here, x_2 is the number of coins taken by player II and y_2 is the total that he guesses.

Notice that infeasible strategies, like guessing a total that is less than what one has taken in hand, are included in the strategy spaces. In the analysis of morra we did not take this kind of strategies into account. Here we do, and there is a reason for this difference. It is easy to see that infeasible strategies cannot help a player in morra, since the players' decisions are made simultaneously. Misleading the opponent doesn't make sense. In coin games, however, infeasible strategies could be useful for player I, at least in theory. If the game is advantageous for player II, then it may be interesting for player I to mislead his opponent by guessing a total of coins that cannot be correct, given his own hand. In this way, he could try to reduce player II's probability of guessing the right sum. Although he thereby reduces his own probability of guessing right to zero, the combined effect might be in his advantage. For this reason we include infeasible strategies in the strategy spaces. However, we will show that for each $C_{m,n}$ we can find optimal behavioural strategies for both players in which the infeasible strategies are unused.

Let us give a short overview of the organization of the remainder of this section. We start by introducing a graphical model for (m, n)-coin games in section 3.1. In section 3.2, we present the equilibria for a large class of $C_{m,n}$, all games with $m \geq n$. Section 3.3 studies the games in which player II has one coin more available than his opponent. The sections 3.4 and 3.5 contain the equilibrium analysis of two boundary cases within the collection of games for which $n > m+2$. In section 3.6, the list of values is completed. This section is devoted to the games in which player II has two coins more than player I. The results of all subsections are summarized and discussed in section 3.7. Included in this summarizing section is Table 3.2, which illustrates the theorems of the preceding sections by listing the values for the (m, n)-coin games with small values of m and n. At some points earlier in the exposition we will refer to this table.

3.1 A graphical model of an (m, n)-coin game

The structure of coin games is more difficult than morra. We will see that for many combinations of m and n, finding the optimal strategies takes some smart construction work. To keep our arguments clear, and to make the constructions and proofs readable, we introduce a graphical representation of a coin game in (x_1, x_2)-diagrams. In such a diagram, it is not too difficult to see what a player can achieve with a specific strategy. To illustrate the interpretation of the diagrams, we compute the expected payoff that results from a specific combination of strategies. Moreover, we will show how to derive for each player a best reply against a given strategy of the opponent.

3.1.1 Representation of strategies in diagrams

Let us introduce the diagrams that we will use to depict strategies for coin games. For the (m, n)-coin game, an (x_1, x_2)-diagram is a grid with $m+1$ columns (corresponding to $x_1 \in [0, m]$) and $n+1$ rows (corresponding to $x_2 \in [0, n]$). In the taking part of the game, player I picks a column and player II picks a row. Then player I guesses a sum y_1, where his guess can depend on x_1. In the (x_1, x_2)-diagram, this choice can be represented by a point in the column that was chosen by player I. On the line with slope -1 that goes through this point are all points in the grid for which $x_1 + x_2 = y_1$. Points on this line cannot be guessed by player II. Player II has to guess a different line with slope -1. For each combination of x_2 (the number of coins in his own hand) and y_1 (the opponent's guess) he has to make such a decision. Different choices of x_2 correspond to different rows, but for each possible value of y_1 we have to draw a separate (x_1, x_2)-diagram to represent a strategy of player II. To describe a behavioural strategy (with mixed decisions per information set), we give the conditional probability with which each of the actions is played.

Let us clarify this description with an example. The diagrams in Figure 3.1 give two graphical representations of one specific behavioural strategy of player I in $C_{1,2}$.

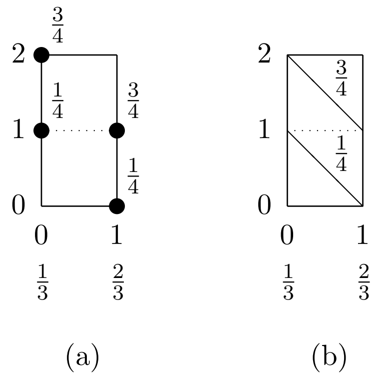

Figure 3.1: Strategy for player I in $C_{1,2}$, represented in two ways in an (x_1, x_2)-diagram.

Figure 3.1(a) gives the general representation for the strategy. This (x_1, x_2)-diagram should be read as follows. Player I picks the left column ($x_1 = 0$) with probability $\frac{1}{3}$ and he picks the right column ($x_1 = 1$) with probability $\frac{2}{3}$. Next, he has to pick y_1. Given $x_1 = 0$, he picks the point $(0, 1)$ (corresponding to $y_1 = 0 + 1 = 1$) with probability $\frac{1}{4}$ and $(0, 2)$ (corresponding to $y_1 = 2$) with probability $\frac{3}{4}$. Similarly, given $x_1 = 1$, player I picks $(1, 0)$ and $(1, 1)$ with probabilities $\frac{1}{4}$ and $\frac{3}{4}$ respectively.

Since the conditional probabilities for the choice of y_1 are the same for $x_1 = 0$ and $x_1 = 1$, we can depict this strategy of player I also a little simpler. This is done in Figure 3.1(b). This figure gives the same probabilities for the choices of the two columns, but summarizes the probabilities for the guessed sum, y_1, in the two lines with slope -1 that are chosen with the probabilities $\frac{1}{4}$ ($y_1 = 1$) and $\frac{3}{4}$ ($y_1 = 2$). Such a representation is only possible if the player's conditional probabilities of guessing y_1 are the same for all x_1 that are chosen with positive probability. For many values of m and n, we present equilibrium strategies for the (m, n)-coin game that can be written in this simple form.

For player II we have also depicted a strategy in $C_{1,2}$ in (x_1, x_2)-diagrams. These diagrams are given in Figure 3.2. We draw one diagram for each possible value of $y_1 \in [0, m + n]$, since the decisions of player II may depend on this value.
In the first place, player II has to pick a number of coins, i.e., he has to choose a row in the grid. A mixed decision is a probability distribution over the rows of the (x_1, x_2)-diagram. Clearly, this distribution cannot depend on y_1, so it is constant over the four diagrams in Figure 3.2. Player II takes one coin with probability $\frac{1}{2}$ and he takes zero or two coins, both with probability $\frac{1}{4}$.

Next, after choosing x_2 and hearing the opponent's guess, y_1, player II has to decide what sum to guess. So for each row in each of the four diagrams, player II can give a probability distribution over the guesses that are interesting for him. In the first diagram, corresponding to $y_1 = 0$, we see that if player II has 2 coins in his hand, he chooses randomly between $y_2 = 2$

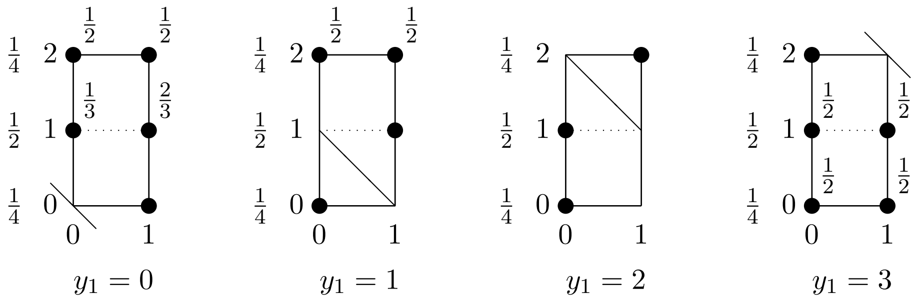

Figure 3.2: Strategy for player II in $C_{1,2}$, represented in four (x_1, x_2)-diagrams.

(the point $(0, 2)$) and $y_2 = 3$ (the point $(1, 2)$). If $x_2 = 1$, he picks $y_2 = 1$ with probability $\frac{1}{3}$ and $y_2 = 2$ with probability $\frac{2}{3}$. For $x_2 = 0$, player II has no choice. He is not allowed to guess the same number as his opponent and we can see in the diagram that $y_2(0, 0) = 1$. We omit the 1, the value of the conditional probability of choosing $y_2(0, 0) = 1$, since it is clear anyway. For $y_1 = 1$, we recognize two of those fixed guesses: $y_2(0, 1) = 0$ and $y_2(1, 1) = 2$. In the diagram that corresponds to $y_1 = 2$, we illustrate how we deal with probability zero: we simply don't draw the dot. Since it is clear now that the probability of choosing $y_2(0, 2) = 0$ must be equal to one, we don't write this number explicitly in the figure.

Note that it is not possible to display all so-called infeasible strategies for the players in the diagrams. For example, to enable player I to guess a sum $y_1 < x_1$, we would have to extend the (x_1, x_2)-diagram at the bottom. Also, to display a strategy in which player II guesses a sum of $m + n$ while he picks $x_2 = 0$ himself, we would have to make an extension of the diagram to the right. As we already mentioned in the introduction of this section, these types of strategies are never needed in equilibria. Therefore, this "flaw" of the diagrams is not a problem. For player II it is immediately clear that there is no point in not trying to win. For player I infeasible strategies could be useful, at least in theory, to try to deceive the opponent with his irrational guess. However, also for the first player these strategies turn out to be redundant when we look for an equilibrium for any $C_{m,n}$.

3.1.2 Expected payoffs

For the combination of the strategies in Figure 3.1 and Figure 3.2, we can compute the expected payoff for player I (and directly derive the expected payoff for player II in this zero-sum game) by summing over all possible combinations of takes and guesses that occur with positive probability. For example, the combination $(x_1, x_2, y_1, y_2) = (0, 0, 1, 0)$ occurs with probability

$$Pr\{x_1 = 0\}Pr\{x_2 = 0\}Pr\{y_1(0) = 1\}Pr\{y_2(0, 1) = 0\} = \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot 1 = \frac{1}{48}.$$

With this combination of takes and guesses player II wins, for $y_2 = x_1 + x_2 = 0$. The payoff for player I is therefore -1. Table 3.1 illustrates the computations that result in the expected payoff of $\frac{11}{288}$ for player I. Only the combinations (x_1, x_2, y_1, y_2) that occur with positive probability are included in the table.

x_1	x_2	y_1	y_2	prob	payoff	prob × payoff
0	0	1	0	$\frac{1}{48}$	-1	$-\frac{1}{48}$
0	0	2	0	$\frac{1}{16}$	-1	$-\frac{1}{16}$
0	1	1	2	$\frac{1}{24}$	1	$\frac{1}{24}$
0	1	2	1	$\frac{1}{8}$	-1	$-\frac{1}{8}$
0	2	1	2	$\frac{1}{96}$	-1	$-\frac{1}{96}$
0	2	1	3	$\frac{1}{96}$	0	0
0	2	2	3	$\frac{1}{16}$	1	$\frac{1}{16}$
1	0	1	1	$\frac{1}{24}$	1	$\frac{1}{24}$
1	0	2	0	$\frac{1}{8}$	0	0
1	1	1	1	$\frac{1}{36}$	0	0
1	1	1	2	$\frac{1}{18}$	-1	$-\frac{1}{18}$
1	1	2	2	$\frac{1}{4}$	1	$\frac{1}{4}$
1	2	1	2	$\frac{1}{48}$	0	0
1	2	1	3	$\frac{1}{48}$	-1	$-\frac{1}{48}$
1	2	2	2	$\frac{1}{16}$	0	0
1	2	2	3	$\frac{1}{16}$	-1	$-\frac{1}{16}$
total				1		$\frac{11}{288}$

Table 3.1: Computing the expected payoff of the combination of strategies in Figures 3.1 and 3.2.

3.1.3 Best replies

We have introduced our graphical representation of strategies for coin games and we have illustrated how to compute the expected payoff that results from a combination of strategies. Since we are going to study equilibria, best replies will play an important role in the remainder of this paper. Let us see how we derive best replies for each player against the given strategy of the opponent.

First, we study the possibilities of player II against the strategy of player I that is depicted in Figure 3.1. In Figure 3.3, this strategy is shown again, but this time the probabilities for taking and guessing are not separated. The probabilities that are given in the diagram, are for the four combinations (x_1, y_1) that are chosen with positive probability. For example, we learn from Figure 3.3 that player I picks the combination $(x_1, y_1) = (0, 1)$ with a probability of $\frac{1}{12}$. This number was found by simply multiplying the probability of taking $x_1 = 0$ coins, $\frac{1}{3}$, and the probability of guessing $y_1 = 1$ with 0 coins in hand, $\frac{1}{4}$.

The easiest way to study the possibilities of player II, is to consider each possible value of

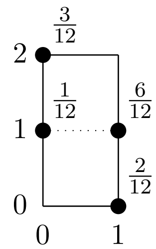

Figure 3.3: Probabilities for (x_1, y_1)-combinations for player I's strategy.

x_2 separately and see what the optimal corresponding choices $y_2(x_2, y_1)$ are. To see what is the best reply, we compare the results for all $x_2 \in [0, n]$. Please observe the following: given the strategy of player I and the choice of x_2 by player II, the probability with which player I wins the play is fixed. Therefore, optimality regarding the selection of $y_2(x_2, y_1)$ only concerns the probability with which player II wins.

Suppose first that player II chooses a strategy with $x_2 = 0$. Then he loses if player I selects one of the points on the corresponding row, $(0, 0)$ and $(1, 0)$. According to Figure 3.3, this happens with probability $0 + \frac{2}{12} = \frac{2}{12}$. What choices of y_2 are optimal for player II, given his choice $x_2 = 0$? He must make a decision for $y_2(0, y_1)$ for each value of y_1 that player I can guess. From Figure 3.3 we know that player I guesses either $y_1 = 1$ or $y_1 = 2$. Let us focus on the case $y_1 = 1$ first. Two of the points in Figure 3.3 that are chosen with positive probability, correspond to $y_1 = 1$: $(0, 1)$ and $(1, 0)$. In the first case, the correct total number of coins taken by the players is $0 + 0 = 0$, in the second case the total is $1 + 0 = 1$. Since $y_1 = 1$, player II is not allowed to guess $y_2 = 1$, so the only choice for $y_2(0, 1)$ with which he can win is 0. His probability of winning is then $\frac{1}{12}$. For $y_1 = 2$, the analysis is slightly more difficult. The points in Figure 3.3 that correspond to this guess are $(0, 2)$ and $(1, 1)$. Given $x_2 = 0$, the correct totals for these points are 0 and 1 respectively. Both totals are allowed as a guess, so player II has a choice. He can select the point on the line $y_1 = 2$ for which the conditional probability that player I chooses it, given $y_1 = 2$, is maximal. This is equivalent to selecting the point on the line $y_1 = 2$ for which the probability shown in Figure 3.3 is maximal. In this case, the optimal choice is $y_2(0, 2) = 1$. With this choice, player II wins with probability $\frac{6}{12}$, the probability with which player I plays $(x_1, y_1) = (1, 2)$. The total probability with which player II wins is now $\frac{1}{12} + \frac{6}{12} = \frac{7}{12}$. Combining this with the probability of player I winning, $\frac{2}{12}$, results in an expected payoff of $\frac{5}{12}$ for player II.

We can apply similar reasoning to strategies of player II with $x_2 = 1$ and $x_2 = 2$ and find that the maximal expected payoffs for player II in these cases are $-\frac{2}{12}$ and $\frac{5}{12}$ respectively. A (but not the unique) best reply of player II against the strategy of player I from Figure 3.1 is therefore the strategy that we discussed, with $x_2 = 0$, $y_2(0, 1) = 0$ and $y_2(0, 2) = 1$. The corresponding expected payoff for player II is $\frac{5}{12}$.

Finding a best reply for player I against player II's strategy from Figure 3.2 is easier. We simply compute the expected payoffs for all $(x_1, y_1) \in S_2$ and compare them. Consider $(x_1, y_1) = (0, 1)$. With this strategy, player I wins with probability $\frac{1}{2}$, the probability that $x_2 = 1$, but he loses with probability $\Pr\{(x_2, y_2(x_2, 1)) = (0, 0)\} + \Pr\{(x_2, y_2(x_2, 1)) = (2, 2)\} = \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{2} = \frac{3}{8}$. His expected payoff with this strategy is therefore $\frac{1}{2} - \frac{3}{8} = \frac{1}{8}$. By computing the expected payoff for all his strategies, we can conclude that the unique best reply of player I is $(x_1, y_1) = (1, 2)$, for which the expected payoff equals $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

In the remainder of section 3, we will describe equilibria for $C_{m,n}$ for all $(m, n) \in \mathbb{N}^2$. The results will be grouped into a number of classes of combinations of m and n. Within each class, the presented equilibrium strategies have a similar structure.

3.2 Fair coin games

Before we start with the analysis of the (m, n)-coin games for which $m \geq n$, we formulate a trivial but helpful result in using the value of $C_{m,n}$ for a certain combination (m, n) to derive bounds for the values of games with a different number of coins for one of the players. The value of $C_{m,n}$ is denoted by $v(C_{m,n})$.

Lemma 3.1 *For all $m, n \in \mathbb{N}$, the following two statements hold:*

(a) $v(C_{m,n}) \leq v(C_{m+1,n})$,

(b) $v(C_{m,n}) \geq v(C_{m,n+1})$.

Proof. The validity of both statements is easily verified by realizing that a player can ignore the extra possibilities he gets by the increase of the number of coins that is available to him. By copying his equilibrium strategy from $C_{m,n}$, player I will be able to guarantee himself at least $v(C_{m,n})$ in the game $C_{m+1,n}$. This is what statement (a) says. Analogous reasoning leads to statement (b). □

As we have already mentioned, Schwartz (1959) has studied the special class of (m, n)-coin games for which $m = n$. He called the games n-coin games.

Proposition 3.2 (Schwartz (1959)) *Let $m \in \mathbb{N}$. Then the (m, m)-coin game is fair, i.e., $v(C_{m,m}) = 0$.*

Proof. We show that $v(C_{m,m}) \geq 0$ and postpone the other half of the proof to (the proof of) Theorem 3.3. Consider the behavioural strategy μ for player I that is shown in Figure 3.4 and defined by the probabilities $\mu(x_1, y_1) = \mu_1(x_1)\mu_2(y_1)$, where

$$\mu_1(x_1) \;=\; \tfrac{1}{m+1} \quad \text{for each } x_1 \in [0, m],$$

$$\mu_2(y_1) \;=\; \begin{cases} 1 & \text{if } y_1 = m, \\ 0 & \text{otherwise.} \end{cases}$$

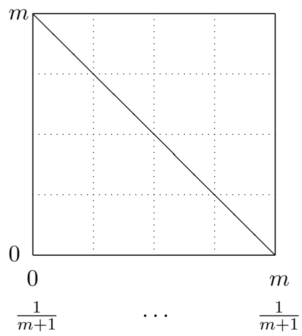

Figure 3.4: An optimal strategy for player I in $C_{m,m}$.

When player I plays according to μ, then his probability of winning is exactly $\frac{1}{1+m}$ against any strategy $(x_2, y_2) \in S_2$ of player II. Player II wins with probability $\frac{1}{m+1}$ if he uses only feasible strategies (i.e., if he puts all of his conditional probability of choosing $y_2(x_2, 1)$ inside the (x_1, x_2)-diagram) and with a lower probability otherwise. Therefore, for any $(x_2, y_2) \in S_2$ the expected payoff of player I is

$$U(\mu, (x_2, y_2)) = \Pr\{\text{I wins}\} - \Pr\{\text{II wins}\} \geq \frac{1}{1+m} - \frac{1}{1+m} = 0.$$

Therefore, $v(C_{m,m}) \geq 0$. □

In the next theorem we show that a much larger class of (m, n)-coin games is fair.

Theorem 3.3 *The m, n-coin game is fair if $m \geq n$.*

Proof. The combination of Lemma 3.1(a) and (the proven part of) Proposition 3.2 already shows that $v(C_{m,n}) \geq 0$. We will define a strategy ν for player II, which guarantees him that he will not have to pay more than zero. In this way we show that $v(C_{m,n}) \leq 0$. Before we can define this strategy, we have to define the sets

$$C(y_1) = [y_1 - n, y_1] \cap [0, m].$$

For a given y_1, $C(y_1)$ is the set of values for x_1 for which (x_1, y_1) is a feasible strategy. We use this set to define a set of points in $\mathbb{N}^2$, $F(x_2, y_1) = \{(a, x_2) \mid a \in C(y_1)\}$. Figure 3.5 illustrates such a set in an (x_1, x_2)-diagram.

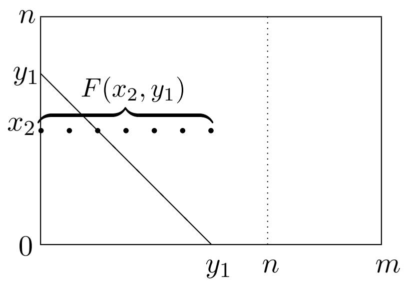

Figure 3.5: The set $F(x_2, y_1)$.

Now we are ready to define the mixed strategy ν for player II, which is determined by the probabilities $\nu(x_2, y_2|y_1) = \nu_1(x_2)\nu_2(y_2|x_2, y_1)$, where

$$\nu_1(x_2) = \frac{1}{n+1} \quad \text{for all } x_2 \in \{0, \dots, n\}$$

and

$$\nu_2(y_2|x_2, y_1) = \begin{cases} \frac{1}{|F(x_2,y_1)|-1} & \text{if } ((y_2 - x_2, x_2) \in F(x_2, y_1) \setminus \{(y_1 - x_2, x_2)\}) \\ & \quad \wedge((y_1 - x_2, x_2) \in F(x_2, y_1)), \\ \frac{1}{|F(x_2,y_1)|} & \text{if } ((y_2 - x_2, x_2) \in F(x_2, y_1)) \wedge ((y_1 - x_2, x_2) \notin F(x_2, y_1)), \\ 1 & \text{if } (x_2 = 0) \wedge (y_1 = 0) \wedge (y_2 = 1), \\ 1 & \text{if } (x_2 = n) \wedge (y_1 = m + n) \wedge (y_2 = m + n - 1), \\ 0 & \text{otherwise.} \end{cases}$$

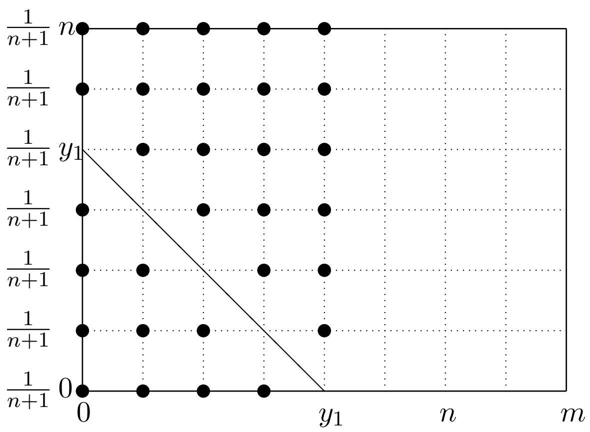

Figure 3.6: Sketch of the structure of an optimal strategy for player II in $C_{m,n}$ with $m \geq n$.

The third and fourth line of the specifications of ν_2 are arbitrary, but necessary for ν to be properly defined. Figure 3.6 shows the structure of ν for a specific y_1. Conditional probabilities for the choice of y_2 are omitted to keep the figure clear. On each x_2-row in the grid, all dots are chosen with equal probability, such that these probabilities sum to one. With infeasible strategies of the form (x_1, y_1) with $x_1 \notin C(y_1)$, player I cannot win, so his expected payoff is non-positive. With a feasible strategy, (x_1, y_1) with $x_1 \in C(y_1)$, the probability that player I wins is $\frac{1}{n+1}$. It is immediately clear from Figure 3.6 that the probability that player II wins against this strategy is

$$\begin{aligned} \Pr\{\text{II wins}\} &= \frac{1}{n+1}\left((|C(y_1)|-1)\frac{1}{|C(y_1)|-1} + ((n+1)-|C(y_1)|)\frac{1}{|C(y_1)|}\right) \\ &= \frac{1}{n+1} + \frac{(n+1)-|C(y_1)|}{(n+1)|C(y_1)|} \geq \frac{1}{n+1}, \end{aligned}$$

with equality for the y_1 for which $[y_1 - n, y_1] \subseteq [0, m]$. As a result,

$$U((x_1, y_1), \nu) = \Pr\{\text{I wins}\} - \Pr\{\text{II wins}\} \leq \frac{1}{1+n} - \frac{1}{1+n} = 0.$$

□

Note that the result of Schwartz (1959), Proposition 3.2, can now be seen as a corollary of Theorem 3.3, since the case $m = n$ clearly is included in the case $m \geq n$. In particular, the strategy ν in the proof of Theorem 3.3 can therefore also be used for the second half of the proof of Proposition 3.2.

Example 3.4 ($\boldsymbol{C_{3,2}}$) In the $(3, 2)$-coin game, the strategy shown in Figure 3.7 is optimal for player II and guarantees that the expected payoff of player I will not be positive. ◁

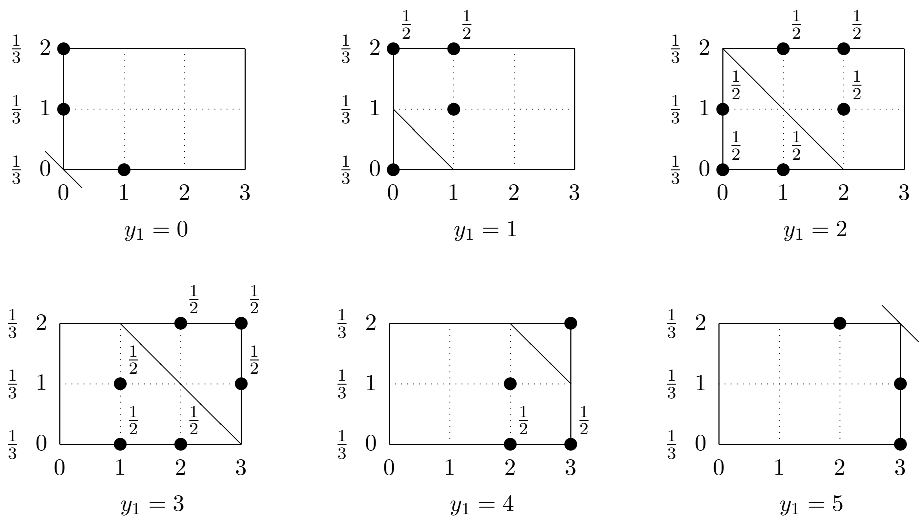

Figure 3.7: An optimal strategy for player II in $C_{3,2}$.

3.3 Games in which player II has one coin more

In the next theorem, we give the value of all coin games in which player II has one coin more than player I.

Theorem 3.5 *Let $m \in \mathbb{N}$ and let $n = m + 1$. Then $v(C_{m,n}) = -\frac{1}{2m+3}$.*

Proof. Consider the strategy μ for player I that is depicted in Figure 3.8. The strategy is defined by the following taking and guessing probabilities: $\mu(x_1, y_1) = \mu_1(x_1)\mu_2(y_1|x_1)$, where

$$\mu_1(x_1) = \begin{cases} \frac{5}{4m+6} & \text{if } x_1 \in \{0, m\}, \\ \frac{4}{4m+6} & \text{if } x_1 \in [1, m-1], \end{cases}$$

$$\mu_2(y_1|x_1) = \begin{cases} \frac{1}{2} & \text{if } (y_1 \in [m, m+1]) \wedge (x_1 \in [1, m-1]), \\ \frac{2}{5} & \text{if } (y_1 = m) \wedge (x_1 = 0), \\ \frac{3}{5} & \text{if } (y_1 = m+1) \wedge (x_1 = 0), \\ \frac{3}{5} & \text{if } (y_1 = m) \wedge (x_1 = m), \\ \frac{2}{5} & \text{if } (y_1 = m+1) \wedge (x_1 = m), \\ 0 & \text{otherwise.} \end{cases}$$

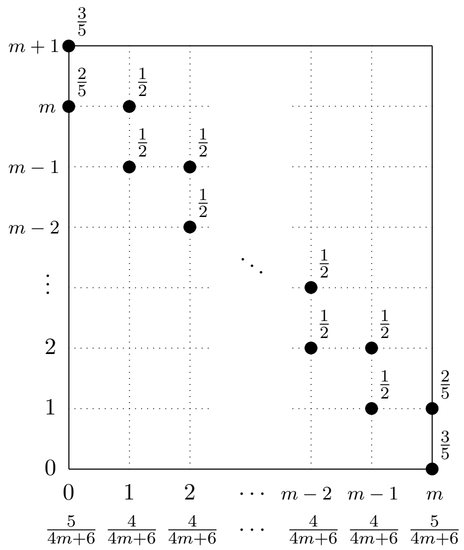

Figure 3.8: Optimal strategy for player I in $C_{m,m+1}$.

Without giving the formal proof of optimality of the μ, we demonstrate how one can quickly check what player II can achieve against this strategy. In our reasoning, we will follow the lines of section 3.1.3. First observe that one can compute the conditional probability that player I has chosen x_1, given that he has guessed a specific y_1. For example,

$$\Pr\{x_1 = 0 | y_1 = m + 1\} = \frac{\frac{3}{5} \cdot \frac{5}{4m+6}}{\frac{3}{5} \cdot \frac{5}{4m+6} + (m-1) \cdot \frac{1}{2} \cdot \frac{4}{4m+6} + \frac{3}{5} \cdot \frac{5}{4m+6}} = \frac{3}{2m+4}.$$

Now, let us see, for example, what player II can achieve against μ by taking $x_2 = m$ and selecting his guesses optimally. Player II knows that he will lose with $x_2 = m$ if his opponent plays $(x_1, y_1) \in \{(0, m), (1, m+1)\}$. Player I will select one of these two strategies with probability $\frac{5}{4m+6} \cdot \frac{2}{5} + \frac{4}{4m+6} \cdot \frac{1}{2} = \frac{2}{2m+3}$. According to μ, player I guesses either $y_1 = m$ or $y_1 = m + 1$. To maximize his winning probabilities, player II has to compute for which $x \in [0, m] \setminus (y_1 - m)$ the probability $\Pr\{x_1 = x | y_1 = m\}$ is maximized. He has to do the same for the probability $\Pr\{x_1 = x | y_1 = m + 1\}$. For the case $y_1 = m$, this conditional probability is maximal for $x_1 = m$, $\Pr\{x_1 = m | y_1 = m\} = \frac{5}{4m+6} \cdot \frac{3}{5} = \frac{3}{4m+6}$. For player II, it is therefore optimal to choose $y_2(m, m) = 2m$. If $y_1 = m + 1$, the maximal probability is assigned to $x_1 = 0$, and it is also equal to $\frac{3}{4m+6}$. So player II should choose $y_2(m, m+1) = m$. If he does this, he will win against μ (with x_2 in his hand) with probability $2 \cdot \frac{3}{4m+6} = \frac{3}{2m+3}$. So the expected payoff for player I will be $\frac{2}{2m+3} - \frac{3}{2m+3} = -\frac{1}{2m+3}$. By considering all other possible values of x_2, we can show that the expected payoff for player I is never lower than $-\frac{1}{2m+3}$.

Next, consider the strategy ν for player II that is shown in (x_1, x_2)-diagrams in Figure 3.9. The taking probabilities can be read directly from the diagrams. We don't explicitly list all underlying guessing probabilities, but we give the idea behind the construction of the strategy-diagrams. Let us fix y_1 for a moment. The corresponding y_1-line crosses at least one of the rows that player II selects with positive probability, say p. The column in which this crossing occurs, corresponds to a value of x_1. With this number of coins in hand, player I wins with probability p. In order to guarantee a value $v < 0$ for player II, the strategy must imply a probability $p + v$ of winning for player II against this combination of x_1 and y_1. This probability should come from the other x_2-rows that are selected with positive probability. In this way we ensure column-wise compensations for each possible value of y_1. This guarantees the value v for player II against any choice of (x_1, y_1) by player I. □

Example 3.6 ($C_{2,3}$) In the $(2, 3)$-coin game, the strategy shown in Figure 3.10 is optimal for player I, while the strategy given in Figure 3.11 is optimal for player II. The value of $C_{2,3}$ is $-\frac{1}{7}$. ◁

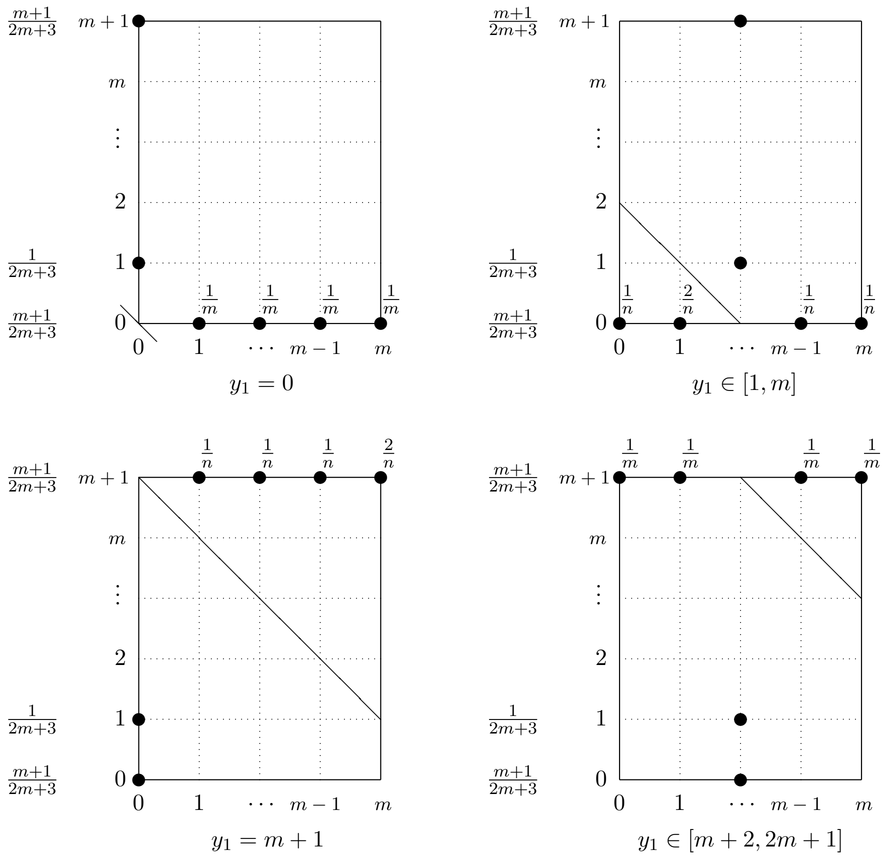

Figure 3.9: Optimal strategy for player II in $C_{m,m+1}$.

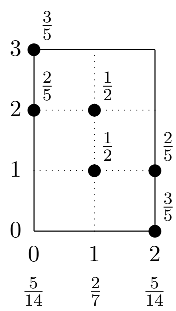

Figure 3.10: An optimal strategy for player I in $C_{2,3}$.

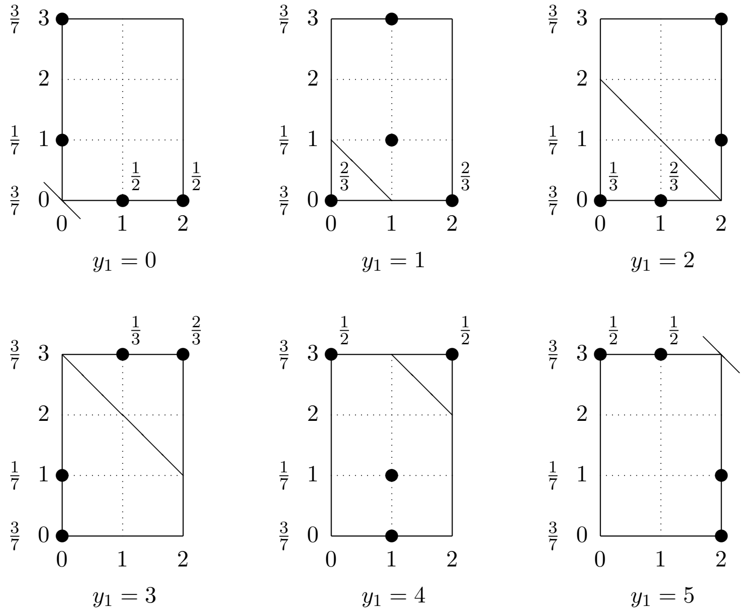

Figure 3.11: An optimal strategy for player II in $C_{2,3}$.

3.4 A special case: $C_{m,n(m,k)}$

The games in the following proposition turn out to be special (boundary) cases (see Table 3.2), with respect to their values, within the collection of (m, n)-coin games with $m < n$. The proposition gives lower bounds for the values for these games. These lower bounds will turn out to be tight later in the paper.

Proposition 3.7 *Let $m \in \mathbb{N}$ and let $k \in \mathbb{N}$ with $k \geq 2$. Define $n(m, k) = k(m + 1) - 1$. Then $v(C_{m,n(m,k)}) \geq \frac{m-n(m,k)}{(m+1)(n(m,k)+1)}$.*

Proof. Consider the behavioural strategy μ for player I that is shown in Figure 3.12 and defined by the probabilities $\mu(x_1, y_1) = \mu_1(x_1)\mu_2(y_1)$, where

$$\mu_1(x_1) = \frac{1}{m+1} \quad \forall\, x_1 \in [0, m],$$

$$\mu_2(y_1) = \begin{cases} \frac{1}{k} & \text{if } y_1 \in \{j(m+1) - 1 \mid j \in [1, k]\}, \\ 0 & \text{otherwise.} \end{cases}$$

The idea behind the strategy is that each x_2-row is covered by exactly one (x_1, y_1) combination, played with probability $\frac{1}{k(m+1)}$. We can apply the same line of reasoning as in the proof of Theorem 3.5, using maximum conditional probabilities of having chosen x_1, given y_1). In this way, the reader can verify that player I loses with the strategy μ with a probability that is at most equal to $\frac{1}{m+1}$, so that μ guarantees the value that is given in Proposition 3.7. □

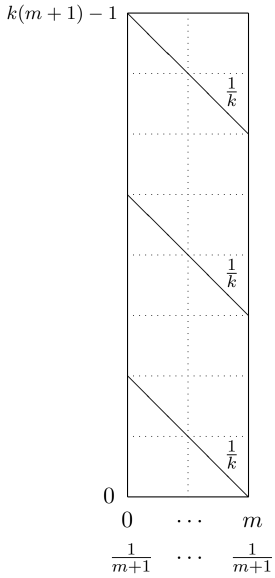

Figure 3.12: An optimal strategy for player I in $C_{m,k(m+1)-1}$ ($k \in \mathbb{N}$, $k \geq 2$).

Example 3.8 **($C_{2,5}$)** In the $(2,5)$-coin game, the strategy shown in Figure 3.13 is optimal for player I and guarantees that his expected payoff will not be smaller than $-\frac{1}{6}$. ◁

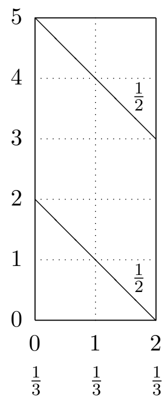

Figure 3.13: An optimal strategy for player I in $C_{2,5}$.

3.5 Another special case: $C_{m,n(m,k-1)+1}$

In this section we study another class of special combinations of m and n. In the games of the next proposition, player II has (roughly speaking) one coin more than in the games of the special case of section 3.4. For this collection of games, which also turn out to form a boundary case (see Table 3.2), we derive an upper value.

Proposition 3.9 *For all $m \in \mathbb{N}$ and all $k \in \mathbb{N}$ with $k \geq 2$, we define $n(m,k) = k(m+1)-1$. Let $k \in \mathbb{N}$ with $k \geq 3$.*[1] *Then*

$$v(C_{m,n(m,k-1)+1}) \leq \frac{m-n(m,k)}{(m+1)(n(m,k)+1)}.$$

Proof. Consider the following mixed strategy ν for player II. Define, for all $(x_2, y_2) \in [0, n(m,k-1)+1] \times [0, m+n(m,k-1)+1]$ and all $y_1 \in [0, m+n(m,k-1)+1]$,

$$\nu(x_2, y_2 | y_1) = \nu_1(x_2)\nu_2(y_2 | x_2, y_1),$$

where

$$\nu_1(x_2) = \begin{cases} \frac{1}{k} & \text{if } x_2 \bmod (m+1) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

[1]Proposition 3.9 concerns the games in which player II has one coin more than in the games of Proposition 3.7. Although we require $k \geq 3$ here, the value of n that we consider is $n(m,k-1)+1$. So, also for the case $k=2$ in Proposition 3.7, the game in which player II has one coin more is included here.

and

$$\nu_2(y_2|x_2, y_1) = \begin{cases} \frac{1}{m} & \text{if } (y_1 \in [x_2, x_2 + m]) \wedge (y_2 \in [x_2, x_2 + m] \setminus \{y_1\}), \\ \alpha & \text{if } (y_1 \notin [x_2, x_2 + m]) \wedge (y_2 \in [x_2, x_2 + m]) \\ & \qquad \wedge (|y_2 - y_1| \bmod (m+1) = 0), \\ \beta & \text{if } (y_1 \notin [x_2, x_2 + m]) \wedge (y_2 \in [x_2, x_2 + m]) \\ & \qquad \wedge (|y_2 - y_1| \bmod (m+1) \neq 0), \\ 0 & \text{otherwise.} \end{cases}$$

Here, $\alpha = \frac{k+m}{(k-1)(m+1)}$ and $\beta = \frac{(k-1)m-(m+1)}{(k-1)m(m+1)}$. It is easy to check that

$$\sum_{x_2 \in [0, n(m,k-1)+1]} \nu_1(x_2) = 1$$

$$\text{and} \quad \sum_{y_2 \in [0, m+n(m,k-1)+1]} \nu_2(y_2|x_2, y_1) = 1 \text{ for all } (x_2, y_1).$$

and that $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$. Thus, ν_1 and ν_2 are well defined probability distributions. Figure 3.14 gives an illustration of the (conditional) probabilities that ν prescribes for a game $C_{m,n(m,k-1)+1}$ for a specific value of y_1. The idea behind the strategy is as follows. The given y_1-line crosses exactly one of the x_2-rows that is chosen with positive probability. The column in which this crossing occurs, indicates with which choice of x_1 player I will win. This winning probability of player I should be made up for by generating a probability of winning for player II in the same column. This compensation is taken care of by the α's. The values of α and β are chosen in such a way that the excess probability of winning for player II is the same in all x_1-columns. Now, let $(x_1, y_1) \in [0, m] \times [0, m + n(m, k-1) + 1]$. Then, if $U(x, y)$ denotes the expected payoff for player I for the (mixed) strategy profile (x, y), we can determine $U((x_1, y_1), \nu)$ by distinguishing two cases:

(i) $|y_1 - x_1| \bmod (m + 1) = 0$ (a positive probability of winning for player I),

(ii) $|y_1 - x_1| \bmod (m + 1) \neq 0$ (probability zero of winning for player I).

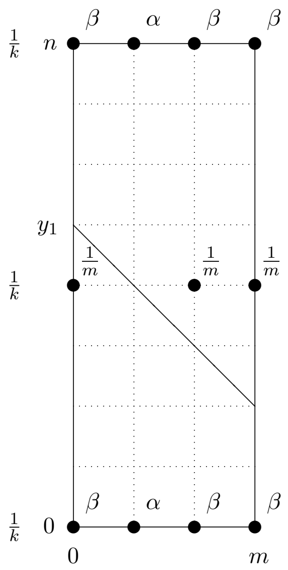

Figure 3.14: An illustration of the strategy ν, which is optimal for player II in $C_{m,n(m,k-1)+1}$.

Case (i):

$$\begin{aligned}
U((x_1,y_1),\nu) &= \nu_1(y_1-x_1) - \sum_{x_2\in[0,(k-1)(m+1)]\setminus(y_1-x_1)} \nu_1(x_2)\nu_2(x_1+x_2|x_2,y_1) \\
&= \frac{1}{k} - \sum_{\substack{x_2:x_2 \bmod (m+1)=0 \\ x_2\neq y_1-x_1}} \nu_1(x_2)\nu_2(x_1+x_2|x_2,y_1) \\
&= \frac{1}{k} - \frac{1}{k}\sum_{\substack{x_2:x_2 \bmod (m+1)=0 \\ x_2\neq y_1-x_1}} \nu_2(x_1+x_2|x_2,y_1) \\
&= \frac{1}{k} - \frac{1}{k}(k-1)\alpha = \frac{1}{k}\left(1-(k-1)\frac{k+m}{(k-1)(m+1)}\right) \\
&= \frac{m+1}{k(m+1)} - \frac{k+m}{k(m+1)} = -\frac{k-1}{k}\frac{1}{m+1} \\
&= \frac{m-n(m,k)}{(m+1)(n(m,k)+1)}
\end{aligned}$$

Case (ii):

$$\begin{aligned}
U((x_1,y_1),\nu) &= -\sum_{x_2\in[0,(k-1)(m+1)]} \nu_1(x_2)\nu_2(x_1+x_2|x_2,y_1) \\
&= -\frac{1}{k}\sum_{x_2:x_2 \bmod (m+1)=0} \nu_2(x_1+x_2|x_2,y_1) \\
&= -\frac{1}{k}\frac{1}{m}-\frac{1}{k}\sum_{\substack{x_2:x_2 \bmod (m+1)=0,\\ y_1\notin[x_2,x_2+m]}} \nu_2(x_1+x_2|x_2,y_1) \\
&= -\frac{1}{k}\frac{1}{m}-\frac{1}{k}(k-1)\beta \\
&= -\frac{1}{k}\left(\frac{1}{m}-(k-1)\frac{(k-1)m-(m+1)}{(k-1)m(m+1)}\right) \\
&= -\left(\frac{(m+1)+(k-1)m-(m+1)}{km(m+1)}\right) = -\frac{k-1}{k}\frac{1}{m+1} \\
&= \frac{m-n(m,k)}{(m+1)(n(m,k)+1)}
\end{aligned}$$

The combination of the payoffs in both cases shows that the (mixed) strategy ν guarantees an expected payoff of $U((x_1,y_1),\nu)=\frac{n(m,k)-m}{(n(m,k)+1)(m+1)}$ for player II. $\square$

Example 3.10 **($C_{2,6}$)** In the $(2,6)$-coin game, the strategy shown in Figure 3.15 is optimal for player II and guarantees that the expected payoff of player I will not be higher than $-\frac{2}{9}$. $\triangleleft$

The $n(m,k)$ from the definition in Proposition 3.9 is exactly the value of n from Proposition 3.7. Therefore, combining these two propositions with Lemma 3.1(b) yields the following result.

Theorem 3.11 *Let $m\in\mathbb{N}$, let $k\in\mathbb{N}$ with $k\geq 3$ and let $n\in[n(m,k-1)+1,n(m,k)]$. Then* $v(C_{m,n})=\frac{m-n(m,k)}{(m+1)(n(m,k)+1)}$.

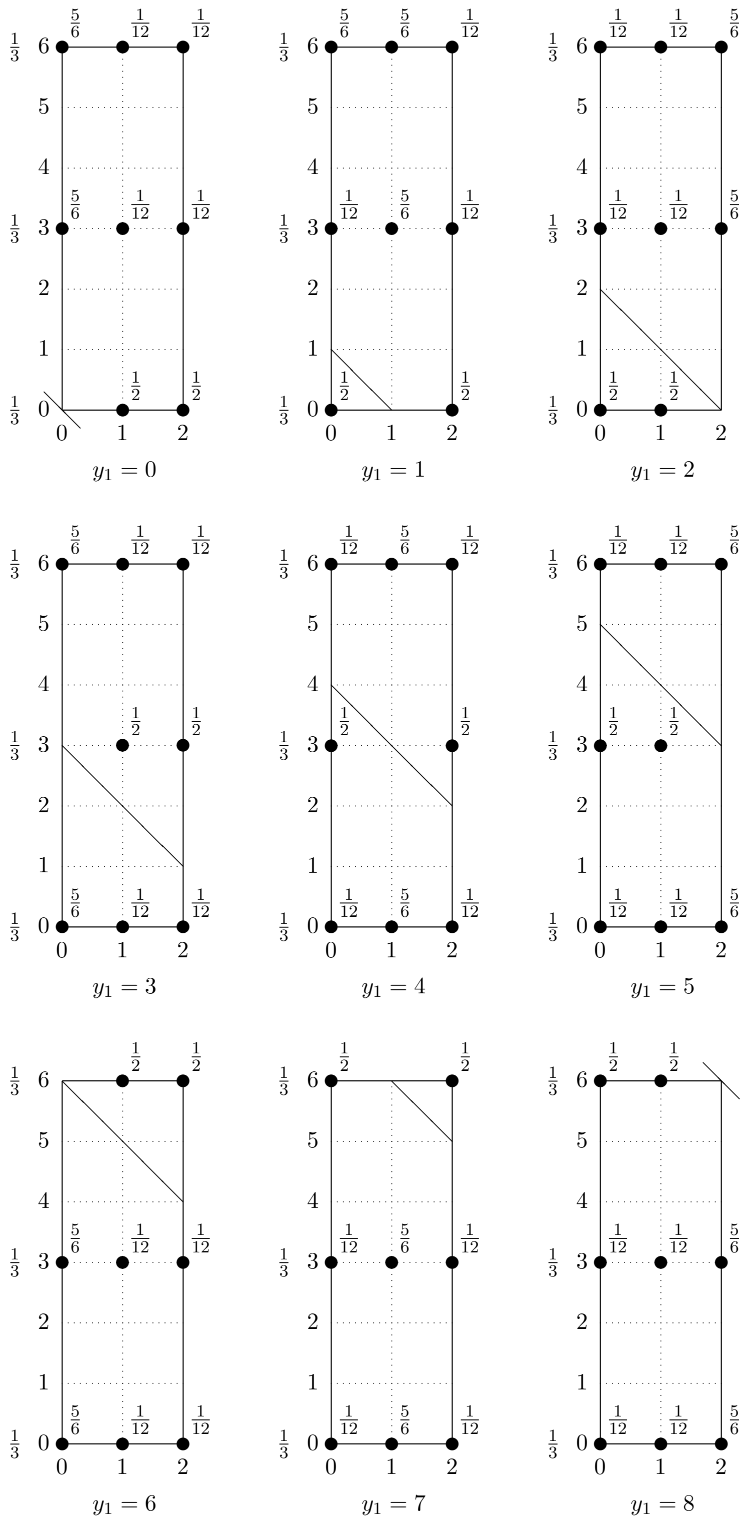

Figure 3.15: An optimal strategy for player II in $C_{2,6}$.

3.6 Games in which player II has two coins more

In the next proposition, we give the value of all coin games in which player II has two coins more than player I.

Proposition 3.12 *Let* $m \in \mathbb{N}$. *Then* $v(C_{m,m+2}) = -\frac{1}{2(m+1)}$.

Proof. We leave it to the reader to verify that the strategy μ for player I that is depicted in Figure 3.16 guarantees the value given in the proposition. The strategy is defined by the following taking and guessing probabilities: $\mu(x_1, y_1) = \mu_1(x_1)\mu_2(y_1)$, where

$$\begin{aligned} \mu_1(x_1) &= \tfrac{1}{m+1} \quad \forall\, x_1 \in [0, m], \\ \mu_2(y_1) &= \begin{cases} \frac{1}{2} & \text{if } \; y_1 \in \{m, m+2\}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

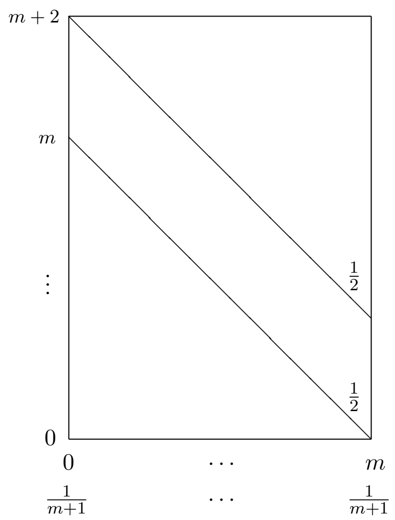

Figure 3.16: Optimal strategy for player I in $C_{m,m+2}$

Next, consider the strategy ν for player II that is shown in (x_1, x_2)-diagrams in Figure 3.17. We don't give a formal description of the taking and guessing probabilities, but we give the intuition behind the construction of the strategy. An y_1-line will intersect at most two of the four rows player II selects with positive probability. The winning probabilities for player I that result from these intersections can be compensated within these two rows (in Figure 3.17, the two dots in the rows for $x_2 \in \{m+1, m+2\}$ do the trick). The remaining rows can be used to generate an excess probability of winning for player II of at least $2 \cdot \frac{1}{4} \cdot \frac{1}{m+1} = \frac{1}{2(m+1)}$. When the y_1-line only crosses of the four rows, then any of the other three rows can be used for compensation. The remaining points on the crossed row can, for example, be selected with equal probability. □

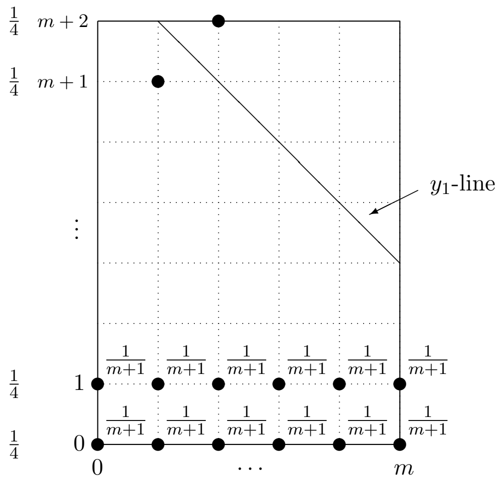

Figure 3.17: The strategy ν, which is optimal for player II in $C_{m,m+2}$.

Example 3.13 ($C_{2,4}$) In the $(2,4)$-coin game, the strategy shown in Figure 3.18 is optimal for player I, while the strategy given in Figure 3.19 is optimal for player II. The value of $C_{2,4}$ is $-\frac{1}{6}$.

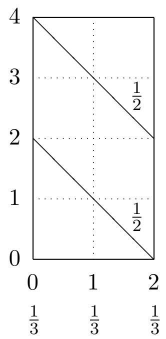

Figure 3.18: An optimal strategy for player I in $C_{2,4}$.

$\triangleleft$

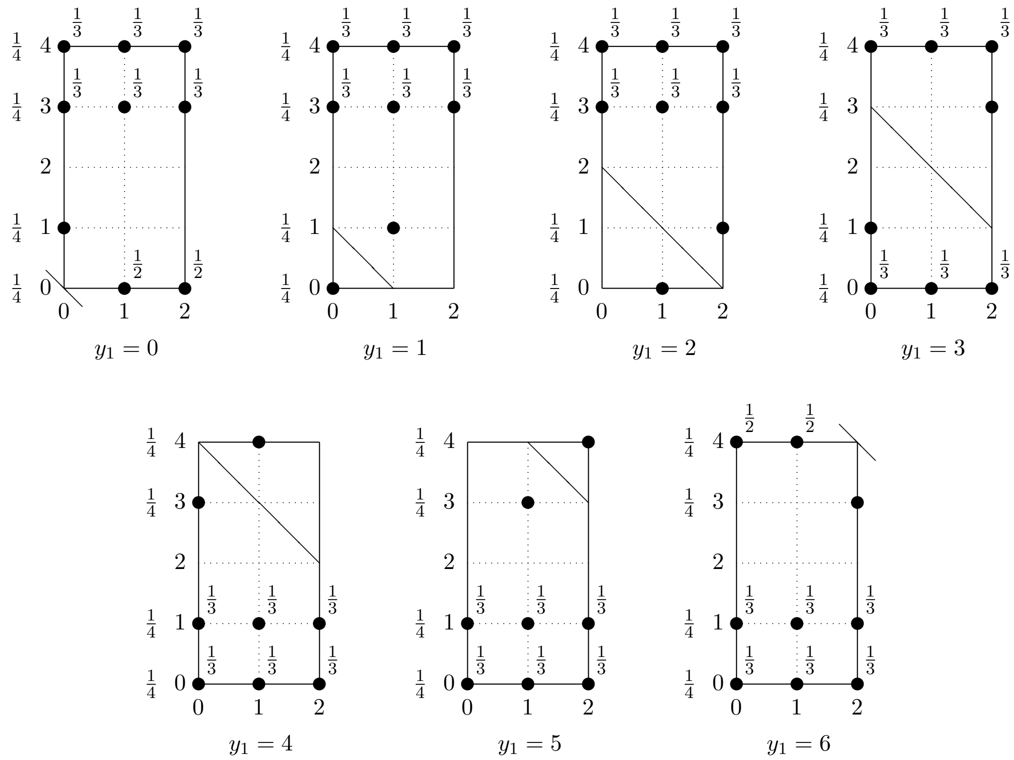

Figure 3.19: An optimal strategy for player II in $C_{2,4}$.

Observe that the value of $C_{m,m+2}$ is exactly the lower bound $\underline{v}$ of the value of $C_{m,2(m+1)-1}$ that we derived in section 3.4:

$$\begin{aligned}\underline{v}(C_{m,2(m+1)-1}) &\overset{\text{Prop 3.7}}{=} \frac{m-n(m,2)}{(m+1)(n(m,2)+1)} = \frac{m-(2(m+1)-1)}{(m+1)2(m+1)} \\ &= -\frac{1}{2(m+1)} \overset{\text{Thm 3.12}}{=} v(C_{m,m+2}).\end{aligned}$$

Therefore, we can combine the results of Theorem 3.12 and Proposition 3.7 and use Lemma 3.1(b) to obtain the following result.

Theorem 3.14 *Let* $m \in \mathbb{N}$ *and let* $n \in [m+2, 2m+1]$. *Then* $v(C_{m,n}) = -\frac{1}{2(m+1)}$.

Although Theorem 3.14 completes our list of values for all (m,n)-coin games (see Table 3.2), we did not yet present optimal strategies for both players for all the games. In particular, for at least one of the players we did not mention how he play optimally in the games $C_{m,n}$ with $k \in \mathbb{N}$ ($k \geq 3$) and $n \in [(k-1)(m+1), k(m+1)-2]$ and in the games $C_{m,n}$ with $m \in \mathbb{N}$ ($m \geq 3$) and $n \in [m+3, 2m+1]$. These are the games for which the values are derived in Theorems 3.11 and 3.14. Following the argument of the proof of Lemma 3.1, an equilibrium strategy for Player II can be copied from a game $C_{m,n}$ with a smaller value of n. Of course, this strategy is not defined for high guesses y_1, since these guesses are not allowed in the game from which player II's strategy is copied. For these values of y_1, player II has to play all feasible guesses with equal probability for each value of x_2 that he takes with positive probability.

For player I, the reader can verify that the strategy with the structure that is displayed in Figure 3.20 is optimal in all these games.

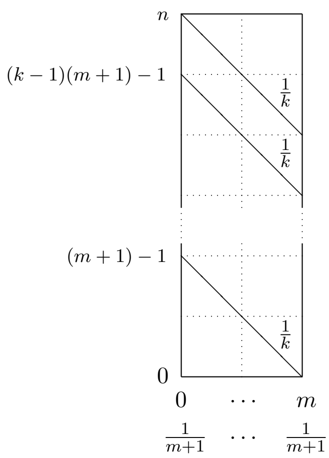

Figure 3.20: An optimal strategy for player I in the remaining (m,n)-coin games ($k = \lceil \frac{n+1}{m+1} \rceil$).

The strategy structure of Figure 3.20 is formally defined by the probabilities $\mu(x_1, y_1) = \mu_1(x_1)\mu_2(y_1)$, where

$$\mu_1(x_1) = \frac{1}{m+1} \quad \forall\, x_1 \in [0, m],$$

$$\mu_2(y_1) = \begin{cases} \frac{1}{k} & \text{if } \; y_1 \in \{j(m+1) - 1 \mid j \in [1, k-1]\} \cup \{n\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $k = \lceil \frac{n+1}{m+1} \rceil$. This strategy is similar to the strategy of player I in the boundary case of section 3.4 (see Figure 3.12). Compared to these strategies, the value of the highest guess is shifted down.

3.7 A summary of the results

In sections 3.2 to 3.6, we have given the values for $C_{m,n}$ for all combinations of m and n. The main results were divided over four theorems (3.3, 3.5, 3.11 and 3.14). Table 3.2 illustrates these theorems by listing the values for the (m, n)-coin games with small values of m and n.

n \ m	1	2	3	4	5	6	7	8	9	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
17	$-\frac{4}{9}$	$-\frac{5}{18}$	$-\frac{1}{5}$	$-\frac{3}{20}$	$-\frac{1}{9}$	$-\frac{2}{21}$	$-\frac{1}{16}$	$-\frac{1}{18}$	$-\frac{1}{20}$	$\cdots$
16	$-\frac{4}{9}$	$-\frac{5}{18}$	$-\frac{1}{5}$	$-\frac{3}{20}$	$-\frac{1}{9}$	$-\frac{2}{21}$	$-\frac{1}{16}$	$-\frac{1}{18}$	$-\frac{1}{20}$	$\cdots$
15	$-\frac{7}{16}$	$-\frac{5}{18}$	$-\frac{3}{16}$	$-\frac{3}{20}$	$-\frac{1}{9}$	$-\frac{2}{21}$	$-\frac{1}{16}$	$-\frac{1}{18}$	$-\frac{1}{20}$	$\cdots$
14	$-\frac{7}{16}$	$-\frac{4}{15}$	$-\frac{3}{16}$	$-\frac{2}{15}$	$-\frac{1}{9}$	$-\frac{2}{21}$	$-\frac{1}{16}$	$-\frac{1}{18}$	$-\frac{1}{20}$	$\cdots$
13	$-\frac{3}{7}$	$-\frac{4}{15}$	$-\frac{3}{16}$	$-\frac{2}{15}$	$-\frac{1}{9}$	$-\frac{1}{14}$	$-\frac{1}{16}$	$-\frac{1}{18}$	$-\frac{1}{20}$	$\cdots$
12	$-\frac{3}{7}$	$-\frac{4}{15}$	$-\frac{3}{16}$	$-\frac{2}{15}$	$-\frac{1}{9}$	$-\frac{1}{14}$	$-\frac{1}{16}$	$-\frac{1}{18}$	$-\frac{1}{20}$	$\cdots$
11	$-\frac{5}{12}$	$-\frac{1}{4}$	$-\frac{1}{6}$	$-\frac{2}{15}$	$-\frac{1}{12}$	$-\frac{1}{14}$	$-\frac{1}{16}$	$-\frac{1}{18}$	$-\frac{1}{20}$	$\cdots$
10	$-\frac{5}{12}$	$-\frac{1}{4}$	$-\frac{1}{6}$	$-\frac{2}{15}$	$-\frac{1}{12}$	$-\frac{1}{14}$	$-\frac{1}{16}$	$-\frac{1}{18}$	$-\frac{1}{21}$	$\cdots$
9	$-\frac{2}{5}$	$-\frac{1}{4}$	$-\frac{1}{6}$	$-\frac{1}{10}$	$-\frac{1}{12}$	$-\frac{1}{14}$	$-\frac{1}{16}$	$-\frac{1}{19}$	0	$\cdots$
8	$-\frac{2}{5}$	$-\frac{2}{9}$	$-\frac{1}{6}$	$-\frac{1}{10}$	$-\frac{1}{12}$	$-\frac{1}{14}$	$-\frac{1}{17}$	0	0	$\cdots$
7	$-\frac{3}{8}$	$-\frac{2}{9}$	$-\frac{1}{8}$	$-\frac{1}{10}$	$-\frac{1}{12}$	$-\frac{1}{15}$	0	0	0	$\cdots$
6	$-\frac{3}{8}$	$-\frac{2}{9}$	$-\frac{1}{8}$	$-\frac{1}{10}$	$-\frac{1}{13}$	0	0	0	0	$\cdots$
5	$-\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{8}$	$-\frac{1}{11}$	0	0	0	0	0	$\cdots$
4	$-\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{9}$	0	0	0	0	0	0	$\cdots$
3	$-\frac{1}{4}$	$-\frac{1}{7}$	0	0	0	0	0	0	0	$\cdots$
2	$-\frac{1}{5}$	0	0	0	0	0	0	0	0	$\cdots$
1	0	0	0	0	0	0	0	0	0	$\cdots$

Table 3.2: Values for $C_{m,n}$ for small values of m and n.

From the table, we can get an idea about what happens if the amount of coins available to one of the players becomes extremely large. This is the subject of the following proposition.

Proposition 3.15 *Let $C_{m,n}$ be an (m,n)-coin game. Then*

$$\lim_{m\to\infty} v(C_{m,n}) = 0,$$

and

$$\lim_{n\to\infty} v(C_{m,n}) = -\frac{1}{m+1}.$$

Proof. The first part of the proposition is trivial. We will prove the second part by using the expression given in Theorem 3.11.

$$\begin{aligned}\lim_{n\to\infty} v(C_{m,n}) &= \lim_{k\to\infty} \frac{m - n(m,k)}{(m+1)(n(m,k)+1)} = \lim_{k\to\infty} \frac{m - (k(m+1)-1)}{(m+1)((k(m+1)-1)+1)} \\ &= \lim_{k\to\infty} \frac{1-k}{k(m+1)-1} = -\frac{1}{m+1}.\end{aligned}$$

□

Comparing the result with Corollary 2.3, we see that the limiting value for the case where the number of coins of player II goes to infinity coincides with the limiting value for this case in morra. From Table 3.2 we can further observe the following interesting facts.

- Although coin games are never symmetric, there is a surprisingly large collection of fair (m,n)-coin games.
- For fixed values of m (and $m < n$), the value $v(C_{m,n})$ is constant for series of $m+1$ values of n. Within this series, player II is not necessarily better off with more coins available. As an example, consider the game $C_{3,5}$. The game becomes more favourable for player II, only if he gets at least three more coins available. One or two extra coins would not help him.
- On the other hand, if $m < n$, player I is always better off with one more coin if he has less coins available than his opponent. Formally, $m < n \Rightarrow v(C_{m,n}) < v(C_{m+1,n})$.
- If $n = m+1$, i.e. if player II has only one more coin available than his opponent, player II cannot take the "regular advantage" that leads to the values for $n \geq m+2$.

4 Concluding remarks

We have studied two classes of two-person take-and-guess games: morra and coin games. In both games, the players first have to take a number of objects and then guess the total number of objects taken by both players. In a game of morra, the players guess simultaneously, while in a coin game player II has to wait for player I's call and is not allowed to guess the same number.

The structure of coin games is less symmetric than the structure of morra. Surprisingly, all coin games in which player I has at least as many objects as player II are fair, while morra is only fair if both players have the same number of fingers available. For all other take-and-guess games in the two classes, the advantage is for the player who has more objects available than his opponent.

Unfair coin games, i.e., (m, n)-coin games with $m < n$, have the same value as (m, n)-morra only in the boundary case of section 3.4, where $n = k(m + 1) - 1$ for some $k \in \mathbb{N}$. For all other unfair combinations of m and n, the (m, n)-coin game is more favourable for player II than (m, n)-morra: $v(C_{m,n}) < v(M_{m,n})$.

Finally, we want to mention three interesting extensions of the analysis in this paper, which are possible subjects for further research. The first extension that deserves attention in the future, is formed by take-and-guess games with more than two players. The winner of such a game receives one unit of all of his opponents. In the case of morra, where there can be multiple winners for the same play, the losers all pay one unit and the winners share the pot equally. A general difficulty in the analysis of games with more than two players, is that optimal play is not defined anymore. Multiple Nash equilibria can exist and the equilibrium strategies are not interchangeable between equilibria. Moreover, the payoffs to the players are not necessarily the same in each equilibrium; there is no such thing as a value in these games.

A second interesting modification of the game would be to make the payoffs dependent of the total number of objects taken by the players. Instead of winning one unit, the winning player receives an amount equal to this total. Guessing higher totals correctly becomes more profitable and at the same time taking higher numbers in hand becomes more risky.

The third and last extension we want to mention, is one that is inspired by the way coin games were played in Dutch bars. Instead of playing one round of the take-and-guess game, the player roles are interchanged after each draw until there is a winner. Such a modification turns the game into a stochastic game, which requires a more sophisticated analysis. Especially for coin games with $m < n$ this change will probably affect the optimal strategies within a round of play too. It might become useful for player I to play infeasible strategies, since apart from winning the game it is interesting now to try to get in the advantageous role of the second player.

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