Spatial Social Networks*

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September 1999

Abstract

We introduce a spatial cost topology in the network formation model analyzed by Jackson and Wolinsky, *Journal of Economic Theory* 71 (1996), 44–74. This cost topology might represent geographical, social, or individual differences. It describes variable costs of establishing social network connections. Participants form links based on a cost-benefit analysis.

We examine the pairwise stable networks within this spatial environment. Incentives vary enough to show a rich pattern of emerging behavior. We also investigate the subgame perfect implementation of pairwise stable and efficient networks. We construct a multistage extensive form game that describes the formation of links in our spatial environment. Finally, we identify the conditions under which the subgame perfect Nash equilibria of these network formation games are stable.

JEL Classification Codes: A14, C70, D20.

Keywords: Social networks; Implementation; Spatial cost topologies.

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*We would like to thank Marco Slikker, Edward Droste, Hans Haller, Dimitrios Diamantaras, and Sudipta Sarangi for discussion and comments on previous drafts of this paper. We also acknowledge Jay Hogan for his programming support.

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1 Introduction

Increasing evidence shows that social capital is an important determinant in trade, crime, education, health care and rural development. Broadly defined, social capital refers to the institutions and relationships that shape a society's social interactions (see Woolcock [25]). Anecdotal evidence for the importance of social capital formation for the well-functioning of our society is provided by Jacobs [14] on page 180: "These [neighborhood] networks are a city's irreplaceable social capital. When the capital is lost, from whatever cause, the income from it disappears, never to return until and unless new capital is slowly and chancily accumulated." Knack and Keefer [16] recently explored the link between social capital and economic performance. They found that trust and civic cooperation have significant impacts on aggregate economic activity. Social networks, especially those networks that take into account the social differences among persons, are the media through which social capital is created, maintained and used. In short, spatial social networks convey social capital. It is our objective to study the formation and the structure of such spatial social networks.

Social networks form as individuals establish and maintain relationships.[1] Being "connected" greatly benefits an individual. Yet, maintaining relationships is costly. As a consequence individuals limit the number of their active relationships. These social-relationship networks develop from the participants' comparison of costs versus benefits of connecting. In the economic analysis and game theoretic literature of network formation, economists have developed theories to study the processes of social link formation and the resulting networks.

One approach in the literature is the formation of social and economic relationships based on cost considerations only, thus neglecting the benefit side of such relationships. Debreu [6], Haller [12], and Gilles and Ruys [10] theorized that costs are described by a topological structure on the set of individuals, being a *cost topology*. Debreu [6] and Gilles and Ruys [10] base the cost topology explicitly on characteristics of the individual agents. Hence, the space in which the agents are located is a topological space expressing individual characteristics. We use the term "neighbors" to describe agents who have similar individual characteristics. The *more similar*

[1] Watts and Strogetz [24] recently showed with computer simulations using deterministic as well as stochastic elements one can generate social networks that are highly efficient in establishing connections between individuals.This refers to the "six degrees of separation" property as perceived in real life networks.

the agents, with regard to their individual characteristics, the *less costly* it is for them to establish relationships with each other. Haller [12] studies more general cost topologies. The papers cited investigate the coalitional cooperation structures that are formed based on these cost topologies. Thus, cost topologies are translated into constraints on coalition formation. Neglecting the benefits from network formation prevents these theories from dealing with the hypothesis that the more *dissimilar* the agents, the more beneficial their interactions might be.

Another approach in the literature emphasizes the benefits resulting from social interaction. The cost topology is a priori given and reduced to a set of constraints on coalition formation or to a given link structure or network. Given these constraints on social interaction, the allocation problem is investigated. For an analysis of constraints on coalition formation and the core of an economy, we refer to, e.g., Kalai et al. [15] and Gilles et al. [9]. Myerson [18] initiated a cooperative game theoretic analysis of the allocation problem under such constraints. For a survey of the resulting literature, we also refer to van den Nouweland [19] and Borm, van den Nouweland and Tijs [4].

Only recently has the focus of the cooperative literature turned to a full cost-benefit analysis of network formation. In 1988, Aumann and Myerson [1] presented an outline of such a research program, however, not until recently has this type of program been initiated. Within the resulting literature we can distinguish three strands: a purely cooperative approach, a purely noncooperative approach, and an approach based both considerations, in particular the equilibrium notion of pairwise stability. A pairwise stable network, introduced by Jackson and Wolinksy [13], is a set of links such that no two individuals would choose to create a link if there is no link between them, and no individual would choose to sever any existing links. Pairwise stability bridges cooperative and non-cooperative elements of link formation because individuals can sever links but pairs must cooperate to form links.

Within the cooperative approach Qin [20] shows that a non-cooperative link formation game, in the spirit of Aumann and Myerson [1], may accompany Myerson's [18] model. In particular, Qin shows this link formation to be a potential game as per Monderer and Shapley [17]. Slikker and van den Nouweland [22] have extended this cooperative game-theoretic line of research. Whereas Qin only considers costless link formation, Slikker and van den Nouweland [22] introduce strictly positive link formation costs. They give a full characterization for three person situations and also reveal that one must make further assumptions in order to analyze situations with more than three players. They conclude that due to the complicated character of the model, further results seem difficult to obtain.

Bala and Goyal [2] and [3] use a purely non-cooperative approach to network formation resulting into so-called *Nash networks*. They assume that each individual player can create a one-sided link with any other player by making the appropriate investment. This concept deviates from the notion of pairwise stability at a fundamental level: a player cannot refuse a connection created by another player, while under pairwise stability both players have to consent explicitly to the creation of a link. Bala and Goyal show that the set of Nash networks is significantly different from the ones obtained by Jackson and Wolinsky [13] and Dutta and Mutuswami [7] based on stronger equilibrium concepts.

Jackson and Wolinsky [13] introduced the notion of a *pairwise stable network*. This equilibrium concept is desirable, although an admittantly weak stability notion, because it relies on a cost-benefit analysis of network formation, allows for both link severance and link formation, and gives some striking results. Jackson and Wolinsky characterized all pairwise stable networks that result within their framework. They prominently feature two network types: the star network and the complete network. Dutta and Mutuswami [7] and Watts [23] refined the Jackson-Wolinsky framework further by introducing other stability concepts and derived implementation results for those different stability concepts.

These contributions to understand network formation are based on specific cost and benefit functions. Slikker [21] further generalizes the cost-benefit approach by using an abstract *reward function* that assigns benefits to individuals in arbitrary communication networks. However, from his analysis, one can conclude, that the use of more abstract reward functions also limits the scope of the conclusions drawn. Slikker limits his analysis to the consideration of the simplest networks — the full communication networks that can be supported as link monotonic allocation schemes.

We intend to extend the Jackson-Wolinsky [13] framework by introducing a *spatial* cost topology, and thus, we incorporate the main hypotheses from Debreu [6] which state that players located closer to one another incur less cost to establish communication. We limit our analysis to the simplest possible implementation of this spatial cost topology within the Jackson-Wolinsky framework. Individuals are located along the real line, and the distance between two individuals determines the cost of establishing a direct link between them. The consequences of this simple extension are profound. A rich structure of equilibrium social networks emerges within our setting, showing the relative strength of the specificity of the model.

We first identify the pairwise stable networks in our spatial extension of the Jackson-Wolinsky framework. We find an extensive typology of stable spatial networks. We mainly distinguish two classes of situations. If costs are *high* in relation to the potential benefits, only the empty network is stable. If costs are *low* in relation to the potential benefits, an array of stable network types emerges. However, we derive that *locally complete networks* are the most prominent stable network type in this spatial environment. In these networks, localities are completely connected. This represents a situation frequently studied and applied in spatial games, as exemplified in the literature on local interaction, e.g., Ellison [8] and Goyal and Janssen [11]. This result also confirms the anecdotal evidence from Jacobs [14] on city life. Furthermore, we note that the networks analyzed by Watts and Strogetz [24] and the notion of the closure of a social network investigated by Coleman [5] also fall within this category of locally complete networks.

Next, we turn to the consideration of Pareto optimal and efficient spatial social networks. A network is *efficient* if the total utility generated is maximal. Pareto optimality leads to an altogether different collection of networks. We show that efficient networks exist that do not have to be pairwise stability.

We present an analysis of the subgame perfect implementation of stable networks by creating an appropriate network formation game. We introduce a class of defined, multi-stage *link formation games* in which all pairs of players sequentially have the potential to form links. The order in which pairs take action is given exogenously.[2] We show that subgame perfect Nash equilibria of such link formation games may consist of only pairwise-stable spatial social networks.

In the next section we present the model. Section 3 is devoted to the characterization of pairwise stable networks. Section 4 examines the question of efficient spatial networks and Section 5 concludes the paper with the implementation results.

[2] A link formation game differs from the network formation game considered by Aumann and Myerson [1] in that each pair of players takes action only once. In the formation game considered by Aumann and Myerson, all pairs that did not form links are asked repeatedly whether they want to form a link or not. See also Slikker and van den Nouweland [22].

2 The Model

We let $N = \{1, 2, ..., n\}$ be the set of players, where $n \geqq 3$. We introduce a spatial component to our analysis by requiring players to have a *fixed* location on the real line $\mathbb{R}$. Player $i \in N$ is located at x_i. Thus, the set $X = \{x_1, ..., x_n\} \subset [0, 1]$ with $x_1 = 0$ and $x_n = 1$ represents the spatial distribution of the players. Throughout the paper we assume that $x_i < x_j$ if $i < j$ and the players are located on the unit interval. This implies that for all $i, j \in N$ the *distance* between i and j is given by $d_{ij} := |x_i - x_j| \leqq 1$.

As remarked in the introduction, the spatial dispersion of the players could be interpreted to represent the social distance between the players. For a seminal discussion of social distance, we refer to Debreu [6].

2.1 Social Networks

Network relations among players are formally represented by graphs where the nodes are identified with the players and in which the edges capture the pairwise relations between these players. These relationships are interpreted as *social* links that lead to benefits for the communicating parties, but on the other hand are costly to establish and to maintain.

We first discuss some standard definitions from graph theory. Formally, a *link* ij is the subset $\{i, j\}$ of N containing i and j. We define $g^N := \{ij \mid i, j \in N\}$ as the collection of all links on N. An arbitrary collection of links $g \subset g^N$ is called an (undirected) *network* on N. The set g^N itself is called the *complete network* on N. Obviously, the family of all possible networks on N is given by $\{g \bigm| g \subset g^N\}$. The number of possible networks is $\sum_{k=1}^{c(n,2)} c(c(n,2), k) + 1$, where for every $k \leqq n$ we define $c\,(n, k) := \frac{n!}{k!(n-k)!}$.

Two networks $g, g' \subset g^N$ are said to be of the same *architecture* whenever it holds that $ij \in g$ if and only if $n - i + 1, n - j + 1 \in g'$. It is clear that this defines an equivalence relation on the family of all networks. Each equivalence class consists exactly of two mirrored networks and will be denoted as an "architecture."[3]

Let $g + ij$ denote the network obtained by adding link ij to the existing network g and $g - ij$ denote the network obtained by deleting link if from the existing network

[3]Bala and Goyal [3] define an architecture as a class of networks that are equivalent for arbitrary permutations. We have limit ourselves to mirror permutations only to preserve the cost topology.

g, i.e., $g + ij = g \cup \{ij\}$ and $g - ij = g \setminus \{ij\}$.

Let $N(g) = \{i \mid ij \in g \text{ for some } j\} \subset N$ be the set of players involved in at least one link and let $n(g)$ be the cardinality of $N(g)$. A *path* in g connecting i and j is a set of distinct players $\{i_1, i_2, ..., i_k\} \subset N(g)$ such that $i_1 = i$, $i_k = j$, and $\{i_1 i_2, i_2 i_3, ..., i_{k-1} i_k\} \subset g$. We call a network *connected* if between any two nodes there is a path. A *cycle* in g is a path $\{i_1, i_2, ..., i_k\} \subset N(g)$ such that $i_1 = i_k$. We call a network *acyclic* if it does not contain any cycles. We define t_{ij} as the number of links in the shortest path between i and j. A *chain* is a connected network composed of exactly one path with a spatial requirement.

Definition 2.1 *A network $g \subset g^N$ is called a* ***chain*** *when (i) for every $ij \in g$ there is no h such that $i < h < j$ and (ii) g is connected.*

Since $i < j$ if and only if $x_i < x_j$, there exists exactly one chain on N and it is given by $g = \{12, 23, \ldots, (n-1)\,n\}$.

Let $i, j \in N$ with $i < j$. We define $i \leftrightarrow j := \{h \in N \mid i \leqq h \leqq j\} \subset N$ as the set of all players that are spatially located between i and j and including i and j. We let $n\,(ij)$ denote the cardinality of the set $i \leftrightarrow j$. Furthermore, we introduce $\ell\,(ij) := n\,(ij) - 1$ as the *length* of the set $i \leftrightarrow j$ or the number of locations between i and j.

Let g be given. Now the set of players $i \leftrightarrow j$ is a *clique* in g if $g^{i \leftrightarrow j} \subset g$ where $g^{i \leftrightarrow j}$ is the complete network on $i \leftrightarrow j$.

Definition 2.2 *A network g is called* ***locally complete*** *when for every $i < j : ij \in g$ implies $i \leftrightarrow j$ is a clique in g.*

Locally complete networks are networks that consist of spatially located cliques. These networks can range in complication from any subnetwork of the chain to the complete network. In a locally complete network, a connected agent will always be connected to at least one of his direct neighbors and belong to a complete subnetwork.

To illustrate the social relevance of locally complete networks we refer to Jacobs [14], who keenly observes the intricacy of social networks that turn city streets, blocks and sidewalk areas into a city neighborhood. Using the physical space of the a city street or sidewalk as an example of the space for the players, the concept of local completeness could be interpreted as each player knowing everyone on their block or their section of the sidewalk. This would be a clique. If one player in the clique was

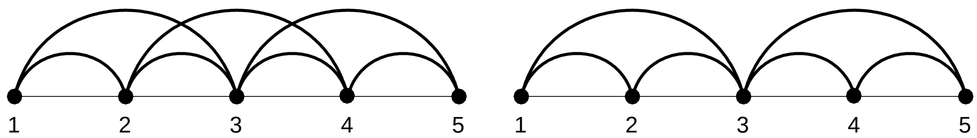

Figure 1: Examples of locally complete networks.

connected to another player on the next block, then each player in the clique would be indirectly connected to everyone on the next block by that single connection. Finally, if we have a connected network, these indirect connections traverse the entire length of the street being the whole space $X \subset [0, 1]$.

Definition 2.3 *Let $i, j \in N$. The set $i \leftrightarrow j \subset N$ is called a **maximal clique** in the network $g \subset g^N$ if it is a clique in g and for every player $h < i$, $h \leftrightarrow j$ is not a clique in g and for every player $h > j$, $i \leftrightarrow h$ is not a clique in g.*

A maximal clique in a certain network is a subset of players that represent a maximal complete subnetwork of that network. For some results in this paper a particular type of locally complete network is relevant.

Definition 2.4 *Let $k \leqq n$. A network g is called **regular of order k** when for every i, $j \in N$ with $\ell(ij) = k$, the set $i \leftrightarrow j$ is a maximal clique.*

Examples of regular networks are the empty network and the chain; the empty network is regular of order zero, while the chain is regular of order one. The complete network is regular of order $n - 1$. We point out that it is not necessary for k to be a divisor of n.

Finally, we introduce the concept of a *star* in which one player is directly connected to all other players and these connections are the only links in the network. Formally, the star with player $i \in N$ as its center is given by $g_i^s = \{ij \mid j \neq i\} \subset g^N$.

To illustrate the concepts defined we refer to Figure 1. The left network is the second order regular network for $n = 5$. The right network is locally complete, but not regular. Similarly we refer Figure 4 on page 14 for a locally complete network which is not regular.

2.2 A Spatial Connections Model

As mentioned in the introduction to this section, a network on the set of players N creates benefits for the players, whether this benefit comes form enhanced communication, production, or friendship. Of interest to us are the incentives to form direct benefits that create and build network connections. In this discussion, it is crucial to base payoffs (utility) on the level of connectedness of each player i. A player's payoffs are explicitly defined on how she is connected to other players in the network. Let for each player $i \in N$ these individual benefits be described by a utility function $u_i : \{g \mid g \subset g^N\} \to \mathbb{R}$ that assigns to every network a (net) benefit for that player.

Following Jackson and Wolinsky [13] and Watts [23] we model the total *value* of a certain network as the sum of all gross benefits of the members in the society with regard to that network. Thus, the total value of a network $g \subset g^N$ is given by

$$v(g) = \sum_{i \in N} u_i(g). \tag{1}$$

This formulation implies that we allow for interpersonal utility comparisons.

We modify the Jackson-Wolinsky *connections model*[4] by incorporating the spatial dispersion of the players into a non-trivial cost topology. This is pursued by replacing the cost concept used by Jackson and Wolinsky with a cost function that varies with the spatial distance between the different players.

Let $c : g^N \to \mathbb{R}_+$ be a general cost function with $c\,(ij) \geqq 0$ being the cost to create or maintain the link $ij \in g^N$. We simplify our notation to $c_{ij} = c\,(ij)$. In the Jackson-Wolinsky connections model the resulting utility function of each player i from network $g \subset g^N$ is now given by

$$u_i\,(g) = \sum_{j \neq i} \delta^{t_{ij}} - \sum_{j:\, ij \in g} c_{ij}, \tag{2}$$

where t_{ij} is the number of links in the shortest path in g between i and j, and $0 < \delta < 1$ is a communication depreciation rate. Clearly this benefit function gives a higher value to more direct connection than to more indirect connections. In this model the parameter δ is a depreciation rate based on network connectedness, not a spatial depreciation rate.

[4]Jackson and Wolinsky discuss two models, the connections model and the co-author model. Both are completely characterized by a specific formulation of the individual utility functions based on the assumptions underlying the sources of the benefits of a social network. Here we only consider the connections model.

Using the Jackson-Wolinsky connections model and a linear cost topology we are now able to re-formulate the utility function for each individual player to arrive at a *spatial connections model*. Recall that $d_{ij} \geqq 0$ is the distance between the players i and j in N. Now, for all $i, j \in N$ we define $c_{ij} := \gamma \cdot d_{ij}$, where $\gamma \geqq 0$ is a common social cost parameter. We further assume that the n individuals are uniformly distributed along the real line segment $[0, 1]$. This implies that $d_{ij} = \frac{\ell(ij)}{n-1}$. By specifying $\gamma = c \cdot (n-1)$, where $c \geqq 0$, we can reformulate the cost of establishing a link between individuals i and j as $c_{ij} = c \cdot \ell(ij)$. Finally, we simplify our analysis by assuming that for each $i \in N$: $w_{ii} = 0$ and $w_{ij} = 1$ if $i \neq j$. This implies that the utility function for $i \in N$ in the Jackson-Wolinsky connections model — given in (2) — reduces to

$$u_i(g) = \sum_{j \neq i} \delta^{t_{ij}} - c \sum_{j:\, ij \in g} \ell(ij). \tag{3}$$

The formulation of the individual benefit functions given in equation (3) will be used throughout the remainder of this paper. For several of our results and examples we make an additional simplifying assumption that $c = \frac{1}{n-1}$, or equivalently $\gamma = 1$.

3 Pairwise Stability in the Spatial Connections Model

The concept of *pairwise stability* was seminally explored by Jackson and Wolinsky [13]. As mentioned in the introduction, this stability concept represents a natural state of equilibrium for certain network formation processes; The formation of a link requires the consent of both parties involved, but severance can be done unilaterally.

Definition 3.1 *A network $g \subset g^N$ is* ***pairwise stable*** *if*

1. *for all $ij \in g$, $u_i(g) \geqq u_i(g - ij)$ and $u_j(g) \geqq u_j(g - ij)$, and*
2. *for all $ij \notin g$, $u_i(g) < u_i(g + ij)$ implies that $u_j(g) > u_j(g + ij)$.*

The spatial aspect of the cost of connecting with other players enables us to identify stable networks with spatially discriminating features. For example, individuals may attempt to maintain a locally complete network but refuse to connect to more distant neighbors. Conversely, individuals may skip over close neighbors to reap the benefits of being connected directly to a more distant neighbor, who might be well connected. We call these networks *non-locally complete*. The star network is a highly organized

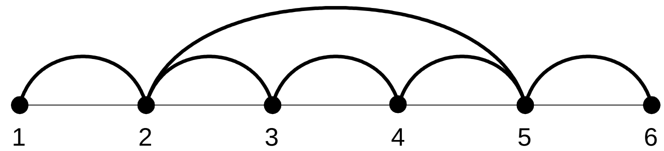

Figure 2: Pairwise stable network for $n = 6$

non-locally complete network. We have selected an example depicted in Figure 2 to illustrate a relatively simple non-locally complete network.

Example 3.2 Let $n = 6$, $c = \frac{1}{n-1} = \frac{1}{5}$, and $\delta = \frac{7}{10}$. Consider the network depicted in Figure 2. This network is pairwise stable for the given values of c and δ.
This pairwise stable network is not locally complete. We observe that players 2 and 5 maintain a link 50% more expensive than a potential link to player 4 or 3 respectively. The pairwise stability of this network hinges on the fact that the direct and indirect benefits, δ and δ^2, are high relative to the cost of connecting. In this example $u_2(g) = 3\delta + 2\delta^2 - 5c$. If player 2 did not support the link $25 \in g$, we would arrive at $u_2(g - 25) = \delta + \sum_{k=1}^{4} \delta^k - 2c$. Obviously, $u_2(g) - u_2(g - 25) = \delta + \delta^2 - \delta^3 - \delta^4 - 3c > 0$. Players are willing to make higher costs to maintain relationships with distant players in order to reap the high benefits from such connections. ■

We investigate which networks are pairwise stable in the spatial connections model. We distinguish two major mutually exclusive cases: $\delta \leqq c$ and $\delta > c$.

Proposition 3.3 *Let* $0 < \delta \leqq c = \frac{1}{n-1}$.

(a) *For $\delta < c$ there exists exactly one acyclic pairwise stable network, the empty network.*

(b) *For $\delta = c$ there exist exactly two acyclic pairwise stable networks, the empty network and the chain.*

We remark that for $\delta < c$ the analysis becomes rather complex if we consider cyclic networks. That is also the reason why in Proposition 3.3 we limit our analysis to the case of acyclic networks. The following example illustrates this.

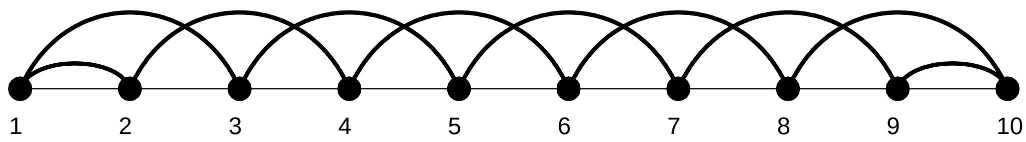

Figure 3: Example of a cyclic pairwise stable network.

Example 3.4 Consider a network g_c for n even, i.e., we can write $n = 2k$. The network g_c is defined as the unique cycle given by

$$g_c = \{12, (n-1)\, n\} \cup \{i\,(i+2) \mid i = 1, \ldots, n-2\}\,.$$

For $k = 5$ the resulting network is depicted in Figure 3.

We investigate for which values of k and (δ, c) with $0 < \delta < c < 1$ the cyclic network g_c is pairwise stable. It is clear that there is only one condition to be considered, namely whether the severance of one of the links of length 2 in g_c is beneficial for one of the players. The net benefit of severing a link of length 2 is computed as

$$\Delta = 2c - \sum_{m=1}^{k-1} \delta^m + \sum_{m=k+1}^{n-1} \delta^m = 2c - \frac{\delta - \delta^k - \delta^{k+1} + \delta^n}{1-\delta} \tag{4}$$

We analyze when $\Delta \leqq 0$. Remark that $\delta - \delta^k - \delta^{k+1} + \delta^n > \delta \left(1 - 2\delta^k\right)$. Now we consider values of (δ, c) such that

$$\delta \frac{1 - 2\delta^k}{1-\delta} = 2c > 2\delta \tag{5}$$

We note that for high enough values of k condition (5) is indeed feasible.
As an example we consider $k = 5$ and $\delta = \sqrt[k-1]{\frac{1}{k}} = \sqrt[4]{\frac{1}{5}} \sim 0.66874$. Then

$$\frac{1-2\delta^k}{1-\delta} = \frac{1 - 2\left(\sqrt[4]{\frac{1}{5}}\right)^5}{1 - \sqrt[4]{\frac{1}{5}}} \sim 2.211$$

and we conclude that condition (5) is indeed satisfied for

$$c = \delta \frac{1-2\delta^k}{2-2\delta} = \sqrt[4]{\frac{1}{5}} \frac{1 - 2\left(\sqrt[4]{\frac{1}{5}}\right)^5}{2 - 2\sqrt[4]{\frac{1}{5}}} \sim 0.739$$

From the analysis above it is clear that for these values the cycle network g_c is indeed pairwise stable. (See Figure 3.) ■

For the case that $\delta > c$ there is a complex array of possibilities.

Proposition 3.5 *For $\delta > c > 0$, every pairwise stable network is connected.*

Proof. Assume there exists a pairwise stable network $g \subset g^N$ that is not connected. Since the network is not connected there will be two direct neighbors, i and j, with the following characteristics: player i is in one connected component of g and player j is in another connected component of g. Since i and j are direct neighbors, the cost to i and j to connect is equal to c. The benefit of connection to each will always be at least the direct benefit of δ. Therefore, both i and j will always want to form a connection since $\delta > c$. ■

For all δ and c we define

$$\hat{n}(c, \delta) := \left\lfloor \frac{\delta}{c} \right\rfloor \tag{6}$$

where $\left\lfloor \frac{\delta}{c} \right\rfloor$ indicates the smallest integer greater than or equal to $\frac{\delta}{c}$.

The next proposition provides a partial characterization of the main pairwise stable network types in a spatial environment.

Proposition 3.6 *Let $\delta > c > 0$ and $c = \frac{1}{n-1}$.*

(a) *For $[\hat{n}(c, \delta) - 1] \cdot c < \delta - \delta^2$ and $\hat{n}(c, \delta) \geqq 3$, there exists a pairwise stable network which is regular of order $\hat{n}(c, \delta) - 1$.*

(b) *For $c > \delta - \delta^2$, there is no pairwise stable network which contains a clique of a size of at least three players.*

(c) *For $c > \delta - \delta^2$ and $\delta < \frac{1}{2}$, the chain is pairwise stable.*

(d) *For $c > \delta - \delta^2$ and $\delta > \frac{\lfloor \frac{n}{2} \rfloor}{n-1}$, there exists a star which is pairwise stable.*

(e) *For $c > \delta - \delta^2$ and $\delta > \frac{1}{2}$, for $n \leqq 5$ the chain is the only regular pairwise stable network, for $n = 6$ there are certain values of δ for which the chain is pairwise stable, and for $n \geqq 7$ the chain is not pairwise stable.*

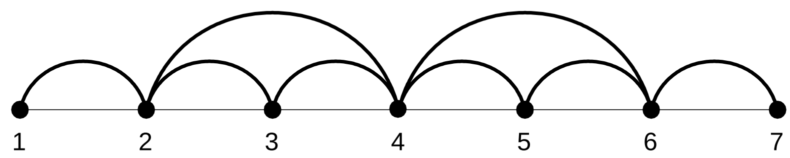

Figure 4: A stable network for $n = 7$.

To illustrate why the restrictions of $\hat{n}(c, \delta) \geqq 3$ and $[\hat{n}(c, \delta) - 1] \cdot c < \delta - \delta^2$ are placed in the formulation of Proposition 3.6(a) we use the following example to show how if one of these restrictions is violated, a pairwise stable network will not be necessarily regular of order $\hat{n}(c, \delta)$.

Example 3.7 Let $n = 7$, $c = \frac{1}{n-1} = \frac{1}{6}$, and $\delta = \frac{1}{2}$. Consider the network depicted in Figure 4. This network is pairwise stable for the given values of c and δ.
Furthermore, we note that this pairwise stable network is locally complete, but it is not regular of any order. With reference to Proposition 3.6(a) we note that $\hat{n}(c, \delta) = 3$. However, $[\hat{n}(c, \delta) - 1] \cdot c = \frac{1}{3} > \delta - \delta^2 = \frac{1}{4}$. Indeed, this confirms assertion (a) in Proposition 3.6. Namely, we identify two maximal cliques of size 2, namely $\{1, 2\}$ and $\{6, 7\}$, and two maximal cliques of the size 3, namely $\{2, 3, 4\}$ and $\{4, 5, 6\}$. ∎

From the propositions formulated in this section we immediately deduce that for large enough populations, there exist appropriate values such that each of the identified architectures of networks is pairwise stable. This is summarized in the following theorem.

Theorem 3.8 (Classification of stable spatial networks) *Let* $c = \frac{1}{n-1}$.

(i) *Suppose that* $n \geqq 4$. *Then,*

  (a) for any $\delta < c$ *the empty network is pairwise stable,*

  (b) for $\delta = c$ *the chain network is pairwise stable,*

  (c) there is some $\delta > c$ *for which the chain network is pairwise stable, and*

  (d) there is some $\delta > c$ *for which a star network is pairwise stable.*

(ii) *Suppose that $n \geqq 7$. Then there exists some $\delta > c$ for which a locally complete network is pairwise stable which is different from the empty network or the chain.*

(iii) *Suppose that $n \geqq 9$. Then there exists some $\delta > c$ for which a regular network of order k is pairwise stable, where $3 \leqq k \leqq n$ is well chosen.*

4 Efficiency in the Spatial Connections Model

Overall efficiency of a network is in the literature usually expressed by the total utility generated by that network. Consequently, a network $g \subset g^N$ is *efficient* if g maximizes the value function $v = \sum_N u_i$ over the set of all potential networks $\{g \mid g \subset g^N\}$, i.e., $v(g) \geqq v(g')$ for all $g' \subset g^N$.

Next we investigate the relationship between efficiency and pairwise stability of networks in our spatial connections model.

Example 4.1 Consider the case that $N = \{1, 2, 3\}$ and let $c > 0$ and $0 < \delta < 1$. We do not make any additional assumptions. (Hence, we do not explicitly require that $c = \frac{1}{n-1} = \frac{1}{2}$.) There are basically five network architectures in the spatial connections model for this case. These five architectures can be represented by the following five networks: $g_1 = g^N$ (representing the complete network), $g_2 = \{12, 23\}$ (representing the chain), $g_3 = \{12, 13\}$ (representing the star with an endpoint player as its center), $g_4 = \{12\}$ (representing the networks with one link only), and $g_5 = \emptyset$ (representing the empty network). For each of these five architectures we compute the resulting utilities as follows:

Network	$u_1(g)$	$u_2(g)$	$u_3(g)$	$v(g) = \sum_i u_i(g)$
$g_1 = g^N$	$2\delta - 3c$	$2\delta - 2c$	$2\delta - 3c$	$6\delta - 8c$
$g_2 = \{12, 23\}$	$\delta + \delta^2 - c$	$2\delta - 2c$	$\delta + \delta^2 - c$	$4\delta + 2\delta^2 - 4c$
$g_3 = \{12, 13\}$	$2\delta - 3c$	$\delta + \delta^2 - c$	$\delta + \delta^2 - 2c$	$4\delta + 2\delta^2 - 6c$
$g_4 = \{12\}$	$\delta - c$	$\delta - c$	0	$2\delta - 2c$
$g_5 = \emptyset$	0	0	0	0

We can show the following four properties:

(i) *The complete network g_1 is pairwise stable and the unique efficient network if $\delta - \delta^2 > 2c$.*

(ii) *The chain* g_2 *is the unique efficient network if* $\delta - \delta^2 < 2c < 2\delta + \delta^2$.

(iii) *The empty network* g_5 *is pairwise stable and the unique efficient network if* $2\delta + \delta^2 < 2c$.

(iv) *The chain* g_2 *is not pairwise stable if* $\delta < c$.

(v) *From properties (ii) and (iv) we conclude that if* $2\delta < 2c < 2\delta + \delta^2$ *there exists an efficient network that is not pairwise stable, being the chain* g_2.

To show property (iv) we remark that the chain g_2 is not pairwise stable if $\delta < c$ since player 2 would sever the link with player 3 to establish network g_4. Thus, for c sufficiently small we have established the existence of an efficient network that is not pairwise stable. ■

Example 4.1 above shows an interesting insight that relates to the insights derived by Jackson and Wolinsky [13] regarding efficient networks.

Lemma 4.2 *The unique efficient network is the empty network if* $\delta + \frac{n-2}{2}\delta^2 < c$. *Furthermore, for certain values of* n *the given bound is sharp.*

Proof. The proof of the first statement immediately follows from Proposition 1(iii) of Jackson and Wolinsky [13], pages 49–50. Here we remark that their connections model is based on lower connection costs and therefore if the empty network is efficient in their setting it certainly is efficient in our spatial setting.
For the second statement we refer to the case with $n = 3$ discussed in Example 4.1. There it is shown that the bound given is exactly reached by the comparison of the empty network with the three player chain — which is equivalent to the three player star. Also note that in cases with $n > 2$ players this bound is therefore again determined by the deviation of these n players forming a star consisting of $n - 1$ links in comparison with the empty network. Thus we conclude that this bound is indeed sharp for the case $n = 3$. ■

Next we consider the star as a special case. In the standard non-spatial connections model for $\delta - \delta^2 < c < \delta$ a star is pairwise stable as well as the unique efficient network. (Jackson and Wolinsky [13], Proposition 1(ii) and Proposition 2(iii).) We show that this is no longer the case in our spatial connections model:

Lemma 4.3 *Let $\delta - \delta^2 < c < \delta$ with $\delta > \frac{\lfloor \frac{n}{2} \rfloor}{n-1}$ and $n > 5$, then any star is not efficient.*

Proof. Without loss of generality we may assume that n is even. We examine the value of two networks: (1) $g^s \subset g^N$ is the star with its center at $\frac{n}{2}$ and (2) $g' \subset g^N$ looks very much like the star described except player located at the far right has not connected to the center but to his direct neighbor. The value of g^s is

$$v\left(g^s\right) = 2(n-1)\delta + (n-1)(n-2)\delta^2 - 4\sum_{k=1}^{\frac{n-1}{2}} kc - 2\binom{n}{2} c. \tag{7}$$

The value of the network g' is

$$v\left(g'\right) = 2(n-1)\delta + (n-2)(n-3)\delta^2 + 2\delta^2 + 2\left(n-3\right)\delta^3 - 4\sum_{k=1}^{\frac{n-1}{2}} kc - 2c. \tag{8}$$

The difference between equation (7) and equation (8) is

$$2(n-3)\delta^2 - 2(n-3)\delta^3 - (n-2)c$$

This is negative when $\delta^2 < \delta^3 + \frac{(n-2)}{2(n-3)}c$. We conclude that the star g^s may not be the network with the highest value. ■

Even for relatively small numbers of players the number of possible networks can be very large, requiring us to use a computer program to calculate the value of all social networks for each n. We limit our computations to $n \leqq 7$ as the number of possible networks for $n = 8$ exceeds 250 million and computing power is limited. Given n, $c = \frac{1}{n-1}$, and δ, Figure 5 summarizes our results. Figure 6 identifies the ranges of δ for which the social networks are both pairwise stable and efficient.

We make some simple observations by comparing Figures 5 and 6.

n=3,4 For $n = 3$ (respectively $n = 4$) and $\delta \geqq 0.4143$ (respectively $\delta \geqq 0.280$) the chain is efficient. For all other values the empty network with $v(g) = 0$ is efficient. Referring to Figure 6, the black areas indicate where there is no pairwise stable network that is efficient as well.

n=5 The network g is efficient for $\delta \in [0.4288, 0.8128]$. However, this network is never pairwise stable for any δ because both players 2 and 4 would increase their utility by $2c - \delta + \delta^3$ if either player severed the link between them. We know this expression is positive as $2c = \frac{1}{2}$ and $\delta - \delta^3 < \frac{1}{2}$.

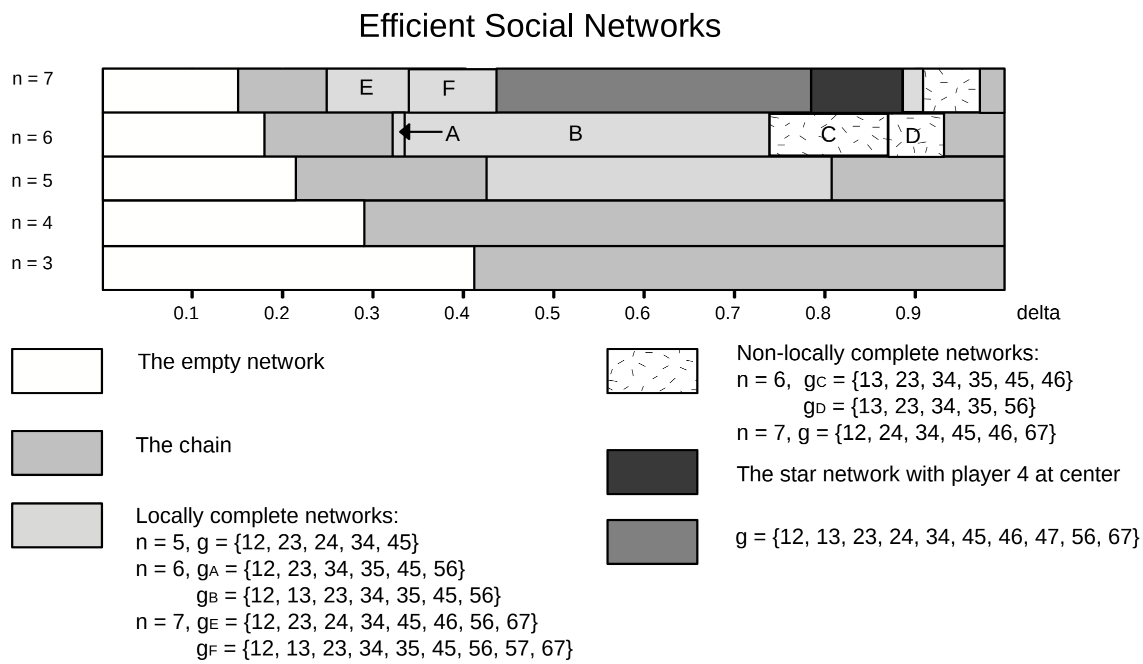

Figure 5: Typology of the efficient networks for $n \leqq 7$.

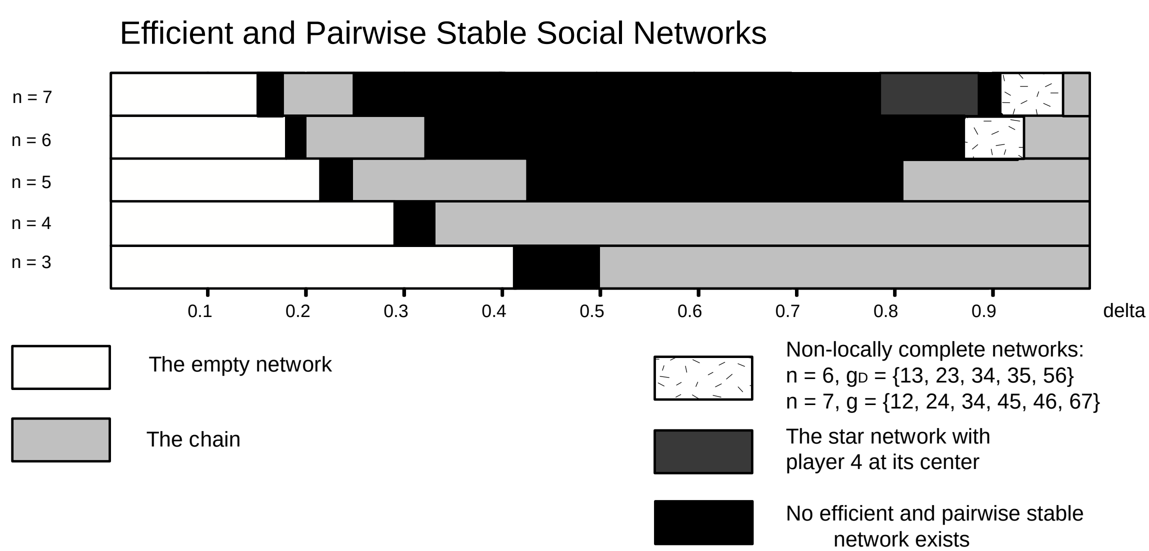

Figure 6: Efficient and pairwise stable networks for $n \leqq 7$.

n=6 The locally complete network g_A is efficient for $\delta \in [0.3142, 0.3375]$. This particular network has one link of length two and the range is indicated by the letter A in Figure 5. The network g_B is efficient for $\delta \in [0.3376, 0.7236]$, but it is not pairwise stable for any δ because there would be one player located at an endpoint with a non-rational connection of length two. If player 3 severs his link with player 1 his utility would increase by $\frac{1}{3} - \delta + \delta^2 > 0$. For $\delta \in [0.7237, 0.8788]$ the non-locally complete network g_C is efficient. The three links $\{34, 35, 45\}$ give this network a locally complete aspect that renders it unstable. The non-locally complete network g_D in Figure 5 is both efficient and pairwise stable for the range $\delta \in [0.8789, 0.9306]$. This network architecture is discussed in the proof of Lemma 4.3.

n=7 The locally complete network g_E is efficient for $\delta \in [0.2468, 0.3480]$. This network is described in Example 3.7 and depicted in Figure 4. Similarly the locally complete network g_F is efficient for $\delta \in [0.3481, 0.4299]$. Neither network is pairwise stable when it is efficient. For $\delta \in [0.4300, 0.7886]$ the union of the chain network and the star g_4^s. This network is never pairwise stable because if either long connection, 14 or 47, were to be severed, player 4's utility would increase as $\delta - \delta^2 < \frac{1}{2}$. The star g_4^s is efficient for $\delta \in [0.7887, 0.8811]$. The star is also pairwise stable because as shown by Proposition 3.6(d). For the relatively small range, $\delta \in [0.8812, 0.9030]$, again the network g_E discussed above is efficient. However, this locally complete network is not pairwise stable for that range. Finally, for $\delta \in [0.9031, 0.9694]$, the network $g = \{12, 24, 34, 45, 46, 67\}$ is efficient and pairwise stable.

We conclude that the empty network is always pairwise stable if it is efficient. For $n \leqq 7$, if the chain is efficient, it is also pairwise stable for $\delta \geqq c = \frac{1}{n-1}$. For relatively high δ, the chain is always efficient.[5] For $n \leqq 7$, a locally complete network with a clique of three or more players is never efficient and pairwise stable for some δ.

[5]The chain is efficient for $n = 5$ if $\delta \in [0.215, 0.4287] \cup [0.8125, 1)$; $n = 6$ if $\delta \in [0.1727, 0.3141] \cup [0.9307, 1)$; and $n = 7$ if $\delta \in [0.1465, 0.2467] \cup [0.9695, 1)$.

5 Implementation of Pairwise Stable Networks

Implementation of pairwise stable networks has been explored in the literature for the Jackson-Wolinsky framework with a binary cost topology. Watts [23] explicitly models the connections model of Jackson and Wolinsky [13] as an extensive form game. In particular, she bases her analysis on the use of myopic players playing the Grim Strategy[6] to illustrate the possibilities the resulting equilibria of such a game. Dutta and Mutuswami [7] look at the relationship between stability and efficiency in great detail, but they use a static, strategic form framework. We make a similar analysis.

In this section we investigate strategic link formation behavior in the spatial connections model. Initially, none of the players are connected. Over multiple playing rounds, players make contact with the other players and determine whether to form a link with each other or not. Exactly one pair of players meets each round — or "stage." Each pair of players meets once and only once in the course of the game.[7] The resulting extensive form game is called the *link formation game*.

Formally, the "order of play" in the link formation game is determined exogenously. Such an "order of play" is represented by a bijection $O : g^N \to \{1, \ldots, c(n,2)\}$ that assigns to every potential pair of players $\{i,j\} \subset N$ a unique index $O_{ij} \in \{1, \ldots, c(n,2)\}$. The set of all orders is denoted by $\mathbb{O}$.

The link formation game has therefore $c(n,2)$ stages. In stage k of the game the pair $\{i,j\} \subset N$ such that $O_{ij} = k$ play a subgame. For any two players, i and j with $i < j$, the choice set facing each player is $A_i\,(ij) = \{C_{ij}, R_{ij}\}$ and $A_j\,(ij) = \{C_{ij}, R_{ij}\}$, where C_{ij} represents the offer to establish the link ij and R_{ij} represents the refusal to establish the link ij given that the network g has been established thus far in the game. Players will form a connection when it is mutually agreed upon, i.e., link ij is

[6] Watts [23] defines the Grim Strategy as follows: Each player agrees to link with the fist two players he meets. Secondly, each player never severs a link as long as all the other players cooperate. However, if player i deviates, then every player $j \neq i$ severs all ties with i and refuses to form any links with i for the rest of the game. Thus, if player i deviates, his payoff will be 0 in all future periods.

[7] We remark that this assumption implies that our link formation game differs considerably from the one formulated in Aumann and Myerson [1]. There the pairs that did not link in previous stages of the game meet again to reconsider their decision. The game continues until a status quo has been reached and no remaining unlinked pairs of players are willing to reconsider. Obviously our structure implies that the "order of play" is crucial, while the Aumann-Myerson structure this is not the case. On the other hand the analysis of our game is more convenient and rather strong results can be derived.

established if and only if both players i and j select action C_{ij}. No connection will be formed if either player refuses to form a connection, i.e., when either one of the players i or j selects R_{ij}. Link formation is permanent; no player can sever the links that were formed during earlier stages of the game. The sequence of actions, recorded as the history of the game, determines in a straightforward fashion the resulting network. We emphasize that all players have complete information in this game.

To complete the description of strategies in the link formation game with order of play $O \in \mathbb{O}$ we introduce the notion of a *(feasible) history*. A history is a listing $h \in H(O) := \cup_{k=1}^{c(n,2)} H_k(O)$, where

$$\begin{aligned} H_k(O) &= X_1(O) \times \cdots \times X_k(O) \text{ with for every } 1 \leqq p \leqq k \\ X_p(O) &= A_i(ij) \times A_j(ij) \text{ for } \{i,j\} \subset N \text{ with } O_{ij} = p. \end{aligned}$$

The history $h = (h_1, \ldots, h_k) \in H_k(O)$ is said to have a length of k, where $h_p \in X_p(O)$ for every $1 \leqq p \leqq k$. A history describes all actions undertaken by the players in the link formation game up till a certain moment in that game. The network $g(h) \in g^N$ corresponding to history $h = (h_1, \ldots, h_k) \in H_k(O)$ is defined as the network that has been formed up till stage k of the link formation game with order O, i.e., $ij \in g(h)$ if and only if $O_{ij} \leqq k$ and $x_{O_{ij}} = (C_{ij}, C_{ij})$. Now we are able to introduce for each player $i \in N$ the *strategy set*

$$S_i = \prod_{ij \in g^N} \prod_{h \in H_{O_{ij}}(O)} A_i(ij). \tag{9}$$

A strategy for player i assigns to every potential link ij of which i is a member, and every possible history of the link formation game up till stage O_{ij} an action. A strategy tuple in the link formation game is now given by $a \equiv (a_1, \ldots, a_n) \in S := \prod_{i \in N} S_i$. With each strategy $a \in A$ we can define the resulting network as $g_a \subset g^N$. Furthermore, player i receives a payoff $u_i(g_a)$ for every strategy tuple $a \in A$.

Formally, for any order of play $O \in \mathbb{O}$ the above describes a game tree $\mathcal{G}_O$. This implies that for order $O \in \mathbb{O}$ the *link formation game* Γ_O may be described by the $(2n+2)$-tuple

$$\Gamma_O = (N, \mathcal{G}_O, S_1, \ldots, S_n, u_1, ..., u_n). \tag{10}$$

Since the link formation game is a well-defined extensive form game, we can use the concept of subgame perfection to analyze the formation of networks. The next results investigate the nature of the subgame perfect Nash equilibria of the link formation game developed above.

Our analysis considers mostly the case that $c < \delta$. As shown in Proposition 3.6 there is a wide range of non-trivial pairwise stable networks in this situation. It can be shown that there is a set of efficient and pairwise stable networks can be implemented as subgame perfect equilibria of link formation games. First, we consider the chain.

Theorem 5.1 *For $\frac{1}{2}\delta + \frac{n+1}{2}\delta^2 < c < \delta$ there exists an order of play $O \in \mathbb{O}$ such that the chain can be supported as a subgame perfect Nash equilibrium of the link formation game with order O.*

Our second implementation results addresses the conditions under which regular networks can be implemented as subgame perfect equilibria of the link formation game.

Theorem 5.2 *Let $m \in \{2, \ldots, n-1\}$. Then for (c, δ) satisfying*

$$\frac{1}{m+1}\delta + \left(\frac{n-1}{m+1} + 1\right)\delta^2 < c < \frac{1}{m}\delta - \frac{1}{m}\delta^2 \tag{11}$$

there exists an order of play $O \in \mathbb{O}$ such that the regular network of order m can be supported as a subgame perfect Nash equilibrium of the link formation game with order O.

Remark that the chain is the unique regular network of order one on N. By substituting $m = 1$ into the condition (11) given in Theorem 5.2 we derive a much weaker upper bound for c than given in Theorem 5.1 for the implementation of the chain as a subgame perfect Nash equilibrium of a link formation game.

From these two main implementation results above we are able derive some further conclusions. Our first conclusion concerns the support of the complete network as SPNE of a link formation game. Such a complete network can be supported for high enough benefits in relation to the link costs:

Corollary 5.3 *For $(n-1)\,c < \delta - \delta^2$ and for any order of play $O \in \mathbb{O}$, the complete network g^N can be supported as a subgame perfect Nash equilibrium of the link formation game with order O.*

Proof. The assertion follows from a slight modification of part (1) in the proof of Theorem 5.2 for $m = n - 1$. (Remark that the complete network on N is the unique regular network of order $n-1$.) Here the order of the game is irrelevant, thus showing that any order of play leads to the establishment of the strategy $\widehat{a}$ as given in the proof of Theorem 5.2 as a subgame perfect Nash equilibrium. ■

Our second conclusion from the two implementation theorems is the support of the empty network for low enough benefits in relation to the link costs:

Corollary 5.4 *For $c > \delta + n\delta^2$ there exists an order of play $O \in \mathbb{O}$ such that the empty network $\emptyset$ can be supported as a subgame perfect Nash equilibrium of the link formation game with order O.*

Proof. This assertion follows easily from a slight modification of part (2) in the proofs of both Theorem 5.1 as well as Theorem 5.2. We note that the order constructed in the proofs of both theorems for these parameter values is the same. ■

Finally we consider under which conditions the identified subgame perfect Nash equilibria generate a pairwise stable network. The following corollary of Proposition 3.6 and Theorems 5.1 and 5.2 summarizes some insights:

Corollary 5.5 *The following properties hold:*

(a) *Suppose that $\frac{1}{2}\delta + \frac{n+1}{2}\delta^2 < c = \frac{1}{n-1} < \delta$. Then there exists an order of play $O \in \mathbb{O}$ such that a subgame perfect Nash equilibrium of the link formation game with order O is pairwise stable.*

(b) *Suppose that $\hat{n}(c, \delta) \geqq 3$. If*

$$\frac{1}{\hat{n}(c,\delta)+1}\delta + \frac{n+\hat{n}(c,\delta)}{\hat{n}(c,\delta)+1}\delta^2 < c < \frac{1}{\hat{n}(c,\delta)}\delta - \frac{1}{\hat{n}(c,\delta)}\delta^2 \tag{12}$$

then there exists an order of play $O \in \mathbb{O}$ such that a subgame perfect Nash equilibrium of the link formation game with order O is pairwise stable.

Proof. We show the assertions separately:

(a) First we remark that $\delta > \frac{1}{n-1} > \frac{1}{n+3}$ implies that

$$c > \frac{1}{2}\delta + \frac{n+1}{2}\delta^2 > \delta - \delta^2. \tag{13}$$

Now condition (13) implies that Proposition 3.6(c) holds. Hence, the chain is pairwise stable. But from the condition in the assertion it also follows that Theorem 5.1 holds, implying that there is an order of play O such that the chain is a SPNE of that link formation game.

(b) First we remark that from (12) it follows immediately that $[\hat{n}(c,\delta)-1]\cdot c < \hat{n}(c,\delta)\cdot c < \delta-\delta^2$, and so Proposition 3.6(a) is satisfied. Hence, the regular network of order $\hat{n}(c,\delta)-1$ is pairwise stable. Furthermore, from (12) it follows through Theorem 5.2 that the regular network of order $\hat{n}(c,\delta)-1$ can be supported as a subgame perfect Nash equilibrium for some order of play $O \in \mathbb{O}$ in the link formation game.

This completes the proof of the corollary. ■

When the conditions of either theorem above are not satisfied, the SPNE of the link formation game may not be efficient.

Example 5.6 Consider the case that $n = 5$, $c = \frac{1}{n-1} = \frac{1}{4}$, and $\delta = \frac{5}{8}$. A network that is pairwise stable and not efficient is the star network $g_3^s = \{13, 23, 43, 53\}$. The value of a star is $8\delta + 12\delta^2 - 12c$ and the value of the chain is $8\delta + 6\delta^2 + 4\delta^3 + 2\delta^4 - 8c$. The chain has a higher value. Player 3, in the center of the star would prefer to be in a member of the chain network; but notice since $\delta > \frac{1}{2}$, even the players located at the end points prefer the star network $\left(2\delta^2 - c > \delta^3 + \delta^4\right)$. The order of play is crucial. We must allow pairs 12 and 45 to refuse to connect before player 3 has the opportunity to refuse any connection to the furthest star points. We propose an order of play given by $\{\mathbf{12}, \mathbf{45}, 23, 34, \mathbf{15}, \mathbf{14}, \mathbf{25}, \mathbf{24}, 13, 35\}$ to guarantee that the star forms with the center at 3. The pairs emphasized in the proposed ordering will not form a link because both players will refuse to make the connection to guarantee the that the network that each of them prefers to form will indeed form. ■

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Appendix: Proofs of the main results

Proof of Proposition 3.3. In this proof we have to introduce some auxiliary notions. We define a path $\{i_1, \ldots, i_m\} \subset N(g)$ in the network $g \subset g^N$ to be *terminal* if $\#\{i_m j \in g \mid j \in N(g)\} = 1$ and for every $k = 2, \ldots, m-1$ it holds that $\#\{i_k j \in g \mid j \in N(g)\} = 2$. We also say that player i_1 *anchors* this terminal path.

We consider the two assertions stated in the proposition separately:

(a) Let $\emptyset \subset g^N$ represent the empty network on N. For any two players the cost of connecting is at least c. The benefit of connection to each is equal to the direct benefit of connection δ. Since $\delta < c$, no two players would attempt to add a link. So, the empty network $\emptyset$ is pairwise stable.

We now consider a network $g \subset g^N$ that is assumed to be pairwise stable as well as acyclic. Hence, in the network $g \subset g^N$ there is at least one player $i \in N(g) \neq \emptyset$ such that $\#\{ij \in g \mid j \in N(g) \setminus \{i\}\} = 1$. Clearly since $\delta < c$, player $j \neq i$ with $ij \in g$ is better off by severing the link with i. Thus, g cannot be pairwise stable. Therefore we conclude that any pairwise stable network has to be empty.

(b) It is obvious that both the empty network and the chain on N are pairwise stable given that $\delta = c$.

Next let $g \subset g^N$ be pairwise stable, non-empty, as well as acyclic. We first show that g is connected.

Suppose to the contrary that g is not connected. Then there will be two direct neighbors, i and j, with the following characteristics: player i is in a non-empty connected component of g of size at least 2 and player j is in another connected component of g. (Here we remark that $\{j\}$ is a trivially connected component of any network in which j is not connected to any other individual.) Since i and j are direct neighbors, the cost to i and j to connect is equal to c. The net benefit for i of making a connection to j is then at least $\delta - c = 0$. The net benefit for j for making a connection to i is at least $\delta + \delta^2 - c = \delta^2 > 0$. Therefore, g is not pairwise stable, since both i and j will want to form a connection. This contradicts our hypothesis and therefore g has to be connected.

Next we show that g is the chain. Suppose to the contrary that g is not the chain. From the assumptions it can easily be derived that there exists a player $i \in N$ with $\#\{ij \in g \mid j \in N(g)\} \geqq 3$.

First, we show that there is no player $j \in N$ with $ij \in g$, $\ell(ij) \geqq 2$, and the link ij is the initial link in a terminal path in g that is anchored by player i. Suppose to the contrary that such a player j exists and that the length of this terminal path is m. Then the net benefit for player i to sever the link ij is at least

$$2c - \sum_{k=1}^{m} \delta^k = 2\delta - \frac{\delta - \delta^{m+1}}{1-\delta} = \frac{\delta - 2\delta^2 + \delta^{m+1}}{1-\delta} > \delta \frac{1-2\delta}{1-\delta} \geqq 0$$

since $\delta = c = \frac{1}{n-1} \leqq \frac{1}{2}$. Thus, we conclude that player i is better off by severing the link ij. Hence, there is no player $j \in N$ with $ij \in g$, $\ell(ij) \geqq 2$, and the link ij is the initial link in a terminal path in g that is anchored by player i.

From this property it follows that the *only* case not covered is that $n \geqq 6$ and there exists a player j with $ij \in g$, $\ell(ij) \geqq 3$, $\#\{jh \in g \mid h \in N(g)\} = 3$, and that the two other links at j have length 1 that are connected to terminal paths,

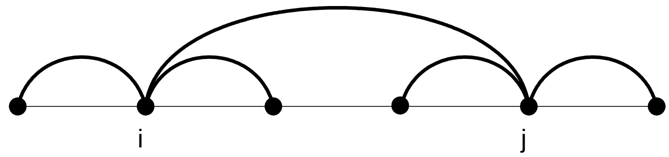

Figure 7: Case for $\ell(ij) = 3$.

respectively of length m_1 and m_2. The smallest (partial) network satisfying this case is depicted in Figure 7 and is the situation with $n = 6$ and $\ell(ij) = 3$. In the general case, the maximal net benefit of agent i to sever the link ij can be estimated at

$$\begin{aligned} 2c - \delta - \sum_{k=2}^{m_1} \delta^k - \sum_{k=2}^{m_2} \delta^k &= \delta - \frac{\delta^2 - \delta^{m_1+1}}{1-\delta} - \frac{\delta^2 - \delta^{m_2+1}}{1-\delta} \\ &= \delta \frac{1 - 3\delta + \delta^{m_1+1} + \delta^{m_2+1}}{1-\delta} \\ &> \delta \frac{1-3\delta}{1-\delta} \end{aligned}$$

Since $n \geqq 6$ it follows immediately that $\delta \leqq \frac{1}{5}$, and thus the term above is positive.
This shows that the network g indeed cannot be pairwise stable. This implies that every non-empty acyclic pairwise stable network has to be the chain.

This completes the proof of Proposition 3.3. ■

Proof of Proposition 3.6. Let $g \subset g^N$ be pairwise stable.

(a) Consider $g \subset g^N$ on N to be regular of order $\hat{n}(c,\delta) - 1$. Then the maximal net benefit of severing a link $ij \in g$ within a clique in g would be $c_{ij} + \delta^2$. Since $c_{ij} \leqq [\hat{n}(c,\delta) - 1] \cdot c < \delta - \delta^2$, it holds that $\delta > c_{ij} + \delta^2$ and, so, no player would be willing to sever a link. An additional link would form if $c_{ij} \leqq \delta - \delta^{\tilde{n}}$, where $\tilde{n}$ represents the value of an indirect connection lost due to a shorter path being created when a new link is created in a connected network. Since the network is connected, if a player were to add a link, his net benefit would be composed of three parts: The benefit of the new link and possibly higher indirect connections, the loss of indirect connections replaced by a shorter path created by the new link, and the cost of maintaining the link. $\delta^{\tilde{n}}$ represents the value of an indirect connection lost due to a shorter path being created when a new link is created. If more than one indirect connection is replaced by a shorter path, we use the convention of ranking the benefits $\delta^{\tilde{n}}$ by decreasing the superscript $\tilde{n}$.[8] We know that $c_{ij} \geqq \hat{n}(c,\delta) \cdot c > [\hat{n}(c,\delta) - 1] \cdot c$ because the location for any player that i could form an additional link with would lie beyond the maximal clique. Using the definition of $\hat{n}(c,\delta)$, we know that $c_{ij} \geqq \delta$. Therefore no player will try to form an additional link outside the maximal clique. Hence, g is pairwise stable.

(b) Suppose $g^{i \leftrightarrow j} \subset g \subset g^N$ with $\ell(ij) \geqq 2$. If player i severed one of his connections to a player within the clique $i \leftrightarrow j$, the resulting benefits from replacing a direct with an indirect connection are $\delta^2 + c > \delta$, the right hand side being the benefit of a direct link. Therefore, player i will sever one of his connections. This shows that a locally complete network with cliques of at least 3 members are not pairwise stable, thus showing the assertion.

(c) Suppose g is the chain on N. The net benefit to any player severing a link with their nearest neighbor would be at most $c - \delta < 0$. Therefore no player will sever a link.
A player $i \in N$ will connect to a player j with $\ell(ij) = 2$ only if $2c \leqq \delta - \delta^{\tilde{n}}$. Because $\delta < \frac{1}{2}$ and $\delta > c > \delta - \delta^2$, we know $2c > \delta \geqq \delta - \delta^{\tilde{n}}$. Thus, player i will not make such a connection.
Next consider j with $\ell(ij) \geqq 3$. Player i will make a link with j if the net benefit of such a connection is positive. Let $\ell(ij) = k$. For k odd, the net benefit for player i connecting to player j is

$$\delta + 2\sum_{l=2}^{\frac{k-1}{2}} \delta^l + \delta^{\frac{k-1}{2}+1} \quad - \sum_{m=\tilde{n}-(k-2)}^{\tilde{n}} \delta^m - kc.$$

For k even, the net benefit for player i connecting to player j is

$$\delta + 2\sum_{l=2}^{\frac{k}{2}} \delta^l \quad - \sum_{m=\tilde{n}-(k-2)}^{\tilde{n}} \delta^m - kc.$$

We proceed with a proof of induction with regard to the parameter $k \geqq 3$. When $k = 3$, the net benefit expression above simplifies to $\delta + \delta^2 - \delta^{\tilde{n}} - \delta^{\tilde{n}-1} - 3c$. If $3c + \delta^{\tilde{n}} + \delta^{\tilde{n}-1} < \delta + \delta^2$, player i would consider making a link with player j. This expression is never true for $\delta < \frac{1}{2}$ and $c > \delta - \delta^2$. For higher values of k the positive elements of the net benefit value increase by less than δ^2 and the

[8] A player's net benefit from the addition of a new link is composed of three parts: The benefit of the new link and possibly higher indirect connections, the loss of indirect connections replaced by a shorter path created by the new link, and the cost of maintaining the link. $\delta^{\tilde{n}}$ represents the value of an indirect connection lost due to a shorter path being created when a new link is created. If more than one indirect connection is replaced by a shorter path, we use the convention of ranking the benefits $\delta^{\tilde{n}}$ by decreasing the superscript $\tilde{n}$.For example, a benefit of δ^5 and δ^4 can be denoted by $\delta^{\tilde{n}}$ and $\delta^{\tilde{n}-1}$.

negative elements increase by c. As $c > \delta^2$, the net benefit function decreases with respect to k. Thus for any $k \geqq 3$, player i will not consider creating a link with player j.
Thus, we have shown that no player will sever or add a link when g is chain on N and, therefore, the chain is pairwise stable.

(d) Let g be a star on N with the central player located at $\lfloor \frac{n}{2} \rfloor$. Refer to all players except the center as "points." The benefit of maintaining a connection to the center for all points is $\delta + (n-2)\delta^2$. The maximal cost of any connection in this star is $\lfloor \frac{n}{2} \rfloor \cdot c = \frac{\lfloor \frac{n}{2} \rfloor}{n-1} < \delta$. Thus, no player will sever a connection, not even the center. The net benefit of adding an additional connection for a player is $\delta - \delta^2 < c$. Thus, the star with the central player located at $\lfloor \frac{n}{2} \rfloor$ is pairwise stable.

(e) From assertion (b) shown above, it follows that any pairwise stable network $g \subset g^N$ does not contain a clique of at least three players. This implies that the chain is the only regular pairwise stable network to be investigated. Let g be the chain on N. First note that since $c < \delta$ no player has an incentive to sever a link in g. We will discuss three subcases, $n \geqq 7$, $n = 6$, and $n \leqq 5$.

(1) Assume $n \geqq 7$. Select two players i and j, $i < j$, who are neither located at the end locations of the network nor direct neighbors. Also assume that $\ell(ij) = 3$. If i were to connect to a player j the minimum net benefit of such a connection to either i or j would be $\delta + \delta^2 - \delta^3 - \delta^4 - 3c$. The maximal cost of connection c_{ij} when $\ell(ij) = 3$ is $\frac{1}{2}$ since $c = \frac{1}{n-1} \leqq \frac{1}{6}$. Since $\delta > \frac{1}{2}$, the minimum benefit, $\delta + \delta^2 - \delta^3 - \delta^4$, of such a connection is greater than the maximal cost. Thus, the additional connection will be made.[9] Also, note that player i is not connected to j's neighbor to the left. This player has essentially been skipped over by player i. Nor does player i have any incentive to form a link with the player that was skipped over. A connection to this player would cost $2c$, and the benefit would only be $\delta - \delta^2$. Thus, the chain is not pairwise stable.
(2) Assume $n = 6$. From assertion (b) shown above, we need only to examine two situations of link addition for two players i and j: a) $\ell(ij) = 3$ and $1 \neq i \neq n$, and b) $\ell(ij) \geqq 3$, $i = 1$ or $i = n$.

a) Select two players i and j with i not located at the end of the network, i.e., $1 \neq i \neq n$, and $\ell(ij) = 3$. If i were to connect to a player j the cost of such a connection would be $3c = \frac{3}{5}$ and the net benefit of this connection would be

[9] Because $\frac{1}{6} \geqq c > \delta - \delta^2$, and $\delta > \frac{1}{2}$, we know that $\delta + \delta^2 - \delta^3 - \delta^4$ has a minimum value of $\left(\frac{1}{2} + \frac{1}{6}\sqrt{3}\right) + \left(\frac{1}{2} + \frac{1}{6}\sqrt{3}\right)^2 - \left(\frac{1}{2} + \frac{1}{6}\sqrt{3}\right)^3 - \left(\frac{1}{2} + \frac{1}{6}\sqrt{3}\right)^4$ which is approximately equal to 0.53. Here we note that this minimum is attained in a corner solution determined by the constrained $\delta - \delta^2 < c$.

$\delta+\delta^2-\delta^3-\delta^4$. Because $c>\delta-\delta^2$, and $\delta>\frac{1}{2}$, we know that $\delta+\delta^2-\delta^3-\delta^4$ has a minimum value which is less than $\frac{3}{5}$.[10]

b) Select two players i and j with i, $\ell(ij)\geqq 3$, $i=1$ or $i=n$. If player j were to connect to player i the minimum cost of a connection would be $3c$ or $\frac{3}{5}$. The minimal net benefit of such connection would be $\delta-\delta^{\tilde{n}}$ where $\tilde{n}\in\{3,4,5\}$. Since $c>\delta-\delta^2$, we know that $3c>\delta-\delta^{\tilde{n}}$. We can conclude that a link to an end agent will never be stable from such a distance.

We thus conclude that for δ such that $\delta+\delta^2-\delta^3-\delta^4\leqq\frac{3}{5}$ the chain is pairwise stable and for some values of δ a non-locally complete network is stable.

(3) Assume $n\leqq 5$. Select three players i, j and k, where $i<j<k$ and j,k with $\ell(ij)\geqq 2$. We know $ij\notin g$ and $ik\notin g$. Suppose that $i=1$. If player j were to make a new connection with player i, the maximum net benefit of such a connection to player j would be $\delta-\delta^2-2c<0$. For player k we have that $\ell(ik)\geqq 3$, so, the net benefit of such a connection for player k would be at most $\delta-\delta^3-3c<0$. If the player at the opposite end of the network linked with player i the net benefit would always be negative.[11] We conclude that no player would decide to connect with a player at either end points of the chain. From this it can easily be concluded that a similar argument can be applied to the other players for the case $n=5$. (Note that the cases $n\leqq 4$ are trivially excluded.) Therefore, no player will form an additional link, and we conclude that the chain is pairwise stable.

This completes the proof of Proposition 3.6. ■

Proof of Theorem 5.1. First remark that the condition put on the variables c and δ indeed is feasible.

Now we partition the set of potential links g^N into n subsets $\{G_2,\ldots,G_n\}$ where we define for $k\in\{2,\ldots,n\}$

$$G_k:=\left\{ij\in g^N\mid n(ij)=k\right\}$$

We now consider the (partially reversed) order $\widetilde{O}:=\left(\widetilde{G}_2,\widetilde{G}_n,\widetilde{G}_{n-1},\ldots,\widetilde{G}_3\right)\in\mathbb{O}$, where $\widetilde{G}_k$ is an enumeration of G_k, $k=2,\ldots,n$. We now show that the chain is a subgame perfect Nash equilibrium of the link formation game corresponding to the order $\widetilde{O}$. For that purpose we apply backward induction to this link formation game. We define the strategy tuple $\widehat{a}$ by $\widehat{a}_i(ij,h)=C_{ij}$ $\left(\text{where } h\in H\left(\widetilde{O}\right)\right)$ if and only

[10] Because $\frac{1}{5}\geqq c>\delta-\delta^2$, and $\delta>\frac{1}{2}$, we know that the polynomial $\delta+\delta^2-\delta^3-\delta^4$ has a minimum value given by $\left(\frac{1}{2}+\frac{1}{10}\sqrt{5}\right)+\left(\frac{1}{2}+\frac{1}{10}\sqrt{5}\right)^2-\left(\frac{1}{2}+\frac{1}{10}\sqrt{5}\right)^3-\left(\frac{1}{2}+\frac{1}{10}\sqrt{5}\right)^4$ which is approximately equal to 0.594. Again this minimum is determined by the constraint $\delta-\delta^2<c$.

[11] For $n=3$, $\delta-\delta^2-2c<0$. For $n=4$, $\delta-\delta^3-3c<0$. For $n=5$, $\delta+\delta^2-\delta^3-\delta^4\ -4c<0$. ($\delta+\delta^2-\delta^3-\delta^4$ is maximized at $\delta=\frac{1}{8}+\frac{1}{8}\sqrt{17}$ at a value of approximately 0.62 and $4c=1$)

if $ij \in G_1$.[12] It is immediately clear that the resulting network $g_{\widehat{a}}$ is the chain on N. We proceed to show that $\widehat{a}$ is indeed a best response to any history in the link formation game, following the backward induction method.
(1) When any player i is paired with a player j where $n(ij) = 2$, i.e., $ij \in G_2$, both players will choose to make a connection because those connections will always have a positive net benefit because $c < \delta$. This is independent of the number of links made in the previous stages of the game. We conclude that when $n(ij) = 2$, the history in the link formation game with order $\widetilde{O}$ does not affect the willingness to make this connection.
Thus, it remains to check the pairs in $g^N \setminus G_2$:
(2) Let $k \in \{2, \ldots, n-1\}$ and $i, j \in N$ with $i < j$ be such that $n(ij) = k + 1 \geqq 3$ and let $h \in H_{\widetilde{O}_{ij}}\left(\widetilde{O}\right)$ be an arbitrary history of the link formation game up till stage $\widetilde{O}_{ij}$. Then given the backward induction hypothesis that in later stages no links will be formed, the network $g(h)$ only consists of links of length 1 and links of lengths k and higher.[13] This implies that player j can be connected at most with two direct neighbors with links of length 1 and at most with $(n - k + 1)$ players with links of length of at least k. So, an upper bound for the net benefits $U_i(ij)$ for player i of creating a direct link with player j can be estimated at

$$\begin{aligned} U(ij) &\leqq \delta + (n-k+3)\,\delta^2 - kc \\ &\leqq \delta + (n-k+3)\,\delta^2 - 2c \\ &< \delta + (n+1)\,\delta^2 - 2\left(\frac{1}{2}\delta + \left(\frac{n+1}{2}\right)\delta^2\right) \\ &= 0 \end{aligned}$$

We conclude that player i will not have any incentives to create a link with player j in the link formation game with order $\widetilde{O}$.
Thus we conclude from (1) and (2) above that the strategy $\widehat{a}$ is indeed a subgame perfect Nash equilibrium of the link formation game with order $\widetilde{O}$. This shows that the chain can be supported as a subgame perfect Nash equilibrium for the parameter values described in the assertion. ■

Proof of Theorem 5.2. First we remark that (11) stated in Theorem 5.2 is indeed a feasible condition on the parameters c and δ. Namely, this holds for low enough values of δ; to be exact $\delta < (m(n+m) + m + 1)^{-1}$.
Now we partition the set of potential links g^N into $n-m$ subsets $\{G_0, G_{m+1}, \ldots, G_n\}$

[12] Hence, this strategy prescribes that all links are formed in the first $(n-1)$ stages of the game corresponding to all pairs in G_1. Furthermore, irrespective of the history in the link formation game up till that moment there are no links formed in the final $C(n, 2) - (n-1)$ stages of the link formation game corresponding to the pairs in $G_2, \ldots, G_n$.
[13] Remark that the *length* of the link $ij \in g^N$ is given by $n(ij) - 1$.

where we define

$$\begin{aligned} G_0 &= \left\{ij \in g^N \mid n(ij) \leqq m+1\right\} \\ G_k &= \left\{ij \in g^N \mid n(ij) = k\right\} \text{ where } k \in \{m+2, \ldots, n\} \end{aligned}$$

We now consider the order $\widetilde{O} := \left(\widetilde{G}_0, \widetilde{G}_n, \widetilde{G}_{n-1}, \ldots, \widetilde{G}_{m+2}\right) \in \mathbb{O}$, where $\widetilde{G}_k$ is an enumeration of G_k, $k = 0, m{+}2, \ldots, n$. We now show that the regular network of order m is a subgame perfect Nash equilibrium of the link formation game corresponding to the order $\widetilde{O}$. For that purpose we apply backward induction to this link formation game.

We define the strategy tuple $\widehat{a}$ by $\widehat{a}_i(ij, h) = C_{ij}$ $\left(\text{where } h \in H\left(\widetilde{O}\right)\right)$ if and only if $ij \in G_0$.[14] From this definition it is clear that the resulting network $g_{\widehat{a}}$ is the unique network on N that is regular of order m. We proceed to show that the strategy described by $\widehat{a}$ is indeed a best response to any history in the link formation game, following the backward induction method.

(1) When any player i is paired with a player j where $ij \in G_0$, i.e., $n(ij) \leqq m+1$, both players will choose to make a connection because those connections will always have a positive net benefit because a lower bound for the net benefit of such a link is given by $\delta - \delta^2 - [n(ij) - 1] \cdot c \geqq \delta - \delta^2 - m \cdot c > 0$ from the right-hand side of condition (11). This is independent of the number of links made in the previous or later stages of the game. Hence, we conclude that if $n(ij) \leqq m+1$, the history in the link formation game with order $\widetilde{O}$ does not affect the willingness to make the connection ij.

Next, we proceed by checking the remaining pairs:

(2) Let $k \in \{m+1, \ldots, n-1\}$ and $i, j \in N$ with $i < j$ be such that $n(ij) = k+1 \geqq m+2$ and let $h \in H_{\widetilde{O}_{ij}}\left(\widetilde{O}\right)$ be an arbitrary history of the link formation game up till stage $\widetilde{O}_{ij}$. Then given the backward induction hypothesis that in later stages no links will be formed, the network $g(h)$ only consists of links of length less than $m+1$ and links of lengths k and higher. This implies that player j can be connected to at most $2m$ players with links of length m or less and to at most with $(n-k+1)$ players with links of length k and higher. So, an upper bound for the net benefits $U_i(ij)$ for player i of creating a direct link with player j can be constructed to be

$$\begin{aligned} U_i(ij) &\leqq \delta + (n-k+2m+1)\,\delta^2 - kc \\ &\leqq \delta + (n+m)\,\delta^2 - (m+1)c \\ &< \delta + (n+m)\,\delta^2 - (m+1)\left(\frac{1}{m+1}\delta + \left(\frac{n-1}{m+1} + 1\right)\delta^2\right) = 0. \end{aligned}$$

[14] Hence, this strategy prescribes that all links are formed in the first $|G_0|$ stages of the game corresponding to all pairs in G_0. Furthermore, irrespective of the history in the link formation game up till that moment there are no links formed in the final $C(n,2) - |G_0|$ stages of the link formation game corresponding to the pairs in $G_{m+1}, \ldots, G_n$. Obviously the outcome of this strategy is that $ij \in g_{\widehat{a}}$ if and only if $n(ij) \leqq m$.

We conclude that player i will not have any incentives to create a link with player j in the link formation game with order $\widetilde{O}$.
Thus we conclude from (1) and (2) above that the strategy $\widehat{a}$ is indeed a subgame perfect Nash equilibrium of the link formation game with order $\widetilde{O}$. This shows that the regular network of order m can be supported as such for the parameter values described in the assertion. ■