

Center



# Discussion Paper

No. 2005–120

**ON THE MATRIX INEQUALITY  $(I + x)^{-1} \leq I$ .**

By Jacob Engwerda

November 2005

ISSN 0924-7815

On the matrix inequality  $(I + X)^{-1} \leq I$ .

Jacob Engwerda  
Tilburg University  
Dept. of Econometrics and O.R.  
P.O. Box: 90153, 5000 LE Tilburg, The Netherlands  
e-mail: engwerda@uvt.nl

November, 2005

**Abstract:** In this note we consider the question under which conditions all entries of the matrix  $I - (I + X)^{-1}$  are nonnegative in case matrix  $X$  is a real positive definite matrix. Sufficient conditions are presented as well as some necessary conditions. One sufficient condition is that matrix  $X^{-1}$  is an inverse  $M$ -matrix. A class of matrices for which the inequality holds is presented.

**Keywords:** Positive matrices, positive definite matrices, inverse  $M$ -matrix problem.

**Jel-codes:** C00.

## 1 Introduction

In various scientific areas like e.g. economics and biology naturally the study of positive systems arises. To analyze these kind of systems the theory of nonnegative matrices plays an important role. This theory is well-documented in e.g. the seminal work of Berman and Plemmons [1].

In this note we study the problem under which conditions, entry-wise, the inequality

$$(I + X)^{-1} \leq I \tag{1}$$

holds if  $X$  is a positive definite matrix. This question e.g. recently arose in [3] in the equivalent formulation of this problem (see below) under which conditions on the positive definite matrix  $X$ , matrix  $(I + X)^{-1}$  is a so-called diagonally dominant Stieltjes matrix. Before we present formal definitions we introduce some notation.

### Notation 1.1

- $A \geq B$  denotes the entrywise inequality for  $A$  and  $B$ ,  $a_{ij} \geq b_{ij}$ .
- $A \succ (\succeq)B$  means that matrix  $A - B$  is positive (semi-)definite.
- $Z_n$  denotes the set of real  $n \times n$  matrices with nonpositive off-diagonal entries.
- $I$  and  $0$  denote the  $n \times n$  identity and zero matrix, respectively.
- $e_i$  is the  $i^{th}$  standard basisvector in  $\mathbb{R}^n$  and  $e = (1, 1, \dots, 1)^T$ . □

Next we recall some definitions.

### Definition 1.2

- $X$  is called an  $M$ -matrix if  $X \in Z_n$ , its inverse exists and  $X^{-1} \geq 0$ .
- $X$  is called a *Stieltjes matrix* if  $X$  is a symmetric  $M$ -matrix.
- a symmetric matrix  $X$  is called *diagonally dominant* if  $|x_{ii}| \geq \sum_{j=1, j \neq i}^n |x_{ij}|$ , for all  $i = 1, \dots, n$ . □

From e.g. [1, pp.141] we have the next result.

### Lemma 1.3

- 1) *Symmetric  $M$ -matrices are positive definite.*
- 2) *If  $X \succ 0$  and  $X \in Z_n$  then  $X$  is an  $M$ -matrix.* □

The following results, which can be verified e.g. by using the definition of matrix inverse, will be used in the next section.

**Lemma 1.4**

1) If  $A$  and  $C$  are nonsingular  $m \times m$  and  $n \times n$  matrices and  $A + BCD$  is invertible, then

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1}DA^{-1}. \quad (2)$$

2) If  $XY = 0$  and  $I + X$  is invertible, then  $(I + X)^{-1}Y = Y$ .

3) If  $XY = YX = 0$  and all next inverse matrices exist, then  $(I + X + Y)^{-1} = (I + X)^{-1} + (I + Y)^{-1} - I$ .

□

**2 Main Results**

We start this section with an equivalent statement of the problem. The proof of this result uses the next well-known matrix identity (which follows directly from (2), provided  $X$  has no eigenvalues  $-1$  and  $0$ )

$$I - (I + X)^{-1} = (I + X^{-1})^{-1}. \quad (3)$$

**Theorem 2.1**

Let  $X \succ 0$ . Then, (1) holds if and only if  $(I + X)^{-1}$  is a Stieltjes matrix.

Proof:

$\Rightarrow$  Since  $I - (I + X)^{-1} \geq 0$  it is obvious that  $(I + X)^{-1}_{ij} \leq 0$ ,  $i \neq j$ . Furthermore, since  $X \succ 0$ ,  $(I + X)^{-1} \succ 0$  too. From Lemma 1.3, item 2), it follows then that  $(I + X)^{-1}$  is an  $M$ -matrix.

$\Leftarrow$  By assumption  $(I + X)^{-1}_{ij} \leq 0$ ,  $i \neq j$ . So,  $(I - (I + X)^{-1})_{ij} \geq 0$ ,  $i \neq j$ . Since  $I - (I + X)^{-1} = (I + X^{-1})^{-1} \succ 0$  it follows that also  $(I - (I + X)^{-1})_{ii} > 0$ . From which the conclusion is now obvious.

□

**Corollary 2.2**

Let  $X \succ 0$ . Then

1) (1) holds only if  $X \geq 0$ .

2) (1) holds if and only if  $X = Y - I$  for some positive definite matrix  $Y \succ I$  which has the property that  $Y^{-1}$  is Stieltjes.

Proof:

1) According Theorem 2.1 (1) holds only if  $(I + X)^{-1}$  is a Stieltjes matrix. But this implies that  $I + X \geq 0$ . From this it is obvious that  $X_{ij} \geq 0$ ,  $i \neq j$ . Since by assumption  $X \succ 0$  it follows that also  $X_{ii} \geq 0$ , which completes the proof.

2) " $\Rightarrow$ " According Theorem 2.1,  $Y^{-1} := (I + X)^{-1}$  is a Stieltjes matrix. Obviously,  $X = Y - I$ . Since, by assumption,  $X \succ 0$  it is clear that  $Y \succ I$ .

" $\Leftarrow$ " Follows directly by Theorem 2.1 from the fact that  $Y^{-1} = (I + X)^{-1}$  is Stieltjes. □

Another sufficient condition which immediately results from equality (3) is

**Theorem 2.3**

Let  $X \succ 0$  and assume that  $X^{-1}$  is a Stieltjes matrix. Then

1) (1) holds.

2) if  $X^{-1}$  is, moreover, diagonally dominant then  $(I + X)^{-1}$  is diagonally dominant too.

Proof:

1) If  $X^{-1}$  is a Stieltjes matrix,  $I + X^{-1}$  is a Stieltjes matrix too. So  $(I + X^{-1})^{-1} \geq 0$ . Using (3) the result follows then.

2) First note that  $I - (I + X)^{-1} \succ 0$ . From item 1) it follows that  $(I + X)^{-1} \in Z_n$ . Consequently,  $I - (I + X)^{-1} \geq 0$ . Using this and the assumption that  $X^{-1}e \geq 0$  it follows then immediately that

$$(I + X)^{-1}e = (I - (I + X)^{-1})X^{-1}e \geq 0.$$

□

#### Remark 2.4

From Theorem 2.3 it follows that all known conditions in literature that suffice to conclude from the positiveness of matrix  $X$  that  $X^{-1}$  is an  $M$ -matrix are also sufficient to conclude that (1) holds. So, e.g., if  $X \geq 0$  either is

- totally nonnegative and  $\det(X_{[i,j]}) = 0$  for  $i + j = 2k$ , where  $k$  is a positive integer and  $i \neq j$  (see Markham [5]);

- scaled to have unit diagonal elements and off-diagonal elements which satisfy  $0 < y \leq X_{ij} \leq x < 1$  and  $s$  defined by  $x^2 = sy + (1 - s)y^2$  satisfies  $s \leq \frac{1}{n-2}$  (assuming  $n \geq 3$ , see [8]);

-  $X_{ij} \geq \min(X_{ik}, X_{kj})$  for all  $i, j, k$  and  $X_{ii} > X_{ij}$  for all  $i, j$ ,  $i \neq j$ , (the so-called ultrametric matrices see (Martínez et al. [6]). These matrices can equivalently be characterized as  $X = \sum_{i=1}^{2n-1} \tau_i u_i u_i^T$ , where the nonzero vectors  $u_i$  have only entries 0 and 1,  $\tau_i \geq 0$  and always  $\tau_i > 0$  if  $u_i$  contains only one nonzero entry. These vectors  $u_i$  are determined from the  $2n - 1$  vertices of the with  $X$  associated rooted tree. (see Nabben et al. [7]).

So, in all of these cases we can conclude that (1) is satisfied too. Moreover, notice that the inverse of an ultrametric matrix is diagonally dominant (see [7] again). □

Lemma 2.5, below, contains on the one hand a result which will be used in the ensuing proofs and on the other hand some observations concerning diagonally dominant matrices.

#### Lemma 2.5

*Assume  $X$  is an invertible matrix and  $D$  is a positive diagonal matrix. Consider  $P := (X + D)^{-1}$ .*

1) *If  $X^{-1}$  is diagonally dominant, then  $P$  is diagonally dominant.*

2) *If  $X^{-1}$  is a Stieltjes matrix, then  $P$  is a Stieltjes matrix.*

3) *If  $X^{-1}$  is a diagonally dominant Stieltjes matrix, then  $(X + \alpha ee^T)^{-1}$  is a diagonally dominant Stieltjes matrix for all  $\alpha \geq 0$ .*

Proof:

1) First notice that

$$(X + D)^{-1} = D^{-1} - D^{-1}(D^{-1} + X^{-1})^{-1}D^{-1}. \quad (4)$$

Next consider

$$H := \begin{pmatrix} D^{-1} + X^{-1} & D^{-1} \\ D^{-1} & D^{-1} \end{pmatrix}$$

Due to our assumptions, it is easily verified that  $H$  is diagonally dominant. From e.g. Lei et al. [4] (see also Carlson et al. [2]) we conclude then that the Schur complement of  $H$ , which equals (4), is also diagonally dominant.

2) Since by assumption  $X^{-1}$  is a Stieltjes matrix, by Lemma 1.3, item 1),  $X^{-1} \succ 0$ . From this it is obvious that also  $P \succ 0$ . So, the diagonal entries of  $P$  are positive.

Furthermore since, by assumption, both  $X$  and  $D$  are a positive matrix also  $X + D$  is a positive matrix. Next we consider the off-diagonal entries of  $P$ . Since both  $D^{-1}$  and  $X^{-1}$  are Stieltjes matrices, also  $D^{-1} + X^{-1}$  is a Stieltjes matrix. So, in particular, all entries of  $(D^{-1} + X^{-1})^{-1}$  are positive. From (4) it is obvious then that all off-diagonal entries of  $P$  are negative. Since we already argued above that  $P$  is positive definite, Lemma 1.3 shows that  $P$  is a Stieltjes matrix.

3) Clearly, for every  $\alpha \geq 0$ ,  $(X + \alpha ee^T)^{-1} \succ 0$ . That this matrix is then a Stieltjes matrix follows from the fact that

$$\begin{aligned} (X + \alpha ee^T)^{-1} &= X^{-1} - \alpha X^{-1} e (\alpha e^T X^{-1} e + 1)^{-1} e^T X^{-1} \\ &= X^{-1} - \frac{\alpha}{\beta} y y^T, \end{aligned}$$

where  $\beta := \frac{1}{\alpha e^T X^{-1} e + 1} > 0$  and  $y = X^{-1} e \geq 0$  (since  $X^{-1}$  is diagonally dominant).

Furthermore it follows from the above identity that

$$\begin{aligned} (X + \alpha ee^T)^{-1} e &= X^{-1} e - \frac{\alpha}{\beta} y y^T e \\ &= \left(1 - \frac{\alpha y^T e}{\beta}\right) y \\ &= \frac{1}{\beta} y \geq 0. \end{aligned}$$

□

From items 1) and 3) of Lemma 2.5 we conclude in particular that if  $(I + X)^{-1}$  is diagonally dominant then also for an arbitrary positive diagonal matrix  $D$  and  $\alpha \geq 0$  the matrix  $(I + X + D + \alpha ee^T)^{-1}$  is diagonally dominant.

### Proposition 2.6

Let  $X \succ 0$ . Assume that (1) holds, then

$$(\alpha I + X^{-1})^{-1} \geq 0, \text{ for all } \alpha \in [0, 1].$$

Proof:

For  $\alpha = 0$  the result follows from Corollary 2.2, and for  $\alpha = 1$  by assumption.

Next consider for  $\alpha \in (0, 1)$ ,

$$(\alpha I + X^{-1})^{-1} = \frac{1}{\alpha} \left( I - \frac{1}{\alpha} \left( \frac{1}{\alpha} I + X \right)^{-1} \right).$$

Notice that  $\frac{1}{\alpha} I + X = \left(\frac{1}{\alpha} - 1\right) I + I + X$ , where  $D := \left(\frac{1}{\alpha} - 1\right) I$  is a positive diagonal matrix and  $(I + X)^{-1}$  is a Stieltjes matrix (see Theorem 2.1). So, according Lemma 2.5,  $\left(\frac{1}{\alpha} I + X\right)^{-1}$  is a Stieltjes matrix. Therefore

$$\left( I - \frac{1}{\alpha} \left( \frac{1}{\alpha} I + X \right)^{-1} \right)_{ij} \geq 0, \quad i \neq j. \quad (5)$$

Since  $(\alpha I + X^{-1})^{-1} \succ 0$  it follows that (5) also holds for  $i = j$ . □

**Remark 2.7**

From [1, Theorem 6.2.4] we have that if  $(\alpha I + A)^{-1} \geq 0 \forall \alpha \geq 0$ ,  $A$  is a nonsingular  $M$ -matrix. From this we therefore conclude that if  $X \succ 0$  and  $(I + X^{-1})^{-1} \geq 0$ , but  $X^{-1} \notin Z_n$  necessarily there exists an  $\alpha > 1$  such that  $(\alpha I + X^{-1})^{-1} \not\geq 0$ .  $\square$

Next, we present a class of matrices for which the inequality (1) is satisfied. The matrices are constructed recursively. Lemma 2.8, below, presents the initialization of this process, whereas Theorem 2.9 indicates how after the initialization the matrices are inductively constructed.

**Lemma 2.8**

Consider  $x_i, y_j \in \mathbb{R}^n$ ,  $i = 1, \dots, k, j = 1, \dots, l$ ,  $x_i, y_i \geq 0$ , with  $x_i^T x_j = 0, y_i^T y_j = 0$   $i \neq j$  and  $x_i^T y_j = 0$ . Let  $X_i := x_i x_i^T$ ,  $Y_i := y_i y_i^T$ ,  $x := \sum_{i=1}^k \rho_i x_i$ ,  $y := \sum_{i=1}^l \tau_i y_i$ ,  $X := \sum_{i=1}^k \alpha_i X_i$  and  $Y := \sum_{i=1}^l \beta_i Y_i$ . Then for all  $\alpha_i, \beta_i, \rho_i, \tau_i, \mu_i \geq 0$ ,

1)

$$(I + X)^{-1} = I - \sum_{i=1}^k \frac{\alpha_i}{1 + \alpha_i x_i^T x_i} x_i x_i^T \in Z_n, \quad (6)$$

and  $(I + X)^{-1} x \geq 0$ .

2)  $(I + X + \alpha_{k+1} x x^T)^{-1} \in Z_n$  and  $(I + X + \alpha_{k+1} x x^T)^{-1} x \geq 0$ .

3) With  $z := \mu_1 x + \mu_2 y$ , for all  $\gamma \geq 0$ ,

$$Z^{-1} := (I + X + \alpha_{k+1} x x^T + Y + \beta_{l+1} y y^T + \gamma z z^T)^{-1} \in Z_n,$$

and  $Z^{-1} z \geq 0$ .

Proof:

1) Obviously,  $I + \sum_{i=1}^r \alpha_i x_i x_i^T \succ 0$  and thus invertible. Straightforward calculations show that

$$\begin{aligned} (I + \sum_{i=1}^r \alpha_i x_i x_i^T) (I - \sum_{i=1}^k \frac{\alpha_i}{1 + \alpha_i x_i^T x_i} x_i x_i^T) &= I + \sum_{i=1}^r \alpha_i x_i x_i^T - \sum_{i=1}^k \frac{\alpha_i}{1 + \alpha_i x_i^T x_i} x_i x_i^T \\ &\quad - \sum_{i=1}^k \frac{\alpha_i^2 x_i^T x_i}{1 + \alpha_i x_i^T x_i} x_i x_i^T \\ &= I. \end{aligned}$$

Which shows the correctness of the inversion formula. From the fact that  $x_i \geq 0$  it follows then from (6) that all nondiagonal entries of  $(I + \sum_{i=1}^k \alpha_i X_i)^{-1}$  are nonpositive, whereas its diagonal entries are positive since  $(I + \sum_{i=1}^r \alpha_i x_i x_i^T)^{-1} \succ 0$ .

Finally, it follows directly from (6) and the fact that  $x_i^T x_j = 0$  that

$$\begin{aligned}
(I + X)^{-1}x &= x - \sum_{i=1}^k \frac{\alpha_i}{1 + \alpha_i x_i^T x_i} x_i x_i^T x \\
&= x - \sum_{i=1}^k \frac{\alpha_i \rho_i}{1 + \alpha_i x_i^T x_i} x_i x_i^T x_i \\
&= \sum_{i=1}^k \rho_i \left(1 - \frac{\alpha_i x_i^T x_i}{1 + \alpha_i x_i^T x_i}\right) x_i \\
&= \sum_{i=1}^k \frac{\rho_i}{1 + \alpha_i x_i^T x_i} x_i \geq 0.
\end{aligned}$$

2) Using (2) again we have that

$$(I + X + \alpha_{k+1} x x^T)^{-1} = (I + X)^{-1} - \frac{\alpha_{k+1}}{1 + \alpha_{k+1} x^T (I + X)^{-1} x} (I + X)^{-1} x x^T (I + X)^{-1}.$$

From item 1) we have that  $(I + X)^{-1} \in Z_n$  and  $(I + X)^{-1}x \geq 0$ . Using this, and the fact that  $(I + X + \alpha_{k+1} x x^T)^{-1} \succ 0$ , it is obvious then that  $(I + X + \alpha_{k+1} x x^T)^{-1} \in Z_n$ .

Furthermore it follows that

$$\begin{aligned}
(I + X + \alpha_{k+1} x x^T)^{-1}x &= (I + X)^{-1}x - \frac{\alpha_{k+1}}{1 + \alpha_{k+1} x^T (I + X)^{-1} x} (I + X)^{-1} x x^T (I + X)^{-1}x \\
&= \frac{1}{1 + \alpha_{k+1} x^T (I + X)^{-1} x} (I + X)^{-1}x \geq 0.
\end{aligned}$$

3) First note that  $(X + \alpha_{k+1} x x^T)(Y + \beta_{l+1} y y^T) = (Y + \beta_{l+1} y y^T)(X + \alpha_{k+1} x x^T) = 0$ . So, by Lemma 1.4 item 3) and Lemma 2.8 item 2),

$$\begin{aligned}
V^{-1} &:= (I + X + \alpha_{k+1} x x^T + Y + \beta_{l+1} y y^T)^{-1} \\
&= (I + X + \alpha_{k+1} x x^T)^{-1} + (I + Y + \beta_{l+1} y y^T)^{-1} - I \in Z_n.
\end{aligned}$$

Furthermore we conclude from this expression and Lemma 1.4, item 2), and Lemma 2.8, item 2), respectively, that

$$\begin{aligned}
V^{-1}z &= (I + X + \alpha_{k+1} x x^T)^{-1}z + (I + Y + \beta_{l+1} y y^T)^{-1}z - z \\
&= \mu_2 y + \mu_1 (I + X + \alpha_{k+1} x x^T)^{-1}x + \mu_1 x + \mu_2 (I + Y + \beta_{l+1} y y^T)^{-1}y - (\mu_1 x + \mu_2 y) \\
&= \mu_1 (I + X + \alpha_{k+1} x x^T)^{-1}x + \mu_2 (I + Y + \beta_{l+1} y y^T)^{-1}y \geq 0.
\end{aligned}$$

From (2) it follows then (using the fact that  $Z^{-1} \succ 0$ ) that

$$Z^{-1} = V^{-1} - \frac{\gamma}{1 + \gamma z^T V^{-1} z} V^{-1} z z^T V^{-1} \in Z_n.$$

Finally,

$$\begin{aligned}
Z^{-1}z &= V^{-1}z - \frac{\gamma}{1 + \gamma z^T V^{-1} z} V^{-1} z z^T V^{-1} z \\
&= \frac{1}{1 + \gamma z^T V^{-1} z} V^{-1}z \geq 0.
\end{aligned}$$



□

Theorem 2.9 is a straightforward generalization of Lemma 2.8, item 3). Since the proof is a direct copy of this part of the lemma, we skip it.

**Theorem 2.9** *Assume  $X, Y \in \mathbb{R}^{n \times n}$  are nonnegative positive semi-definite matrices and  $x, y \in \mathbb{R}^n$  nonnegative vectors such that*

1)  $XY = YX = 0$ ,  $Xy = Yx = 0$  and  $x^T y = 0$ ;

2) for all  $\alpha, \beta \geq 0$ ,  $(I + X + \alpha xx^T)^{-1}$ ,  $(I + Y + \alpha yy^T)^{-1} \in Z_n$ ;

3) for all  $\alpha, \beta \geq 0$ ,  $(I + X + \alpha xx^T)^{-1}x$  and  $(I + Y + \alpha yy^T)^{-1}y$  are nonnegative.

Then, with  $z := \mu_1 x + \mu_2 y$  and  $Z^{-1} := (I + X + \alpha xx^T + Y + \beta yy^T + \gamma zz^T)^{-1}$ , for all  $\alpha, \beta, \gamma, \mu_i \geq 0$ ,  $Z^{-1} \in Z_n$  and  $Z^{-1}z \geq 0$ . □

The next algorithm shows how one can construct a set of inverse  $M$ -matrices satisfying (1). The proof that indeed all matrices generated in this way satisfy (1) follows directly from Lemma 2.8 and Theorem 2.9.

### Algorithm 2.10

*Step 1* Choose sets of orthogonal vectors  $\mathcal{X}_1$  and  $\mathcal{Y}_i$ ,  $i = 1, \dots, m$  which are mutual orthogonal too.

*Step 2* Choose arbitrary nonnegative vector  $x_1 \in \text{Span}\mathcal{X}_1$  and  $y_i \in \text{Span}\mathcal{Y}_i$ , respectively. Let  $z_1$  be an arbitrary nonnegative linear combination of  $x_1$  and  $y_1$ .

Assume  $x_{1j} \in \mathcal{X}_1, j = 1, \dots, k$  and  $y_{ij} \in \mathcal{Y}_i, j = 1, \dots, l_i$ . Let, for arbitrary nonnegative  $\alpha_j, \beta_{ij}, \gamma_i$ ,  $X_{[1]} := \sum_{j=1}^k \alpha_j x_{1j} x_{1j}^T + \alpha_{k+1} x_1 x_1^T$ ,  $Y_{[i]} := \sum_{j=1}^{l_i} \beta_{ij} y_{ij} y_{ij}^T + \beta_{l_i+1} y_i y_i^T$  and  $Z_1 := X_{[1]} + Y_{[1]} + \gamma_1 z_1 z_1^T$ .

*Step 3* Next calculate recursively for  $t = 1$  to  $m$

$$\begin{aligned} x_{t+1} &:= z_t \\ X_{[t+1]} &:= X_{[t]} + Y_{[t]} + \alpha_{k+t+1} x_t x_t^T, \\ z_{t+1} &:= \mu_1^{t+1} x_{t+1} + \mu_2^{t+1} y_{t+1} \text{ and} \\ Z_{t+1} &:= X_{[t+1]} + Y_{[t+1]} + \gamma_{t+1} z_{t+1} z_{t+1}^T. \end{aligned}$$

*Step 4* Then,  $Z_{m+1}$  is such that  $(I + Z_{m+1})^{-1} \in Z_n$ . □

### Remark 2.11

1) Obviously, in the above algorithm the choice and numbering of the sets  $\mathcal{X}_1$  and  $\mathcal{Y}_i$  is arbitrary. Consequently, all matrices that can be obtained from the sets  $\mathcal{X}_1$  and  $\mathcal{Y}_i$  that satisfy inequality (1) are permutations of matrix  $Z_{m+1}$  calculated in Step 4 of the algorithm.

2) By considering in the above algorithm for each set the natural basis vector  $e_i$  we obtain the set of all matrices that can be generated using Theorem 2.9 that satisfy the inequality (1). Example 2.13, below, gives an illustration in case  $n = 3$ . □

**Corollary 2.12** *With the notation of Lemma 2.8 and Theorem 2.9 let  $\mathcal{X}_1 := e_1$  and  $\mathcal{Y}_i := e_{i+1}, i = 1, \dots, n-1$ . Choose  $\rho_i = \tau_i = \mu_i = 1$ . Then, using Algorithm 2.10, it follows that any ultrametric matrix satisfies the inequality (1). □*

The next example provides a complete description of all  $3 \times 3$  inverse  $M$ -matrices that can be generated from Theorem 2.9. In particular it shows that the set of Stieltjes matrices which can be generated using Theorem 2.9 contains also matrices that are not ultrametric.

**Example 2.13**

Consider  $\mathcal{X}_1 := \{e_1\}$  and  $\mathcal{Y}_i := \{e_{i+1}\}, i = 1, 2$ , with  $e_i \in \mathbb{R}^3$ . Then for every  $\alpha_i, \beta_i, \mu_i, \rho_i, \gamma \geq 0$ , the next matrix is an inverse  $M$ -matrix

$$\begin{aligned} I + X &= I + \alpha_1 e_1 e_1^T + \alpha_2 e_2 e_2^T + \alpha_3 (\rho_1 e_1 + \rho_2 e_2) (\rho_1 e_1 + \rho_2 e_2)^T + \beta_1 e_3 e_3^T + \\ &\quad \gamma (\mu_1 \rho_1 e_1 + \mu_1 \rho_2 e_2 + \mu_2 e_3) (\mu_1 \rho_1 e_1 + \mu_1 \rho_2 e_2 + \mu_2 e_3)^T \\ &= \begin{pmatrix} 1 + \alpha_1 + (\alpha_3 + \gamma \mu_1^2) \rho_1^2 & (\alpha_3 + \gamma \mu_1^2) \rho_1 \rho_2 & \gamma \mu_1 \rho_1 \mu_2 \\ (\alpha_3 + \gamma \mu_1^2) \rho_1 \rho_2 & 1 + \alpha_2 + (\alpha_3 + \gamma \mu_1^2) \rho_2^2 & \gamma \mu_1 \rho_2 \mu_2 \\ \gamma \mu_1 \rho_1 \mu_2 & \gamma \mu_1 \rho_2 \mu_2 & 1 + \beta_1 + \gamma \mu_2^2 \end{pmatrix}. \end{aligned} \quad (7)$$

By considering above the case  $\alpha_i = 0, \gamma = 1, \mu_1 = 1, \rho_1 = 1, \rho_2 = 2, \mu_2 = 3$  we see that

$$I + X := \begin{pmatrix} 2 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 10 \end{pmatrix},$$

is an inverse  $M$ -matrix too. It is easily verified that  $I + X$  is not ultrametric.

Finally notice that all other inverse  $M$ -matrices that can be generated using Algorithm 2.10 are permutations of (7), i.e. are of the form  $P(I + X)P^T$  where  $P$  is a permutation matrix.  $\square$

**Example 2.14**

Consider the nonnegative vectors  $v_i \in \mathbb{R}^{n_i}, n_i \geq 1, i = 1, \dots, k$ , with  $\sum_{i=1}^k n_i = n$ . Let  $w_j^T := (v_1^T, v_2^T, \dots, v_j^T, 0) \in \mathbb{R}^n, j = 1, \dots, k$ .

Then, for all  $\lambda_i \geq 0, (I + X)^{-1}$  is a Stieltjes matrix if

$$X = \sum_{i=1}^k \lambda_i w_i w_i^T.$$

This follows, using the notation from Lemma 2.8 and Theorem 2.9, by considering  $\mathcal{X}_1 := \{w_1\}$  and  $\mathcal{Y}_j := \{y_j\}$ , where  $y_j = (0, v_{j+1}^T, 0), j = 1, \dots, k - 1$ , respectively, with  $\rho_i = \tau_i = \mu_i = 1$ .  $\square$

Unfortunately, Algorithm 2.10 does not generate all nonnegative positive definite matrices for which the inequality (1) holds. This is shown in the next example.

**Example 2.15** Consider

$$X := \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Then  $X \succ 0$  and  $(I + X)^{-1} \in \mathcal{Z}_3$ . However, the equation (see (7))

$$\begin{pmatrix} \alpha_1 + (\alpha_3 + \gamma \mu_1^2) \rho_1^2 & (\alpha_3 + \gamma \mu_1^2) \rho_1 \rho_2 & \gamma \mu_1 \rho_1 \mu_2 \\ (\alpha_3 + \gamma \mu_1^2) \rho_1 \rho_2 & \alpha_2 + (\alpha_3 + \gamma \mu_1^2) \rho_2^2 & \gamma \mu_1 \rho_2 \mu_2 \\ \gamma \mu_1 \rho_1 \mu_2 & \gamma \mu_1 \rho_2 \mu_2 & \beta_1 + \gamma \mu_2^2 \end{pmatrix} = X$$

has not a solution with  $\alpha_i, \beta_i, \mu_i, \rho_i, \gamma \geq 0$ . For, from the (1,3) and (2,3) entries it follows that  $\rho_1 = \rho_2$ . Consequently, according the (1,2) entry,  $(\alpha_3 + \gamma\mu_1^2)\rho_1^2 = 2$ . However this implies that the (1,1) entry does not have an appropriate solution for  $\alpha_1$ .

In a similar way it can be shown that also for the permuted matrices of (7) no appropriate solution exists.  $\square$

We conclude this section with the observation that, if we have a set of complementary matrices,  $X_i$ , for which  $(I + X_i)^{-1}$  are Stieltjes, also the convex hull of these matrices has this property. The exact statement follows in Theorem 2.17 below. To prove this theorem we consider the following lemma.

**Lemma 2.16**

- 1) Assume that  $(I + VV^T)^{-1}$  is a Stieltjes matrix, then  $V(I + V^TV)^{-1}V^T \geq 0$ .
- 2) If  $(I + X)^{-1}$  is a Stieltjes matrix then for all  $0 \leq \alpha \leq 1$  also  $(I + \alpha X)^{-1}$  is a Stieltjes matrix.
- 3) Assume  $X \succeq 0$  and  $Y \succeq 0$  are such that both  $(I + X)^{-1}$  and  $(I + Y)^{-1}$  are Stieltjes matrices and  $XY = 0$ . Then  $(I + X + Y)^{-1}$  is a Stieltjes matrix too.

Proof:

1) From (2) we have

$$(I + VV^T)^{-1} = I - V(I + V^TV)^{-1}V^T.$$

From this it follows immediately that

$$(V(I + V^TV)^{-1}V^T)_{ij} = (I - (I + VV^T)^{-1})_{ij} \geq 0, \text{ if } i \neq j.$$

Since  $V^TV + I \succ 0$  it follows that also the diagonal entries of the above mentioned matrix are nonnegative, which proves the claim.

- 2) Note that  $(I + \alpha X)^{-1} = \frac{1}{\alpha}(I + X + (\frac{1}{\alpha} - 1)I)^{-1}$ . Lemma 2.5, item 2), yields then the result.
- 3) Let  $Y =: VV^T$ . Using Lemma 1.4 we have

$$\begin{aligned} (I + X + Y)^{-1} &= (I + X)^{-1} - (I + X)^{-1}V(V^T(I + X)^{-1}V + I)^{-1}V^T(I + X)^{-1} \\ &= (I + X)^{-1} - V(V^TV + I)^{-1}V^T. \end{aligned}$$

From item 1) the conclusion follows then immediately.  $\square$

**Theorem 2.17**

Let  $X_i \succ 0$ ,  $i = 1, \dots, k$ , be such that  $(I + X_i)^{-1}$  are Stieltjes matrices and  $X_i X_j = 0$ ,  $i \neq j$ . Then, for all  $0 \leq \alpha_i \leq 1$ ,  $(I + \sum_{i=1}^k \alpha_i X_i)^{-1}$  is a Stieltjes matrix.  $\square$

The proof of this theorem follows by a simple induction argument using Lemma 2.16, items 2) and 3). Finally notice that Lemma 2.16, item 2), does not hold for  $\alpha > 1$  as the next example shows. Furthermore, a second example shows that item 3) also breaks down in general if  $XY \neq 0$ .

**Example 2.18**

1) Let

$$X^{-1} := \begin{pmatrix} 9 & 1 & -2.1 \\ 1 & 2 & -1 \\ -2.1 & -1 & 1 \end{pmatrix}.$$

Then  $X \succ 0$  and  $X > 0$ . Furthermore,  $(I + X)^{-1} \in Z_3$  whereas  $(I + 4X)^{-1} \notin Z_3$ .  
 2) Let

$$X := \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } Y := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Then  $X \succ 0$ ,  $Y \succ 0$ ,  $(I + X)^{-1} \in Z_3$ ,  $(I + Y)^{-1} \in Z_3$ , but  $(I + X + Y)^{-1} \notin Z_3$ . □

### 3 Concluding remarks

In this note we considered the question under which conditions the inequality (1) holds in case  $X$  is a positive definite matrix. We showed that this is the case if and only if matrix  $(I + X)^{-1}$  is a Stieltjes matrix. A necessary condition for this is that matrix  $X$  is nonnegative. A sufficient condition is that matrix  $X^{-1}$  is a Stieltjes matrix. Furthermore we presented conditions under which matrix  $(I + X)^{-1}$  is diagonally dominant.

Next we derived a class of matrices, including the ultrametric matrices, that satisfy (1). An open problem is to find a characterization of all matrices  $X$  for which (1) holds.

Finally we showed that in case a set of matrices  $X_i$  are complementary, and  $(I + X_i)^{-1}$  are Stieltjes, then also the convex combination of these matrices has this property. Unfortunately, this result does not hold in general in case the matrices are not complementary.

### References

- [1] Berman A. and Plemmons R.J., 1994, *Nonnegative Matrices in the Mathematical Sciences*, SIAM, Philadelphia.
- [2] Carlson D. and Markham T., Schur complements of diagonally dominant matrices, *Czech. Math. J.* bf 29 (1979), 246-251.
- [3] Engwerda J.C. and Douven R.C., 2005, On the sensitivity matrix of the Nash bargaining solution, *CentER Discussion Paper*, Internal report no. 2005-35.
- [4] Lei T.-G, Woo C.-W, Liu J.-Z. and Zhang F., On the Schur complement of diagonally dominant matrices, *Proceedings of the SIAM Conference on Applied Linear Algebra*, July 2003, Williamsburg VA, CP 13.
- [5] Markham T.L., 1972, Nonnegative matrices whose inverses are  $M$ -matrices, *Proceedings of the American Mathematical Society*, vol.36, no.2, pp.326-330.
- [6] Martínez S., Michon G. and San Martín J., 1994, Inverses of ultrametric matrices are of Stieltjes type, *SIAM Journal on Matrix Analysis and Applications*, vol.15, pp.98-106.
- [7] Nabben R. and Varga R.S., 1994, A linear algebra proof that the inverse of a strictly ultrametric matrix is a strictly diagonally dominant Stieltjes matrix, *SIAM Journal on Matrix Analysis and Applications*, vol.15, pp.107-113.

- [8] Willoughby R.A., 1977, The inverse  $M$ -matrix problem, *Linear Algebra and its Applications*, vol.18, pp.75-94.