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Symmetric convex games and stable structures

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Abstract

We study the model of link formation that was introduced by Aumann and Myerson (1988) and focus on symmetric convex games with transferable utilities. We answer an open question in the literature by showing that in a specific symmetric convex game with six players a structure that results in the same payoffs as the full cooperation structure can be formed according to a subgame perfect Nash equilibrium.

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1 Introduction

Since the organization of individual agents in a network has an important role in the determination of the outcome of social and economic interaction there has been a huge amount of literature on networks. The game-theoretical point of view on networks has received a lot of attention in the last decade. Myerson (1977) provides a first major contribution on cooperative games and networks by introducing an allocation rule for cooperative games supplemented with a network structure. Subsequently, Aumann and Myerson (1988) apply this allocation rule to study the formation of networks by means of a game in extensive form. However, it lasted until the work of Jackson and Wolinsky (1996), in which they study the tension between efficiency and stability, that the analysis of the formation of networks became a focal point. This has up to now already resulted in a vast amount of literature. A survey of the literature that used formal game theoretic reasoning to study the formation of networks can be found in Dutta and Jackson (2003).

Jackson and Wolinsky (1996) departed from the basic framework studied by Myerson (1977) and Aumann and Myerson (1988) by starting with a value function that is defined on networks directly rather than a characteristic function that is defined on coalitions. Though many subsequent papers deal with value functions, cooperative games proved to be relevant from different points of view (see e.g., Jackson and van den Nouweland (2004) and Jackson (2004)). In the current paper we will stick to the original setting of Aumann and Myerson (1988). For their model of network/link formation in extensive form they note that for superadditive games partial cooperation might result according to the subgame perfect Nash equilibrium concept. For a similar model of link formation in strategic form, however, it was shown by Dutta et al. (1998) that according to several equilibrium refinements only networks with the same payoffs as the full cooperation structure will result.

The issue whether in a specific setting equilibria lead to structures with the same payoff as the complete structure was studied by Slikker and Norde (2004). They consider the model in extensive form of Aumann and Myerson (1988) for symmetric convex games and showed that with at most five players the full cooperation structure can be formed in equilibrium. Additionally, they showed for *strictly* convex symmetric games with at most 5 players that all structures that can be formed in equilibrium result in the same payoffs as the full cooperation structure. Furthermore, they showed that this last result cannot be extended to cooperative games with 6 players. They consider a symmetric convex game with 6 players and show that networks can result in which two players receive strictly less than they would according to the full cooperation structure and four players receive strictly more. In fact, any pair of players can be exploited, independent of the rule of order. With respect to this game Slikker and Norde (2004) conclude with the following question:

> "Does a subgame perfect Nash equilibrium that results in a structure payoff equivalent to the full cooperation structure exist?"

In the current paper we will answer this question to the affirmative for all initial orders of the pairs of players.

The setup of this paper is as follows. In Section 2 we provide some preliminaries on games and networks. The model of Aumann and Myerson (1988) and some notation and results of Slikker and Norde (2004) can be found in Section 3. Finally, Section 4 contains the main result of this paper.

2 Preliminaries

This section contains game and graph-theoretical notations and definitions that are used throughout.

A *cooperative game* is a pair (N, v) where $N = \{1, \ldots, n\}$ is a finite set (of players) and $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ the characteristic function which assigns to any coalition $S \subseteq N$ a value $v(S)$ which represents the worth of coalition S. A cooperative game (N, v) is *superadditive* if for all $T_1 \subseteq N$ and all $T_2 \subseteq N \backslash T_1$ it holds that

$$v(T_1) + v(T_2) \leq v(T_1 \cup T_2). \tag{1}$$

A cooperative game (N, v) is *convex* if for all $i \in N$ and all $T_1 \subseteq T_2 \subseteq N$ with $i \in T_1$ it holds that

$$v(T_1) - v(T_1 \backslash \{i\}) \leq v(T_2) - v(T_2 \backslash \{i\}). \tag{2}$$

So, a game is convex if the marginal contribution of a player to any coalition is less than his marginal contribution to a larger coalition. A cooperative game is *strictly convex* if (2) holds with strict inequality for all $i \in N$ and all $T_1 \subset T_2 \subseteq N$ with $i \in T_1$.

Let N be a set of players and let $R \in 2^N \backslash \{\emptyset\}$. The *unanimity game* (N, u_R) is the game with $u_R(S) = 1$ if $R \subseteq S$ and $u_R(S) = 0$ otherwise (see Shapley (1953)). Every game (N, v) can be written as a linear combination of unanimity games in a unique way, i.e., $v = \sum_{R \in 2^N \backslash \{\emptyset\}} \lambda_R(v) u_R$. The *Shapley value* $\Phi(N, v)$ of a game (N, v) is now easily described by[1]

$$\Phi_i(N, v) = \sum_{R \subseteq N:\ i \in R} \frac{\lambda_R(v)}{|R|} \text{ for all } i \in N.$$

A (*communication*) *graph* is a pair (N, L) where the set of vertices N represents the set of players and the set of edges L represents the set of bilateral (communication) links. Two players i and j are *directly connected* iff $\{i, j\} \in L$. For notational convenience we usually denote ij rather than $\{i, j\}$ Two players i and j are *connected* (directly or indirectly) iff $i = j$ or there exists a path between players i and j. The notion of connectedness induces a partition of the player set into communication components, where two players are in the same *communication component* if and only if they are connected. The set of communication components will be denoted by N/L. The component $C \in N/L$ containing player $i \in N$ will be denoted by $C_i(L)$. Furthermore, denote the subgraph on the vertices in coalition $S \subseteq N$ by $(S, L(S))$, where $L(S) = \{ij \in L \mid \{i, j\} \subseteq S\}$, and the partition of S into communication components according to graph $(S, L(S))$ by S/L. Furthermore, define $L^N = \{ij \mid \{i, j\} \subseteq N\}$. Finally, the set of all undirected graphs with vertex set N will be denoted by UG^N.

[1] $|R|$ denotes the cardinality of a set R.

Myerson (1977) studied communication situations (N, v, L) where (N, v) is a cooperative game and (N, L) a communication graph. An allocation rule for communication situations is a function γ that assigns a payoff vector $\gamma(N, v, L)$ to any communication situation (N, v, L). Myerson (1977) introduced an allocation rule based on the *graph-restricted game* (N, v^L), where

$$v^L(S) = \sum_{C \in S/L} v(C) \text{ for all } S \subseteq N.$$

So, a coalition is split into communication components and the value of this coalition in the graph-restricted game is then defined as the sum of the values of the communication components in the original game. The allocation rule introduced by Myerson (1977) is the Shapley value of the game (N, v^L) and is usually referred to as the *Myerson value* of communication situation (N, v, L). Notation:

$$\mu(N, v, L) = \Phi(N, v^L).$$

The analysis in this paper concentrates on symmetric games. A game (N, v) is symmetric if there exist $v_1, v_2, \ldots, v_{|N|} \in \mathbb{R}$ such that $v(S) = v_{|S|}$ for all $S \in 2^N \backslash \{\emptyset\}$. So, in a symmetric game the value of a coalition only depends on its size.

3 A model of link formation

In this section we shortly recall the model of link formation studied by Slikker and Norde (2004) and their main results. The model under consideration is (a variant of) the model of link formation introduced by Aumann and Myerson (1988).

Let (N, v) be a cooperative game with $|N| \geq 2$ and let γ be an allocation rule for communication situations. Let σ be an exogenously given order of pairs of players. Formally, $\sigma : L^N \to \left\{1, 2, \ldots, \binom{n}{2}\right\}$ is a bijection where $\sigma(ij) = k$ denotes that pair ij is in position k. We will denote the link formation game in extensive form determined by cooperative game (N, v), allocation rule γ, and initial order σ by $\Delta^{lf}(N, v, \gamma, \sigma)$. The game starts with no links formed. The first pair of players according to σ gets the opportunity to form a link. This link is actually formed if and only if both players agree on forming this link. If a link is formed, it cannot be broken in a further stage of the game. After a pair of players decided on whether or not to form a link, the next pair of players according to σ who did not form a link with each other yet, gets the opportunity to do so. After the last pair of players in the order has had the opportunity to form a link, the first pair of players in the order who did not form a link with each other yet, gets a new opportunity to form the link between them. The process stops when, after the last link that has been formed, all pairs of players who have not formed a link with each other yet, have had a final opportunity to do so and declined this offer, i.e., in all these pairs at least one of the players refuses to form the link. Throughout the process of link formation the entire history of acceptances and rejections is known to all players. This process results in a set of links, which represents in conjunction with the player set an undirected graph. We will denote this set of links by L. The payoffs to the players are then determined by the allocation rule, i.e., if (N, L) is formed player $i \in N$ receives

$$\gamma_i(N, v, L).$$

In the original model of Aumann and Myerson (1988) player i receives his Myerson value $\mu_i(N, v, L)$. We will restrict ourselves to the Myerson value in this paper as well.

Aumann and Myerson (1988) already argued that since the game of link formation is of perfect information it has subgame perfect Nash equilibria. Furthermore, they note that the order in which two players in a pair decide whether or not to form a link has no influence (on the outcome of the game). Either order leads to the same outcome as simultaneous choice.

Following Slikker and Norde (2004) we, with a slight abuse of notation, sometimes refer to a decision of a link where we actually mean the decisions of the players in the (potential) link. Consider such a decision of a link and assume that strategies are fixed after the decision of this link. If both players (weakly) prefer to form the link then we call their choice to form subgame perfect. Furthermore, if at least one player (weakly) prefers not to form the link then we call their choice not to form subgame perfect. We remark that if links play subgame perfect then one can easily determine subgame perfect play of the players that results in the same outcome. With respect to the notation of subgame perfect Nash equilibria in subgames we follow Slikker and Norde (2004) as well. Let γ be an allocation rule for communication situations and let (N, v) be a cooperative game. A link formation game in extensive form in which the links in A have already been formed is denoted by $\Delta^{lf}(N, v, \gamma, \sigma, A)$, with $A \subset L^N$ a set of links, and $\sigma : L^N \backslash A \to \left\{1, 2, \ldots, \binom{n}{2} - |A|\right\}$ an order of the pairs of players who did not form a link with each other yet. If L is the set of links that have been formed in the game then player $i \in N$ receives $\gamma_i(N, v, L \cup A)$. We denote the set of subgame perfect Nash equilibria of $\Delta^{lf}(N, v, \gamma, \sigma)$ by SPNE($\Delta^{lf}(N, v, \gamma, \sigma)$) and, similarly, the set of subgame perfect Nash equilibria of $\Delta^{lf}(N, v, \gamma, \sigma, A)$ by SPNE($\Delta^{lf}(N, v, \gamma, \sigma, A)$). By $\sigma_{A,ij}$ we denote the order restricted to $L^N \backslash A$ that results when the links in A have been formed and $ij \in A$ is the link in A that has been formed last. Then $\Delta^{lf}(N, v, \gamma, \sigma_{A,ij}, A)$ is a subgame of $\Delta^{lf}(N, v, \gamma, \sigma)$. Furthermore, for all $k \in \{0, \ldots, \binom{n}{2} - |A| - 1\}$ we have that $\Delta^{lf}(N, v, \gamma, \sigma_{A,ij}, A, k)$ is a subgame of $\Delta^{lf}(N, v, \gamma, \sigma)$ where after the last link in A has been formed, k pairs of players have had the opportunity to form a link and have refused to do so.

If there exists a subgame perfect Nash equilibrium that results in graph (N, L) then this graph is called a *perfect equilibrium graph.* We denote the set of perfect equilibrium graphs in $\Delta^{lf}(N, v, \gamma, \sigma)$ by PEG(N, v, γ, σ). Similarly, we denote the set of graphs that result according to subgame perfect equilibria in a subgame $\Delta^{lf}(N, v, \gamma, \sigma, A)$ by PEG$(N, v, \gamma, \sigma, A)$

In case there is no ambiguity about the underlying game (N, v) or the allocation rule γ we will sometimes simply omit it. So, for example, a link formation game can be denoted by $\Delta^{lf}(N, v, \gamma, \sigma)$, $\Delta^{lf}(\gamma, \sigma)$, or even $\Delta^{lf}(\sigma)$. A similar remark holds for all other notations that include an underlying cooperative game and/or an allocation rule.

The main results of Slikker and Norde (2004) deal with symmetric convex games. They show that if the Myerson value is applied as an allocation rule then with at most five players the full cooperation structure results according to a subgame perfect Nash equilibrium. Moreover, if the game is strictly convex then every subgame perfect Nash equilibrium results in a structure that is payoff equivalent to the full cooperation structure. Subsequently, they study a symmetric game with six players that is strictly convex and show that there exists a subgame perfect Nash equilibrium that results in an incomplete structure in which two players are worse off than in the full cooperation structure, whereas four players are better off. Independent of the initial order any pair of players can end up being exploited. This game (N, v) with six players is extensively studied in this paper and fixed by player set $N = \{1, 2, 3, 4, 5, 6\}$ and characteristic function v described by

$$v(T) = \begin{cases} 0 & \text{if } |T| \leq 1; \\ 60 & \text{if } |T| = 2; \\ 180 & \text{if } |T| = 3; \\ 360 & \text{if } |T| = 4; \\ 600 & \text{if } |T| = 5; \\ 1800 & \text{if } T = N. \end{cases} \tag{3}$$

This characteristic function can alternatively be described by $v = 60 \sum_{i,j \in N: i \neq j} u_{\{i,j\}} + 900 u_N$. Recall that two graphs (N_1, L_1) and (N_2, L_2) are *isomorphic* if there is a one-to-one correspondence between the vertices in N_1 and those in N_2 with the property that two vertices in N_1 are connected directly in (N_1, L_1) if and only if the corresponding vertices in N_2 are connected directly in (N_2, L_2). An overview of the payoffs to the players in communication situations (N, v, L), with (N, v) as described above and (N, L) any of the 156 non-isomorphic graphs with six vertices, is taken from Slikker and Norde (2004) and can be found in appendix B. We will refer to the graph with number i in appendix B by (N, L^i).

The following notation is in line with Slikker and Norde (2004) as well. Note that in any graph isomorphic to (N, L^{146}) two players are *exploited* by the others, i.e., these two players receive 298 only, whereas both of them would receive 300 in the full cooperation structure. Furthermore, note that any exploited player is connected with exactly two not-exploited players besides the other exploited player. The graph isomorphic to (N, L^{146}) with players i and j exploited and player i additionally connected to players r and t is denoted by $G^{i,j}_{r,t}$. Furthermore, we denote $G^{i,j} = \{G^{i,j}_{r,t} \mid r, t \in N \backslash \{i, j\}\}$, the set of graphs isomorphic to (N, L^{146}) with players i and j exploited. Note that $G^{i,j} = G^{j,i}$. We denote that a link l precedes l' according to σ by $l \prec_\sigma l'$. If l and l' are such that $\sigma(l) = \sigma(l') - 1$ we say that l is *right in front of* l' (according to σ).

An intermediate result of Slikker and Norde (2004), lemma 5.1, is used extensively. We recall this lemma below.

Lemma 3.1 (lemma 5.1 of Slikker and Norde (2004))
Let (N, v) be the 6-person game described by (3), let $\mathcal{L}$ be the set of graphs that are payoff equivalent to (N, L^N) or isomorphic to (N, L^{146}), let $L \subset L^N$, and let σ be any order of the links in $L^N \backslash L$. Then $\text{PEG}(N, v, \mu, \sigma, L) \subseteq \mathcal{L}$.

As in Slikker and Norde (2004) we apply a specific format for describing a subgame perfect Nash equilibrium. This is formalized in the following remark. More explanation can be found in example 5.3 of Slikker and Norde (2004).

Remark 3.1 We often use a specific format to describe a subgame perfect Nash equilibrium in some subgame $\Delta^{lf}(\sigma, A)$. This description will be in two parts. The first part contains a description of the choices of the links in the subgames that follow after the first formation of a link in the game under consideration, i.e, the subgames $\Delta^{lf}(\sigma_{A\cup\{l\},l}, A\cup\{l\})$ for any $l \in L^N\backslash A$. The second part contains a description of the choices of the links that have not been described in the first part, i.e, the description of the initial choice of any link l in the subgame that starts after all links that precede l according to σ have not formed. Formally, the second part contains the initial choice of any link $l \in L^N\backslash A$ in the subgame $\Delta^{lf}(\sigma, A, \sigma(l)-1)$. In general the description of a part will distinguish between several cases, depending for example on the position of l in σ. These cases will be presented as subparts. By backward induction it follows that to prove that such a strategy profile constitutes a subgame perfect Nash equilibrium it suffices to show that the choices in part 1 constitute subgame perfect Nash equilibria in their respective subgames and that the initial choices in part 2 are subgame perfect.

4 Stable structures

The main result of this section states that independent of the initial order there exists a subgame perfect Nash equilibrium that attributes the same payoff to all players as they would receive according to the full cooperation structure. The associated graphs are wheels, i.e., structures in which each player is involved in two links and there is a unique path from a player to himself, which involves all players.

First, we introduce some additional definitions. A *wheel* (N, W) is a graph that is isomorphic to $(N, \{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\})$. So, a wheel is a graph with a set of links that together form a cycle that traverses all points in the graph.

In graph (N, L) the link $ab \in L$ is called *physically exploitable* if there exists $G \in G^{a,b}$ with $L \subseteq G$. So, ab is physically exploitable in (N, L) if $ab \in L$ and adding links to (N, L) can result in a graph that exploits a and b (they both receive 298). Stated yet differently, in (N, L) link $ab \in L$ is physically exploitable if there exists a strategy profile in $\Delta^{lf}(L)$ that ends in a graph that exploits both a and b. We stress that according to the definition of physically exploitable this strategy profile is not required to be an equilibrium.

The following lemma shows that in any network any pair of players that are physically exploitable can be exploited according to an SPNE. The proof of this lemma can be found in appendix A.

Lemma 4.1 Let (N, v) be the 6-person game described by (3). For all $L \subseteq L^N$, all links $ab \in L$ that are physically exploitable in (N, L), and all orders σ of $L^N\backslash L$ it holds that $\mathrm{PEG}(N, v, \mu, \sigma, L) \cap G^{a,b} \neq \emptyset$.

The following theorem deals with the stability of wheels.

Theorem 4.1 Let (N, v) be the 6-person game described by (3), let (N, L) be a wheel, and let σ be any order of the links in $L^N\backslash L$. Then $(N, L) \in \mathrm{PEG}(N, v, \mu, \sigma, L)$.

Proof: Without loss of generality assume that $L = \{12, 23, 34, 45, 56, 61\}$. A subgame perfect Nash equilibrium s is described as follows.

1.a. $l \in \{13, 15, 24, 26, 35, 46\}$: denote the elements of l by i and j, then there exists $k \in N \backslash \{i, j\}$ such that $ik \in L$ and $jk \notin L$; fix an SPNE that results in $G^* \in G^{i,k}$ (possible by lemma 4.1).

1.b. $l \in \{14, 25, 36\}$: denote the elements of l by i and j; fix an SPNE that results in $G^* \in G^{i,j}$ (possible by lemma 4.1).

2. $l \in L^N \backslash L$: do not form l.

We will show that s is indeed a subgame perfect Nash equilibrium.

1.a., 1.b.: By construction.

2. If l deviates from s, i.e., forming l rather than not forming it, then at least one of the players in l will receive 298, whereas he will receive 300 according to (N, L), which is formed according to s. This player prefers not to form l.

Consequently, s is a subgame perfect Nash equilibrium implying that

$$(N, L) \in \mathrm{PEG}(N, v, \mu, \sigma, L).$$

This completes the proof. □

Using lemma 4.1 we can also show that any pair of players who have not been formed yet, who according to the current network receive less than when they are exploited, who are last in the order under consideration, and are physically exploitable once their link has been added to the current network, can be exploited according to an SPNE.

Lemma 4.2 Let (N, v) be the 6-person game described by (3). For all $L \subseteq L^N$, all $ij \notin L$ such that $\mu_i(L) \leq 298$, $\mu_j(L) \leq 298$, and such that there exists $G \in G^{i,j}$ with $L \cup \{ij\} \subseteq G$, and all orders σ of $L^N \backslash L$ with ij last it holds that $\mathrm{PEG}(N, v, \mu, \sigma, L) \cap G^{i,j} \neq \emptyset$.

Proof: Let $L \subseteq L^N$, $ij \notin L$ such that $\mu_i(L) \leq 298$, $\mu_j(L) \leq 298$ and such that there exists $G \in G^{i,j}$ with $L \cup \{ij\} \subseteq G$, and σ be an order of $L^N \backslash L$ with ij last.

A subgame perfect Nash equilibrium s that results in some network $G^* \in G^{i,j}$ is described as follows.

1.a. $l = ij$: an SPNE that results in some $G^* \in G^{i,j}$ (possible by lemma 4.1).

1.b. $l \prec_\sigma ij$: any SPNE.

2.a. $l = ij$: form l.

2.b. $l \prec_\sigma ij$: do not form l.

We will show that s is indeed a subgame perfect Nash equilibrium.

1.a., 1.b.: By construction.

2.a.: Suppose l deviates from s, i.e., not forming $l = ij$ rather than forming it. Then (N, L) results and both receive at most 298, whereas they both receive 298 according to $G^* \in G^{i,j}$ which results if they choose forming l. The deviation does not strictly improve the payoff of any of the players in l.

2.b.: Suppose l deviates from s, i.e., forming l rather than not forming. Then, by lemma 3.1 a network isomorphic to (N, L^{146}) or a network payoff equivalent to L^N results and both receive at most 301. Since $l \neq ij$ at least one of the players in l receives 301 if they choose not forming l. Hence, at least one of the players in l does not strictly improve his payoff by deviating.

This completes the proof. $\square$

The following lemma deals with exploiting the link that is right in front of the last link.

Lemma 4.3 Let (N, v) be the 6-person game described by (3). Let $L \subseteq L^N$, $ij, ab \notin L$ such that $\mu_a(L) \leq 298$, $\mu_b(L) \leq 298$, $\mu_i(L \cup \{ab\}) \leq 298$, $\mu_j(L \cup \{ab\}) \leq 298$, and such that there exists $G \in G^{i,j}$ with $L \cup \{ab, ij\} \subseteq G$. Let σ be an order of $L^N \backslash L$ with ab last and ij right in front of ab. Then $\mathrm{PEG}(N, v, \mu, \sigma, L) \cap G^{i,j} \neq \emptyset$.

Proof: A subgame perfect Nash equilibrium s that results in some network $G^* \in G^{i,j}$ is described as follows.

1.a. $l = ab$: an SPNE that results in some $G^* \in G^{i,j}$ (possible by lemma 4.2).

1.b. $l = ij$: an SPNE that results in some $G' \in G^{i,j}$ (possible by lemma 4.1).

1.c. $l \prec_\sigma ab$: any SPNE.

2.a. $l = ab$: form l.

2.b. $l = ij$: do not form l.

2.c. $l \prec_\sigma ij$: do not form l.

We will show that s is indeed a subgame perfect Nash equilibrium.

1.a., 1.b.,1.c.: By construction.

2.a.: Suppose l deviates from s, i.e., not forming $l = ab$ rather than forming it. Then (N, L) results and both receive at most 298, whereas they both receive at least 298 according to $G^* \in G^{i,j}$ which results if they choose forming l. The deviation does not strictly improve the payoff of any of the players in l.

2.b.: Suppose l deviates from s, i.e., forming $l = ij$ rather than not forming it. Then $G' \in G^{i,j}$ results and both receive 298, whereas they both receive 298 as well according to $G^* \in G^{i,j}$ which results if they choose not forming l. The deviation does not strictly improve the payoff of both of the players in l.

2.c.: Suppose l deviates from s, i.e., forming l rather than not forming. Then a network isomorphic to (N, L^{146}) or a network payoff equivalent to L^N results and both receive at most 301. Since $l \neq ij$ at least one of the players in l receives 301 if they choose not forming l. Hence, at least one of the players in l does not strictly improve his payoff by deviating.

This completes the proof. $\square$

The following lemmas deal with specific networks that are subnetworks of a wheel. Consecutively, we deal with networks with five, four, three and two links and provide sufficient conditions for a wheel to result according to a subgame perfect Nash equilibrium when starting with such a network. The type of network that is studied in the first of these lemmas can be found in figure 1 (a).

a: Network with 5 links b: Network with 4 links

Figure 1: Two networks

Lemma 4.4 Let (N, v) be the 6-person game described by (3). Let W be a wheel and let $ij \in W$. Let $L = W \backslash \{ij\}$, and let σ be an order of $L^N \backslash L$. Then $W \in \text{PEG}(N, v, \mu, \sigma, L)$ and there exists $G^* \in G^{i,j}$ such that $G^* \in \text{PEG}(N, v, \mu, \sigma, L)$.

Proof: By lemma 4.1 we conclude that once ij is formed there exists an SPNE that results in some $G^* \in G^{i,j}$. Fix such a G^*. Let $G \in \{W, G^*\}$.

A subgame perfect Nash equilibrium s that results in network G is described as follows.

1.a. $l \succ_\sigma ij$: any SPNE.

1.b. $l = ij$: an SPNE that results in G (possible by definition of G^* above and theorem 4.1).

1.c. $l \prec_\sigma ij$: an SPNE that results in $H \in G^{a,b}$ with $ab \cap l \neq \emptyset$, which is possible by lemma 4.1. To see this, assume without loss of generality that $W = \{12, 23, 34, 45, 56, 16\}$ and $ij = 16$ (see figure 1 (a)). For any $l \in L^N \backslash W$ one can exploit a link l' according to table 1.

l	13	14	15	24	25	26	35	36	46
l'	34	14	15	12	25	26	56	36	34

Table 1: Exploitable links

2.a. $l \succ_\sigma ij$: do not form.

2.b. $l = ij$: form l.

2.c. $l \prec_\sigma ij$: do not form l.

We will show that s is indeed a subgame perfect Nash equilibrium.

1.a., 1.b.,1.c.: By construction.

2.a.: Suppose l deviates from s, i.e., forming l rather than not forming. Then, by lemma 3.1, a graph isomorphic to (N, L^{146}) or payoff equivalent to (N, L^N) results and both players in l receive at most 301. Since at least one of them receives 314 or 349 according to (N, L) (network (N, L) is isomorphic to (N, L^{33})), which results if they choose not forming l, at least one of the players in l does not strictly improve his payoff by deviating.

2.b.: Suppose l deviates from s, i.e., not forming $l = ij$ rather than forming it. Then L results and both receive 237, whereas they both receive at least 298 according to G which results if they choose forming l. The deviation does not strictly improve the payoff of any of the players in l.

2.c.: Suppose l deviates from s, i.e., forming l rather than not forming. Then $H \in G^{a,b}$ with $ab \cap l \neq \emptyset$ results and at least one of the players in l receives 298. Since both of them receive at least 298 if they choose not forming l at least one of the players in l does not strictly improve his payoff by deviating.

This completes the proof. □

The type of network that is studied in the next lemma can be found in figure 1 (b).

Lemma 4.5 Let (N, v) be the 6-person game described by (3). Let W be a wheel and let $ij, ab \in W$ with ij and ab disjunct, and $ia \in W$. Let $L = W \backslash \{ij, ab\}$, and let σ be an order of $L^N \backslash L$ with $ab \prec_\sigma ij$. Then $W \in \text{PEG}(N, v, \mu, \sigma, L)$ and there exists $G^* \in G^{i,j}$ such that $G^* \in \text{PEG}(N, v, \mu, \sigma, L)$.

Proof: By lemma 4.4 we know that once ab is formed there exists an SPNE that results in some $G^* \in G^{i,j}$. Fix such a G^*. Let $G \in \{W, G^*\}$.

A subgame perfect Nash equilibrium s that results in network G is described as follows.

1.a. $l \succ_\sigma ij$: any SPNE.

1.b. $l = ij$: an SPNE that results in $H \in G^{a,b}$ (possible by lemma 4.4).

1.c. $ab \prec_\sigma l \prec_\sigma ij$: an SPNE that results in some $H \in G^{x,y}$ with $xy \cap l \neq \emptyset$, which is possible by lemma 4.1. To see this assume without loss of generality that $W = \{12, 23, 34, 45, 56, 16\}$, $ij = 16$, and $ab = 23$. For any $l \in L^N \backslash W$ one can exploit a link l' according to table 2.

l	13	14	15	24	25	26	35	36	46
l'	13	14	15	24	25	26	56	36	34

Table 2: Exploitable links

1.d. $l = ab$: an SPNE that results in G (possible by definition of G^* above and lemma 4.4).

1.e. $l \prec_\sigma ab$: an SPNE that results in some $H \in G^{x,y}$ with $xy \cap l \neq \emptyset$ (possible by lemma 4.1; see also the description of s, part 1.c).

2.a. $l \succ_\sigma ij$ any subgame perfect choice.

2.b. $l = ij$: form l.

2.c. $ab \prec_\sigma l \prec_\sigma ij$: do not form l.

2.d. $l = ab$: form l.

2.e. $l \prec_\sigma ab$ do not form.

We will show that s is indeed a subgame perfect Nash equilibrium.

1.a., 1.b., 1.c., 1.d., 1.e., 2.a.: By construction.

2.b. Suppose l deviates from s, i.e., not forming $l = ij$ rather than forming it. Then, using lemma 3.1, it follows that (N, L), a network isomorphic to (N, L^{146}), or a network payoff equivalent to (N, L^N) results. In all cases both players receive at most 301, whereas both receive 301 if they choose forming l. Hence, the deviation does not strictly improve the payoff of any of the players in l.

2.c.,2.e: Suppose l deviates from s, i.e., forming l rather than not forming. Then $H \in G^{x,y}$ with $xy \cap l \neq \emptyset$ results and at least one of the players in l receives 298. Since both of them receive at least 298 if they choose not forming l at least one of the players in l does not strictly improve his payoff by deviating.

2.d.: Suppose l deviates from s, i.e., not forming $l = ab$ rather than forming it. Then $H \in G^{a,b}$ results and both players receive 298, whereas they both receive at least 300 if they choose forming l. Hence, the deviation does not strictly improve the payoff of any of the players in l.

This completes the proof. □

Lemma 4.6 Let (N, v) be the 6-person game described by (3). Let W be a wheel and let $ij, ab, cd \in W$ be three pairwise disjunct links. Let $L = W \backslash \{ij, ab, cd\}$, and let σ be an order of $L^N \backslash L$. Then $W \in \mathrm{PEG}(N, v, \mu, \sigma, L)$.

Proof: Without loss of generality assume that $cd \prec_\sigma ab \prec_\sigma ij$.

A subgame perfect Nash equilibrium s that results in network G is described as follows.

1.a. $l \succ_\sigma ab$: any SPNE.

1.b. $l = ab$: an SPNE that results in $H \in G^{c,d}$ (possible by lemma 4.5).

1.c. $cd \prec_\sigma l \prec_\sigma ab$: an SPNE that results in some $H \in G^{x,y}$ with $xy = l$ (possible by lemma 4.1).

1.d. $l = cd$: an SPNE that results in W (possible by lemma 4.5).

1.e. $l \prec_\sigma cd$: an SPNE that results in some $H \in G^{x,y}$ with $xy = l$ (possible by lemma 4.1).

2.a. $l \succ_\sigma ab$ any subgame perfect choice.

2.b. $l = ab$: form l.

2.c. $cd \prec_\sigma l \prec_\sigma ab$: do not form l.

2.d. $l = cd$: form l.

2.e. $l \prec_\sigma cd$ do not form.

We will show that s is indeed a subgame perfect Nash equilibrium.

1.a., 1.b., 1.c., 1.d., 1.e. 2.a.: By construction.

2.b. Suppose l deviates from s, i.e., not forming $l = ab$ rather than forming it. Then, using lemma 3.1, we conclude that (N, L), a network isomorphic to (N, L^{146}), or a network payoff equivalent to (N, L^N) results. In all cases both players receive at most 301, whereas both receive 301 if they choose forming l. Hence, the deviation does not strictly improve the payoff of any of the players in l.

2.c.,2.e: Suppose l deviates from s, i.e., forming l rather than not forming. Then $H \in G^{x,y}$ with $xy = l$ results and both players in l receive 298. Since both of them receive at least 298 if they choose not forming l at least one of the players in l does not strictly improve his payoff by deviating.

2.d.: Suppose l deviates from s, i.e., not forming $l = cd$ rather than forming it. Then $H \in G^{c,d}$ results and both players receive 298, whereas they both receive 300 if they choose forming l. Hence, the deviation does not strictly improve the payoff of any of the players in l.

This completes the proof. □

Lemma 4.7 Let (N, v) be the 6-person game described by (3). Let ij, ab, and cd be three pairwise disjunct links. Let $L = \{ij, ab\}$ and let σ be an order of $L^N \backslash L$ with $\sigma(cd) \geq 11$. Then there exists a wheel W such that $W \in \mathrm{PEG}(N, v, \mu, \sigma, L)$.

Proof: We will distinguish between three cases based on the position of cd in the order: **(i)** $\sigma(cd) = 13$; **(ii)** $\sigma(cd) = 12$; **(iii)** $\sigma(cd) = 11$.

Case (i): A subgame perfect Nash equilibrium s that results in a wheel is described as follows.

1.a. $l = cd$: an SPNE that results in a wheel W (possible by lemma 4.6).

1.b. $l \prec_\sigma cd$: an SPNE that results in some $H \in G^{x,y}$ with $xy = l$ (possible by lemma 4.1).

2.a. $l = cd$: form l.

2.b. $l \prec_\sigma cd$: do not form l.

We will show that s is indeed a subgame perfect Nash equilibrium.

1.a., 1.b.: By construction.

2.a. Suppose l deviates from s, i.e., not forming $l = cd$ rather than forming it. Then (N, L) results and both players in l receive 0, whereas they both receive 300 according to W which results if they choose forming l. The deviation does not strictly improve the payoff of any of the players in l.

2.b. Suppose l deviates from s, i.e., forming l rather than not forming. Then $H \in G^{x,y}$ with $xy = l$ results and both players in l receive 298. Since both of them receive 300 if they choose not forming l at least one of the players in l does not strictly improve his payoff by deviating.

Case (ii): A subgame perfect Nash equilibrium s that results in a wheel is described as follows.

1.a. $l = \sigma^{-1}(13)$: an SPNE that results in some $G \in G^{c,d}$ (possible by lemma 4.2).

1.b. $l = cd$: an SPNE that results in a wheel W (possible by lemma 4.6).

1.c. $l \prec_\sigma cd$: an SPNE that results in some $H \in G^{x,y}$ with $xy = l$ (possible by lemma 4.1).

2.a. $l = \sigma^{-1}(13)$: form l

2.b. $l = cd$: form l.

2.c. $l \prec_\sigma cd$: do not form l.

We will show that s is indeed a subgame perfect Nash equilibrium.

1.a., 1.b., 1.c. By construction.

2.a. Suppose l deviates from s, i.e., not forming l rather than forming it. Then (N, L) results. Both players then receive at most 30 ((N, L) is isomorphic to (N, L^4)), whereas both receive at least 298 if they choose forming l. Hence, the deviation does not strictly improve the payoff of any of the players in l.

2.b. Suppose l deviates from s, i.e., not forming $l = cd$ rather than forming it. Then $G \in G^{c,d}$ results and both receive 298, whereas they both receive 300 according to W which results if they choose forming l. The deviation does not strictly improve the payoff of any of the players in l.

2.c. Suppose l deviates from s, i.e., forming l rather than not forming. Then $H \in G^{x,y}$ with $xy = l$ results and both players in l receive 298. Since both of them receive 300 if they choose not forming l at least one of the players in l does not strictly improve his payoff by deviating.

Case (iii) A subgame perfect Nash equilibrium s that results in a wheel is described as follows.

1.a. $l = \sigma^{-1}(13)$: we distinguish between two cases.

Case I $\{\sigma^{-1}(12), \sigma^{-1}(13)\} = \{kc, kd\}$ for some $k \in N\backslash\{c, d\}$: an SPNE that results in $H \in G^{x,y}$ with $\sigma^{-1}(12) = xy$ and $L \cup \{\sigma^{-1}(12), \sigma^{-1}(13)) \subseteq H$. (possible by lemma 4.2; note that $\sigma^{-1}(12)$ is physically exploitable in $(N, L \cup \{\sigma^{-1}(12), \sigma^{-1}(13)\})$)

Case II Otherwise: an SPNE that results in some $G^* \in G^{c,d}$ (possible by lemma 4.3; note that cd is physically exploitable in $(N, L \cup \{cd, \sigma^{-1}(12), \sigma^{-1}(13)\})$).

1.b. $l = \sigma^{-1}(12)$: an SPNE that results in some $G \in G^{c,d}$ (possible by lemma 4.2).

1.c. $l = cd$: an SPNE that results in a wheel W (possible by lemma 4.6).

1.d. $l \prec_\sigma cd$: an SPNE that results in some $H \in G^{x,y}$ with $xy = l$ (possible by lemma 4.1).

2.a. $l = \sigma^{-1}(13)$: form l.

2.b. $l = \sigma^{-1}(12)$: form l.

2.c. $l = cd$: form l.

2.d. $l \prec_\sigma cd$: do not form l.

We will show that s is indeed a subgame perfect Nash equilibrium.

1.a., 1.b., 1.c.,1.d: By construction.

2.a. Suppose l deviates from s, i.e., not forming l rather than forming it. Then (N, L) results and both players in L receive at most 30 ((N, L) is isomorphic to (N, L^4)), whereas they both receive at least 298 if they choose forming ij. The deviation does not strictly improve the payoff of any of the players in l.

2.b. Suppose l deviates from s, i.e., not forming l rather than forming it. Then $G \in G^{c,d}$ results or $H \in G^{x,y}$ with $\sigma^{-1}(12) = xy$. A network $G \in G^{c,d}$ results if they choose forming l. Since both x and y receive at least as much according to $G \in G^{c,d}$ as according to $H \in G^{x,y}$, a deviation does not strictly improve the payoff of any of the players in l.

2.c. Suppose l deviates from s, i.e., not forming $l = cd$ rather than forming it. Then $G \in G^{c,d}$ results and both receive 298, whereas they both receive 300 according to W which results if they choose forming l. The deviation does not strictly improve the payoff of any of the players in l.

2.d. Suppose l deviates from s, i.e., forming l rather than not forming. Then $H \in G^{x,y}$ with $xy = l$ results and both players in l receive 298. Since both of them receive 300 if they choose not forming l at least one of the players in l does not strictly improve his payoff by deviating.

This completes the proof. □

Before we continue we introduce a concept that describes the distance between two links according to an order of the links in L^N. For any order σ of L^N and any $l^1, l^2 \in L^N$ we define the *distance between* l^1 *and* l^2 as $d^\sigma(l^1, l^2) = |\sigma(l^1) - \sigma(l^2)|$.

The following lemma shows that for any order there exist two disjunct links with a distance between them of at most 3.

Lemma 4.8 Let σ be an order of L^N. Then there exist $l^1, l^2 \in L^N$ with $l^1 \cap l^2 = \emptyset$ and $d^\sigma(l^1, l^2) \leq 3$.

Proof: Let l_i be the link that is in position i according to σ, for $i = 1, \ldots, 15$. If there exists a pair of disjunct links in the set $\{l_1, l_2, l_3, l_4\}$ we are done. So, from now on assume that there is no pair of disjunct links in the set $\{l_1, l_2, l_3, l_4\}$.

Denote $l_1 = xy$. Since $l^2 \cap l^1 \neq \emptyset$ denote without loss of generality $l^2 = xz$. Since l^3 is not disjunct from xy or xz we have that $x \in l^3$ or $l^3 = yz$. In the latter case l^4 will always be disjunct from at least one of the first three links. Hence, without loss of generality denote $l^3 = xw$. Subsequently, we directly conclude that l^4 should contain x as well. Denote $l^4 = xu$. Then either link l_5 is disjunct from at least one of the links l_2, l_3, l_4 or link l_5 contains x and the unique player in $N \backslash \{x, y, z, w, u\}$, say t. Then, however, it cannot be that l_6 is disjunct from xw, xu, and xt since the five links containing player x occupied the first five places in the order. We conclude that l_6 is disjunct from at least one of the links l_3, l_4, l_5

This completes the proof. □

We use this lemma to prove the main result of this paper.

Theorem 4.2 Let (N, v) be the 6-person game described by (3). Let σ be an order of L^N. Then there exist a wheel W such that $W \in \text{PEG}(N, v, \mu, \sigma)$.

Proof: Let ab, cd be two disjunct links with $d^\sigma(ab, cd) \leq 3$ (possible by lemma 4.8). Without loss of generality assume that $0 \leq \sigma(ab) - \sigma(cd) \leq 3$. Let ij be the link that is disjunct from both ab and cd. Let $\tau = \sigma_{\{ij\},ij}$. First we prove a claim.

Claim: There exists a wheel W^* such that $W^* \in \text{PEG}(N, v, \mu, \tau, \{ij\})$.

Proof of claim: Let W be a wheel that can be formed according to an SPNE once ab has been formed additionally (possible by lemma 4.7; note that $\sigma_{\{ij,ab\},ab}(cd) \geq 11$.). A subgame perfect Nash equilibrium s that results in W is described as follows.

1.a. $l \succ_\tau ab$: Let $vw = \sigma^{-1}(\sigma(l) - 1)$. If ij, l, vw do not form a triangle then an SPNE that results in $G \in G^{v,w}$.[2] Otherwise, let $tu = \sigma^{-1}(\sigma(l) - 2)$ and choose an SPNE that results in $H \in G^{t,u}$ (possible by lemmas 4.2 and 4.3; note that tu is physically exploitable in $(N, \{ij, l, vw, tu\})$).

1.b. $l = ab$: an SPNE that results in wheel W (possible by definition of W).

1.c. $l \prec_\tau ab$: an SPNE that results in some $H \in G^{x,y}$ with $xy = l$ (possible by lemma 4.1).

2.a. $l \succ_\tau ab$: form l.

2.b. $l = ab$: form l.

2.c. $l \prec_\tau ab$: do not form l.

We will show that s is indeed a subgame perfect Nash equilibrium.

1.a., 1.b., 1.c.: By construction.

2.a. Forming l is obviously subgame perfect if l is last according to σ. From now on assume that l is not last according to σ. Let $l = xy$. Suppose l deviates from s, i.e., not forming l rather than forming it. Then $G \in G^{x,y} \cup G^{v,w}$ results, whereas $H \in G^{v,w} \cup G^{t,u}$ results if they choose forming l. Obviously, none of the players in xy strictly improves his payoff by deviating if $G \in G^{x,y}$. Furthermore, none of the players in xy strictly improves his payoff by deviating if both $G \in G^{v,w}$ and $H \in G^{v,w}$. This covers all cases because $H \in G^{t,u}$ implies that there is no $G' \in G^{v,w}$ such that $\{ij, vw, xy\} \subseteq G'$. This in turn implies that ij, vw, xy form a triangle. Hence, ij and xy cannot form a triangle with the link that follows xy according to σ. Consequently, if $H \in G^{t,u}$ then $G \in G^{x,y}$.

2.b. Suppose l deviates from s, i.e., not forming $l = ab$ rather than forming it. Then $G \in G^{a,b}$ results and both receive 298, whereas they both receive 300 according to W which results if they choose forming l. The deviation does not strictly improve the payoff of any of the players in l.

2.c. Suppose l deviates from s, i.e., forming l rather than not forming. Then $H \in G^{x,y}$ with $xy = l$ results and both players in l receive 298. Since both of them receive 300 if they choose not forming l at least one of the players in l does not strictly improve his payoff by deviating.

This completes the proof of the claim. We use this claim in the proof of the theorem.

1.a. $l \succ_\sigma ij$: An SPNE that results in $G \in G^{x,y}$ with $\{l, xy\} \subseteq G$, where $xy = \sigma^{-1}(\sigma(l) - 1)$ (possible by lemma 4.2).

1.b. $l = ij$: an SPNE that results in a wheel W (possible by claim above).

1.c. $l \prec_\sigma ij$: an SPNE that results in some $H \in G^{x,y}$ with $xy = l$ (possible by lemma 4.1).

2.a. $l \succ_\sigma ij$: form l.

2.b. $l = ij$: form l.

2.c. $l \prec_\sigma ij$: do not form l.

---

[2] Note that if $\sigma(l) = 2$ then link ab is in first position and no triangle is formed by ij, l and $ab = vw$.

We will show that s is indeed a subgame perfect Nash equilibrium.

1.a., 1.b., 1.c.: By construction.

2.a. Suppose l deviates from s, i.e., not forming l rather than forming it. Then (N, L) results or $H \in G^{x,y}$, where $l = xy$, and both players in l receive at most 298, whereas they both receive at least 298 according to G which results if they choose forming l. The deviation does not strictly improve the payoff of any of the players in l.

2.b. Suppose l deviates from s, i.e., not forming $l = ij$ rather than forming it. Then $G \in G^{i,j}$ results and both receive 298, whereas they both receive 300 according to W which results if they choose forming l. The deviation does not strictly improve the payoff of any of the players in l.

2.c. Suppose l deviates from s, i.e., forming l rather than not forming. Then $H \in G^{x,y}$ with $xy = l$ results and both players in l receive 298. Since both of them receive 300 if they choose not forming l at least one of the players in l does not strictly improve his payoff by deviating.

This completes the proof. $\square$

This paper has been devoted completely to the analysis of a 6-person symmetric convex game, the game described by (3). More specifically, we analyzed the associated link formation games in extensive form. The main conclusion from this paper, cf. theorem 4.2, states that starting with no links formed, any order of the links can result in the same payoff for all players, coinciding with their payoffs according to the full cooperation structure. This adds to the result of Slikker and Norde (2004) who showed that for these situations any pair of players can end up being exploited and, moreover, solves their open question. Whether for any symmetric strictly convex game we can always end up in a structure that results in the same payoffs as the full cooperation structure remains an open question.

References

Aumann, R. and Myerson, R. (1988). Endogenous formation of links between players and coalitions: an application of the Shapley value. In Roth, A., editor, *The Shapley Value*, pages 175–191. Cambridge University Press, Cambridge, United Kingdom.

Dutta, B. and Jackson, M., editors (2003). *Networks and Groups (Models of Strategic Formation)*. Studies in Economic Design. Springer-Verlag.

Dutta, B., van den Nouweland, A., and Tijs, S. (1998). Link formation in cooperative situations. *International Journal of Game Theory*, 27:245–256.

Jackson, M. (2004). Allocation rules for network games. *Games and Economic Behavior* (to appear).

Jackson, M. and van den Nouweland, A. (2004). Strongly stable netwoks. *Games and Economic Behavior* (to appear).

Jackson, M. and Wolinsky, A. (1996). A strategic model of social and economic networks. *Journal of Economic Theory*, 71:44–74.

Myerson, R. (1977). Graphs and cooperation in games. *Mathematics of Operations Research*, 2:225–229.

Shapley, L. (1953). A value for n-person games. In Tucker, A. and Kuhn, H., editors, *Contributions to the Theory of Games II*, pages 307–317. Princeton University Press, Princeton.

Slikker, M. and Norde, H. (2004). Incomplete stable structures in symmetric convex games. *Games and Economic Behavior*, 48(1):171–200.

A Proof of lemma 4.1

Before we prove the lemma we need two additional lemmas.

Lemma A.1 Let (N, v) be the 6-person game described by (3). Let $L \subseteq L^N$, $ab \in L$, and $ij \notin L$ such that
- $ab \cap ij = \emptyset$;
- $\mu_i(L) \leq 301$ and $\mu_j(L) \leq 301$;
- $\text{PEG}(N, v, \mu, \tau, L \cup \{ij\}) \cap G^{a,b} \neq \emptyset$ for any order τ of $L^N \backslash (L \cup \{ij\})$.

Furthermore, let σ be an order of $L^N \backslash L$. Then $\text{PEG}(N, v, \mu, \sigma, L) \cap G^{a,b} \neq \emptyset$.

Proof: A subgame perfect Nash equilibrium s that results in some graph in $G^{a,b}$ is described as follows.
1.a. $l \succ_\sigma ij$: any SPNE.
1.b. $l = ij$: an SPNE that results in some $G \in G^{a,b}$ (possible by assumption).
1.c. $l \prec_\sigma ij$: any SPNE.
2.a. $l \succ_\sigma ij$: any subgame perfect choice.
2.b. $l = ij$: form l.
2.c. $l \prec_\sigma ij$: do not form l.
We will show that s is indeed a subgame perfect Nash equilibrium.
1.a., 1.b.,1.c.,2.a: By construction.
2.b.: Suppose l deviates from s, i.e., not forming $l = ij$ rather than forming it. Then, using lemma 3.1, (N, L), a network isomorphic to (N, L^{146}) or a network payoff equivalent to L^N results and both players in l receive at most 301, whereas they both receive 301 according to $G \in G^{a,b}$ which results if they choose forming l. The deviation does not strictly improve the payoff of any of the players in l.

2.c.: Suppose l deviates from s, i.e., forming l rather than not forming. Then, by lemma 3.1, a network isomorphic to (N, L^{146}) or a network payoff equivalent to L^N results and both receive at most 301. Since $ab \in L$ we have that $l \neq ab$. Consequently, at least one of the players in l receives 301 if they choose not forming l. Hence, at least one of the players in l does not strictly improve his payoff by deviating.

This completes the proof. □

Lemma A.2 Let (N, v) be the 6-person game described by (3). Let $L \subseteq L^N$, $ab \in L$, and $ij \notin L$ such that

- $i \in ab$ and $j \notin ab$;
- $\mu_i(L) \leq 298$ and $\mu_j(L) \leq 301$;
- $\text{PEG}(N, v, \mu, \tau, L \cup \{ij\}) \cap G^{a,b} \neq \emptyset$ for any order τ of $L^N \backslash (L \cup \{ij\})$.
- for all $cd \in L^N \backslash (L \cup \{ij\})$ if holds that either $\mu_c(L) \geq 301$ or $\mu_d(L) \geq 301$ or there exists $z \in N \backslash \{i\}$ with $iz \in L \cup \{cd\}$ and $\text{PEG}(N, v, \mu, \tau, L \cup \{cd\}) \cap G^{i,z} \neq \emptyset$ for any order τ of $L^N \backslash (L \cup \{cd\})$.

Furthermore, let σ be an order of $L^N \backslash L$. Then $\text{PEG}(N, v, \mu, \sigma, L) \cap G^{a,b} \neq \emptyset$.

Proof: A subgame perfect Nash equilibrium s that results in some graph in $G^{a,b}$ is described as follows.

1.a. $l \succ_\sigma ij$: if there exists $z \in N \backslash \{i\}$ with $iz \in L \cup \{l\}$ and $\text{PEG}(N, v, \mu, \sigma_{L \cup \{l\}, l}, L \cup \{l\}) \cap G^{i,z} \neq \emptyset$ then an SPNE that results in some $G^* \in G^{i,z}$; else any SPNE.

1.b. $l = ij$: an SPNE that results in some $G \in G^{a,b}$ (possible by assumption).

1.c. $l \prec_\sigma ij$: any SPNE.

2.a. $l \succ_\sigma ij$: a subgame perfect choice where l chooses not forming l if at least one player is indifferent between forming l and not forming l.

2.b. $l = ij$: form l.

2.c. $l \prec_\sigma ij$: do not form l.

We will show that s is indeed a subgame perfect Nash equilibrium.

1.a., 1.b.,1.c.,2.a: By construction.

2.b.: Suppose l deviates from s, i.e., not forming $l = ij$ rather than forming. Then (N, L) results or a network that results after the formation of some $l'(\succ_\sigma l)$. If (N, L) results neither i nor j improves its payoff.

It remains to consider the case that some network results after the formation of some $l'(\succ_\sigma l)$. A network that is formed after the formation of some $l' \succ_\sigma l$ is called a network of type A if this network belongs to some $G^{i,z}$ with $z \in N \backslash \{i\}$ and $iz \in L \cup \{l'\}$. It is called a network of type B otherwise. Note that if after the formation of l' a network of type B results then, by lemma 3.1, this network is isomorphic to (N, L^{146}) or payoff equivalent to L^N.

Suppose a network of type B will result. Let l'' be the last link according to σ that chooses forming l'' and results in a network of type B. If l'' would have chosen not forming l'' then (N, L) results or a network of type A. In the latter case the players in l'' cannot end up with 298 both since $l'' \notin L \cup \{l^*\}$, where l^* denotes the first link that forms after l'' chose not to form. By assumption we know that according to (N, L) at least one of the players in l'' receives at least 301. So, in both cases at least one of the players receives at least 301 after deviating and at most 301 after forming l''. This is in conflict with the description of s, part 2.a. Hence, if l deviates then the resulting network will not be a network of type B. We conclude that a network of type A results after l deviates. In such a network player i receives 298 and player j receives 301 (since $l = ij \notin L \cup \{l^{**}\}$, where l^{**} denotes the first link that forms after ij has deviated). If ij chooses forming l then i receives 298 and j receives 301 as well. So, neither of the players in l can strictly improve his payoff by deviating.

2.c.: Suppose l deviates from s, i.e., forming l rather than not forming. Then, by lemma 3.1, a network isomorphic to (N, L^{146}) or a network payoff equivalent to L^N results and both receive at most 301. Since $l \neq ab$ at least one of the players in l receives 301 if they choose not forming l. Hence, at least one of the players in l does not strictly improve his payoff by deviating.

This completes the proof. □

Using these lemmas we can prove lemma 4.1.

Proof: Tables 3 to 5 in appendix B provide, up to isomorphisms, for all $L \subseteq L^N$ with $|L| \leq 10$ and its links $ab \in L$ that are physically exploitable a link $ij \in L^N \backslash L$ such that either

- $ab \cap ij = \emptyset$;
- $\mu_i(L) \leq 301$ and $\mu_j(L) \leq 301$;
- ab is physically exploitable in $(N, L \cup \{ij\})$;

all hold or

- $i \in ab$ and $j \notin ab$;
- $\mu_i(L) \leq 298$ and $\mu_j(L) \leq 301$;
- ab is physically exploitable in $(N, L \cup \{ij\})$;
- for all $cd \in L^N \backslash (L \cup \{ij\})$ it holds that either $\mu_c(L) \geq 301$ or $\mu_d(L) \geq 301$ or there exists $z \in N \backslash \{i\}$ with $iz \in L \cup \{cd\}$ and iz physically exploitable in $(N, L \cup \{cd\})$.

all hold.

The proof will be by induction to the number of links. For any L with $|L| \geq 11$ the statement in the lemma is obviously true. Suppose the statement in the lemma is true for all L with $|L| = k + 1$ for some $k \in \{1, \ldots, 10\}$. Let L be such that $|L| = k$ and let $ab \in L$.

By the induction hypothesis and the arguments above, we conclude that there exists a link ij such that either

- $ab \cap ij = \emptyset$;

- $\mu_i(L) \leq 301$ and $\mu_j(L) \leq 301$;
- $\text{PEG}(N, v, \mu, \tau, L \cup \{ij\}) \cap G^{a,b} \neq \emptyset$ for any order τ of $L^N \backslash (L \cup \{ij\})$;

all hold or

- $i \in ab$ and $j \notin ab$;
- $\mu_i(L) \leq 298$ and $\mu_j(L) \leq 301$;
- $\text{PEG}(N, v, \mu, \tau, L \cup \{ij\}) \cap G^{a,b} \neq \emptyset$ for any order τ of $L^N \backslash (L \cup \{ij\})$.
- for all $cd \in L^N \backslash (L \cup \{ij\})$ it holds that either $\mu_c(L) \geq 301$ or $\mu_d(L) \geq 301$ or there exists $z \in N \backslash \{i\}$ with $iz \in L \cup \{cd\}$ and $\text{PEG}(N, v, \mu, \tau, L \cup \{cd\}) \cap G^{i,z} \neq \emptyset$ for any order τ of $L^N \backslash (L \cup \{cd\})$.

all hold.

Lemmas A.1 and A.2 above suffice to prove that the statement in the lemma is true for (N, L) and ab.

This completes the proof. □

B Non-isomorphic graphs with 6 players

This appendix deals with payoffs in communication situations with (N, v) of equation (3) as the underlying game. Tables 3 through 5 provide an overview of these payoffs for all 156 non-isomorphic graphs with 6 players according to the Myerson value. This part of the tables is taken from Slikker and Norde (2004). Furthermore it provides for any physically exploitable link an associated link that is used in the proof of lemma 4.1.

Specifically, in the last column one can find three types of elements in the vectors in the last column. A '0' represents that the corresponding link has not been formed yet. For example, in L^1, the first element is '0' because 12 has not been formed in L^1. An 'x' represents that the corresponding link has been formed but it is not physically exploitable. For example, in L^6 the first element is 'x' because 12 forms a triangle with 13 and 23, so 12 has been formed but cannot be physically exploited. Finally, an 'ij' means that the corresponding link is physically exploitable and ij is the link that is used in the proof of lemma 4.1. Here, we have two cases depending on whether or not ij is disjunct from the corresponding link. As an example for the first case, the first element for L^2 is 34, which is disjunct from the corresponding link 12. Note that players 3 and 4 both receive less than 301 in L^2 and that 12 is physically exploitable in L^3, the network that results if 34 is added to L^2. As an example of the second case, element 14 in the vector associated with L^{33} is 56, which is not disjunct from the corresponding link 46. Note that players 5 and 6 receive less than 301 and 298, respectively. If 56 is added, L^{54} results (a wheel). Obviously, link 46 is physically exploitable in L^{54}. Furthermore, let $cd \in L^N \backslash (L \cup \{ij\})$, i.e., $cd \in \{14, 15, 16, 23, 25, 26, 34, 36, 45\}$. Since players 1, 2, 3, and 4 all receive more than 301 in L^{33} we conclude that $\mu_c(L) \geq 301$ or $\mu_d(L) \geq 301$.

Except for L^{146}, the network in which link 56 is exploited, there are no physically exploitable links in networks with at least 11 links. Therefore, we leave the last column for networks with 11 links or more empty.

number	graph	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	associated link
1	(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)	(0,	0,	0,	0,	0,	0)	(0;0;0;0;0;0;0;0;0;0;0;0;0;0;0)
2	(1,0,0,0,0,0,0,0,0,0,0,0,0,0,0)	(30,	30,	0,	0,	0,	0)	(34;0;0;0;0;0;0;0;0;0;0;0;0;0;0)
3	(1,1,0,0,0,0,0,0,0,0,0,0,0,0,0)	(80,	50,	50,	0,	0,	0)	(34;24;0;0;0;0;0;0;0;0;0;0;0;0;0)
4	(1,0,0,0,0,0,0,0,0,1,0,0,0,0,0)	(30,	30,	30,	30,	0,	0)	(35;0;0;0;0;0;0;0;0;15;0;0;0;0;0)
5	(1,1,1,0,0,0,0,0,0,0,0,0,0,0,0)	(150,	70,	70,	70,	0,	0)	(34;24;23;0;0;0;0;0;0;0;0;0;0;0;0)
6	(1,1,0,0,0,1,0,0,0,0,0,0,0,0,0)	(60,	60,	60,	0,	0,	0)	(x;x;0;0;0;x;0;0;0;0;0;0;0;0;0)
7	(1,1,0,0,0,0,1,0,0,0,0,0,0,0,0)	(115,	115,	65,	65,	0,	0)	(34;25;0;0;0;0;15;0;0;0;0;0;0;0;0)
8	(1,1,0,0,0,0,0,0,0,0,0,0,1,0,0)	(80,	50,	50,	30,	30,	0)	(34;24;0;0;0;0;0;0;0;0;0;0;16;0;0)
9	(1,0,0,0,0,0,0,0,0,1,0,0,0,0,1)	(30,	30,	30,	30,	30,	30)	(35;0;0;0;0;0;0;0;0;15;0;0;0;0;13)
10	(1,1,1,1,0,0,0,0,0,0,0,0,0,0,0)	(240,	90,	90,	90,	90,	0)	(x;x;x;x;0;0;0;0;0;0;0;0;0;0;0)
11	(1,1,1,0,0,1,0,0,0,0,0,0,0,0,0)	(130,	80,	80,	70,	0,	0)	(x;x;25;0;0;x;0;0;0;0;0;0;0;0;0)
12	(1,1,1,0,0,0,1,0,0,0,0,0,0,0,0)	(200,	150,	85,	85,	80,	0)	(34;24;23;0;0;0;16;0;0;0;0;0;0;0;0)
13	(1,1,1,0,0,0,0,0,0,0,0,0,0,0,1)	(150,	70,	70,	70,	30,	30)	(34;24;23;0;0;0;0;0;0;0;0;0;0;0;23)
14	(1,1,0,0,0,1,0,0,0,0,0,0,1,0,0)	(60,	60,	60,	30,	30,	0)	(x;x;0;0;0;x;0;0;0;0;0;0;16;0;0)
15	(1,1,0,0,0,0,1,0,0,1,0,0,0,0,0)	(90,	90,	90,	90,	0,	0)	(35;25;0;0;0;0;15;0;0;15;0;0;0;0;0)
16	(1,1,0,0,0,0,1,0,0,0,1,0,0,0,0)	(162,	142,	142,	77,	77,	0)	(34;25;0;0;0;0;15;0;0;0;14;0;0;0;0)
17	(1,1,0,0,0,0,1,0,0,0,0,0,0,0,1)	(115,	115,	65,	65,	30,	30)	(34;25;0;0;0;0;15;0;0;0;0;0;0;0;14)
18	(1,1,0,0,0,0,0,0,0,0,0,0,1,1,0)	(80,	50,	50,	80,	50,	50)	(34;24;0;0;0;0;0;0;0;0;0;0;16;15;0)
19	(1,1,1,1,1,0,0,0,0,0,0,0,0,0,0)	(500,	260,	260,	260,	260,	260)	(x;x;x;x;x;0;0;0;0;0;0;0;0;0;0)
20	(1,1,1,1,0,1,0,0,0,0,0,0,0,0,0)	(220,	100,	100,	90,	90,	0)	(x;x;x;x;0;x;0;0;0;0;0;0;0;0;0)
21	(1,1,1,1,0,0,0,1,0,0,0,0,0,0,0)	(455,	335,	255,	255,	255,	245)	(x;x;x;x;0;0;0;0;34;0;0;0;0;0;0)
22	(1,1,1,0,0,1,1,0,0,0,0,0,0,0,0)	(95,	95,	85,	85,	0,	0)	(x;x;x;0;0;x;x;0;0;0;0;0;0;0;0)
23	(1,1,1,0,0,1,0,1,0,0,0,0,0,0,0)	(165,	165,	100,	85,	85,	0)	(x;x;26;0;0;x;0;16;0;0;0;0;0;0;0)
24	(1,1,1,0,0,1,0,0,0,0,0,0,1,0,0)	(180,	95,	95,	150,	80,	0)	(x;x;25;0;0;x;0;0;0;0;0;0;16;0;0)
25	(1,1,1,0,0,1,0,0,0,0,0,0,0,0,1)	(130,	80,	80,	70,	30,	30)	(x;x;25;0;0;x;0;0;0;0;0;0;0;0;24)
26	(1,1,1,0,0,0,1,1,0,0,0,0,0,0,0)	(400,	400,	250,	250,	250,	250)	(34;45;35;0;0;0;34;34;0;0;0;0;0;0;0)
27	(1,1,1,0,0,0,1,0,0,1,0,0,0,0,0)	(178,	113,	113,	88,	108,	0)	(34;24;23;0;0;0;16;0;0;16;0;0;0;0;0)
28	(1,1,1,0,0,0,1,0,0,0,1,0,0,0,0)	(412,	327,	327,	250,	242,	242)	(45;45;56;0;0;0;46;0;0;0;45;0;0;0;0)
29	(1,1,1,0,0,0,1,0,0,0,0,0,0,0,1)	(389,	359,	247,	247,	319,	239)	(34;46;36;0;0;0;34;0;0;0;0;0;0;0;34)
30	(1,1,0,0,0,1,0,0,0,0,0,0,1,1,0)	(60,	60,	60,	80,	50,	50)	(x;x;0;0;0;x;0;0;0;0;0;0;16;15;0)
31	(1,1,0,0,0,0,1,0,0,1,0,0,0,0,1)	(90,	90,	90,	90,	30,	30)	(35;25;0;0;0;0;15;0;0;15;0;0;0;0;14)
32	(1,1,0,0,0,0,1,0,0,0,1,0,1,0,0)	(120,	120,	120,	120,	120,	0)	(34;25;0;0;0;0;15;0;0;0;14;0;16;0;0)
33	(1,1,0,0,0,0,1,0,0,0,1,0,0,1,0)	(349,	349,	314,	314,	237,	237)	(56;56;0;0;0;0;56;0;0;0;56;0;0;56;0)
34	(1,1,1,1,1,1,0,0,0,0,0,0,0,0,0)	(480,	270,	270,	260,	260,	260)	(x;x;x;x;x;x;0;0;0;0;0;0;0;0;0)
35	(1,1,1,1,0,1,1,0,0,0,0,0,0,0,0)	(185,	115,	105,	105,	90,	0)	(x;x;x;x;0;x;x;0;0;0;0;0;0;0;0)
36	(1,1,1,1,0,1,0,0,1,0,0,0,0,0,0)	(420,	350,	270,	255,	255,	250)	(x;x;x;x;0;x;0;0;34;0;0;0;0;0;0)
37	(1,1,1,1,0,1,0,0,0,0,0,0,1,0,0)	(200,	100,	100,	100,	100,	0)	(x;x;x;x;0;x;0;0;0;0;0;0;x;0;0)
38	(1,1,1,1,0,1,0,0,0,0,0,0,0,1,0)	(435,	265,	265,	335,	255,	245)	(x;x;x;x;0;x;0;0;0;0;0;0;0;25;0)
39	(1,1,1,1,0,0,0,0,1,0,0,1,0,0,0)	(436,	286,	286,	258,	258,	276)	(x;x;x;x;0;0;0;0;34;0;0;24;0;0;0)
40	(1,1,1,0,0,1,1,0,0,1,0,0,0,0,0)	(90,	90,	90,	90,	0,	0)	(x;x;x;0;0;x;x;0;0;x;0;0;0;0;0)
41	(1,1,1,0,0,1,1,0,0,0,1,0,0,0,0)	(118,	118,	173,	103,	88,	0)	(x;x;x;0;0;x;x;0;0;0;16;0;0;0;0)
42	(1,1,1,0,0,1,1,0,0,0,0,0,0,0,1)	(95,	95,	85,	85,	30,	30)	(x;x;x;0;0;x;x;0;0;0;0;0;0;0;34)
43	(1,1,1,0,0,1,0,1,0,0,0,1,0,0,0)	(350,	350,	350,	250,	250,	250)	(x;x;56;0;0;x;0;46;0;0;0;45;0;0;0)
44	(1,1,1,0,0,1,0,1,0,0,0,0,1,0,0)	(131,	131,	106,	116,	116,	0)	(x;x;26;0;0;x;0;16;0;0;0;0;16;0;0)
45	(1,1,1,0,0,1,0,1,0,0,0,0,0,1,0)	(377,	342,	265,	327,	247,	242)	(x;x;35;0;0;x;0;36;0;0;0;0;0;35;0)
46	(1,1,1,0,0,1,0,0,0,0,0,0,1,1,0)	(380,	260,	260,	400,	250,	250)	(x;x;25;0;0;x;0;0;0;0;0;0;26;25;0)
47	(1,1,1,0,0,1,0,0,0,0,0,0,1,0,1)	(369,	257,	257,	359,	319,	239)	(x;x;26;0;0;x;0;0;0;0;0;0;26;0;26)
48	(1,1,1,0,0,0,1,1,0,1,0,0,0,0,0)	(366,	366,	281,	253,	281,	253)	(34;45;36;0;0;0;34;34;0;46;0;0;0;0;0)
49	(1,1,1,0,0,0,1,0,0,1,0,1,0,0,0)	(132,	112,	112,	112,	132,	0)	(34;24;23;0;0;0;16;0;0;16;0;16;0;0;0)
50	(1,1,1,0,0,0,1,0,0,1,0,0,0,1,0)	(392,	280,	280,	332,	272,	244)	(36;26;23;0;0;0;36;0;0;26;0;0;0;23;0)
51	(1,1,1,0,0,0,1,0,0,1,0,0,0,0,1)	(360,	288,	288,	252,	360,	252)	(34;24;23;0;0;0;34;0;0;24;0;0;0;0;23)
52	(1,1,1,0,0,0,1,0,0,0,1,0,0,0,1)	(374,	297,	297,	254,	289,	289)	(34;24;23;0;0;0;34;0;0;0;24;0;0;0;23)

Table 3: Payoffs for game of equation (3) and associated links, used in the proof of lemma 4.1, part 1

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number	graph	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	associated link
53	(1,1,0,0,0,1,0,0,0,0,0,0,1,1,1)	(60,	60,	60,	60,	60,	60)	(x;x;0;0;0;x;0;0;0;0;0;0;x;x;x)
54	(1,1,0,0,0,0,1,0,0,0,1,0,0,1,1)	(300,	300,	300,	300,	300,	300)	(34;25;0;0;0;0;15;0;0;0;14;0;0;15;14)
55	(1,1,1,1,1,1,1,0,0,0,0,0,0,0,0)	(445,	285,	275,	275,	260,	260)	(x;x;x;x;x;x;x;0;0;0;0;0;0;0;0)
56	(1,1,1,1,1,1,0,0,0,0,0,0,1,0,0)	(460,	270,	270,	270,	270,	260)	(x;x;x;x;x;x;0;0;0;0;0;0;x;0;0)
57	(1,1,1,1,0,1,1,1,0,0,0,0,0,0,0)	(135,	135,	110,	110,	110,	0)	(x;x;x;x;0;x;x;x;0;0;0;0;0;0;0)
58	(1,1,1,1,0,1,1,0,1,0,0,0,0,0,0)	(370,	370,	275,	275,	255,	255)	(x;x;x;x;0;x;x;0;x;0;0;0;0;0;0)
59	(1,1,1,1,0,1,1,0,0,1,0,0,0,0,0)	(180,	110,	110,	110,	90,	0)	(x;x;x;x;0;x;x;0;0;x;0;0;0;0;0)
60	(1,1,1,1,0,1,1,0,0,0,1,0,0,0,0)	(138,	123,	123,	108,	108,	0)	(x;x;x;x;0;x;x;0;0;0;x;0;0;0;0)
61	(1,1,1,1,0,1,1,0,0,0,0,1,0,0,0)	(373,	288,	358,	273,	255,	253)	(x;x;x;x;0;x;x;0;0;0;0;25;0;0;0)
62	(1,1,1,1,0,1,1,0,0,0,0,0,0,0,1)	(400,	280,	270,	270,	335,	245)	(x;x;x;x;0;x;x;0;0;0;0;0;0;0;34)
63	(1,1,1,1,0,1,0,0,1,0,0,1,0,0,0)	(431,	291,	291,	258,	258,	271)	(x;x;x;x;0;x;0;0;x;0;0;x;0;0;0)
64	(1,1,1,1,0,1,0,0,1,0,0,0,1,0,0)	(400,	350,	270,	265,	265,	250)	(x;x;x;x;0;x;0;0;34;0;0;0;x;0;0)
65	(1,1,1,1,0,1,0,0,1,0,0,0,0,1,0)	(389,	304,	276,	289,	258,	284)	(x;x;x;x;0;x;0;0;34;0;0;0;0;35;0)
66	(1,1,1,1,0,1,0,0,0,0,0,0,0,1,1)	(416,	268,	268,	286,	286,	276)	(x;x;x;x;0;x;0;0;0;0;0;0;0;25;24)
67	(1,1,1,1,0,0,0,0,1,0,0,1,0,1,0)	(391,	283,	283,	283,	259,	301)	(x;x;x;x;0;0;0;0;34;0;0;24;0;23;0)
68	(1,1,1,0,0,1,1,0,0,1,0,0,0,0,1)	(90,	90,	90,	90,	30,	30)	(x;x;x;0;0;x;x;0;0;x;0;0;0;0;15)
69	(1,1,1,0,0,1,1,0,0,0,1,0,1,0,0)	(117,	117,	127,	127,	112,	0)	(x;x;x;0;0;x;x;0;0;0;16;0;16;0;0)
70	(1,1,1,0,0,1,1,0,0,0,1,0,0,1,0)	(293,	293,	355,	355,	252,	252)	(x;x;x;0;0;x;x;0;0;0;16;0;0;15;0)
71	(1,1,1,0,0,1,1,0,0,0,1,0,0,0,1)	(285,	285,	387,	267,	332,	244)	(x;x;x;0;0;x;x;0;0;0;16;0;0;0;16)
72	(1,1,1,0,0,1,0,1,0,0,0,1,1,0,0)	(308,	308,	360,	285,	285,	254)	(x;x;56;0;0;x;0;46;0;0;0;46;46;0;0)
73	(1,1,1,0,0,1,0,1,0,0,0,0,1,1,0)	(309,	301,	273,	371,	291,	255)	(x;x;26;0;0;x;0;36;0;0;0;0;26;35;0)
74	(1,1,1,0,0,1,0,1,0,0,0,0,0,1,1)	(317,	317,	274,	299,	299,	294)	(x;x;35;0;0;x;0;34;0;0;0;0;0;35;34)
75	(1,1,1,0,0,1,0,0,0,0,0,0,1,1,1)	(380,	260,	260,	380,	260,	260)	(x;x;25;0;0;x;0;0;0;0;0;0;x;x;x)
76	(1,1,1,0,0,0,0,1,1,0,1,1,0,0,0)	(368,	308,	308,	256,	280,	280)	(45;45;56;0;0;0;0;46;45;0;46;45;0;0;0)
77	(1,1,1,0,0,0,0,1,1,0,1,0,0,1,0)	(324,	324,	288,	288,	288,	288)	(34;45;36;0;0;0;0;34;34;0;34;0;0;34;0)
78	(1,1,1,0,0,0,0,1,0,0,1,0,0,1,1)	(321,	286,	286,	293,	321,	293)	(34;24;23;0;0;0;0;34;0;0;24;0;0;23;23)
79	(1,1,1,1,1,1,1,1,0,0,0,0,0,0,0)	(395,	305,	280,	280,	280,	260)	(x;x;x;x;x;x;x;x;0;0;0;0;0;0;0)
80	(1,1,1,1,1,1,1,0,0,1,0,0,0,0,0)	(440,	280,	280,	280,	260,	260)	(x;x;x;x;x;x;x;0;0;x;0;0;0;0;0)
81	(1,1,1,1,1,1,1,0,0,0,1,0,0,0,0)	(398,	293,	293,	278,	278,	260)	(x;x;x;x;x;x;x;0;0;0;x;0;0;0;0)
82	(1,1,1,1,1,1,1,0,0,0,0,0,0,0,1)	(425,	285,	275,	275,	270,	270)	(x;x;x;x;x;x;x;0;0;0;0;0;0;0;x)
83	(1,1,1,1,0,1,1,1,0,1,0,0,0,0,0)	(130,	130,	115,	115,	110,	0)	(x;x;x;x;0;x;x;x;0;x;0;0;0;0;0)
84	(1,1,1,1,0,1,1,1,0,0,0,1,0,0,0)	(311,	311,	366,	278,	278,	256)	(x;x;x;x;0;x;x;x;0;0;0;45;0;0;0)
85	(1,1,1,1,0,1,1,0,1,1,0,0,0,0,0)	(365,	365,	280,	280,	255,	255)	(x;x;x;x;0;x;x;0;x;x;0;0;0;0;0)
86	(1,1,1,1,0,1,1,0,1,0,1,0,0,0,0)	(311,	381,	296,	278,	276,	258)	(x;x;x;x;0;x;x;0;x;0;x;0;0;0;0)
87	(1,1,1,1,0,1,1,0,1,0,0,0,0,0,1)	(327,	327,	281,	281,	292,	292)	(x;x;x;x;0;x;x;0;x;0;0;0;0;0;34)
88	(1,1,1,1,0,1,1,0,0,1,0,0,0,0,1)	(395,	275,	275,	275,	335,	245)	(x;x;x;x;0;x;x;0;0;x;0;0;0;0;26)
89	(1,1,1,1,0,1,1,0,0,0,1,0,1,0,0)	(124,	119,	119,	119,	119,	0)	(x;x;x;x;0;x;x;0;0;0;x;0;x;0;0)
90	(1,1,1,1,0,1,1,0,0,0,1,0,0,1,0)	(316,	298,	293,	363,	275,	255)	(x;x;x;x;0;x;x;0;0;0;x;0;0;25;0)
91	(1,1,1,1,0,1,1,0,0,0,0,1,0,1,0)	(375,	287,	300,	300,	258,	280)	(x;x;x;x;0;x;x;0;0;0;0;25;0;25;0)
92	(1,1,1,1,0,1,1,0,0,0,0,1,0,0,1)	(332,	296,	314,	278,	291,	289)	(x;x;x;x;0;x;x;0;0;0;0;25;0;0;25)
93	(1,1,1,1,0,1,0,0,1,0,0,1,1,0,0)	(411,	291,	291,	268,	268,	271)	(x;x;x;x;0;x;0;0;x;0;0;x;x;0;0)
94	(1,1,1,1,0,1,0,0,1,0,0,1,0,1,0)	(386,	288,	288,	283,	259,	296)	(x;x;x;x;0;x;0;0;x;0;0;x;0;25;0)
95	(1,1,1,1,0,1,0,0,1,0,0,0,1,1,0)	(332,	309,	278,	309,	278,	294)	(x;x;x;x;0;x;0;0;35;0;0;0;x;35;0)
96	(1,1,1,1,0,1,0,0,1,0,0,0,0,1,1)	(332,	304,	280,	286,	286,	312)	(x;x;x;x;0;x;0;0;34;0;0;0;0;35;34)
97	(1,1,1,1,0,0,0,0,1,0,0,1,0,1,1)	(328,	286,	286,	286,	286,	328)	(x;x;x;x;0;0;0;0;x;0;0;x;0;x;x)
98	(1,1,1,0,0,1,1,0,0,0,1,0,1,0,1)	(285,	285,	303,	303,	368,	256)	(x;x;x;0;0;x;x;0;0;0;16;0;16;0;16)
99	(1,1,1,0,0,1,1,0,0,0,1,0,0,1,1)	(291,	291,	316,	316,	293,	293)	(x;x;x;0;0;x;x;0;0;0;16;0;0;15;15)
100	(1,1,1,0,0,1,0,1,0,0,0,1,1,1,0)	(298,	305,	305,	318,	287,	287)	(x;x;56;0;0;x;0;16;0;0;0;15;16;15;0)
101	(1,1,1,0,0,1,0,1,0,0,0,0,1,1,1)	(312,	312,	276,	312,	312,	276)	(x;x;36;0;0;x;0;36;0;0;0;0;x;x;x)
102	(1,1,1,0,0,0,0,1,1,0,1,1,1,0,0)	(308,	308,	308,	284,	308,	284)	(46;46;46;0;0;0;0;46;46;0;46;46;46;0;0)
103	(1,1,1,1,1,1,1,1,1,0,0,0,0,0,0)	(330,	330,	285,	285,	285,	285)	(x;x;x;x;x;x;x;x;x;0;0;0;0;0;0)
104	(1,1,1,1,1,1,1,1,0,1,0,0,0,0,0)	(390,	300,	285,	285,	280,	260)	(x;x;x;x;x;x;x;x;0;x;0;0;0;0;0)

Table 4: Payoffs for game of equation (3) and associated links, used in the proof of lemma 4.1, part 2

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number	graph	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	associated link
105	(1,1,1,1,1,1,1,1,0,0,0,1,0,0,0)	(336,	316,	301,	283,	283,	281)	(x;x;x;x;x;x;x;x;0;0;0;x;0;0;0)
106	(1,1,1,1,1,1,1,0,0,1,0,0,0,0,1)	(420,	280,	280,	280,	270,	270)	(x;x;x;x;x;x;x;0;0;x;0;0;0;0;x)
107	(1,1,1,1,1,1,1,0,0,0,1,0,1,0,0)	(384,	289,	289,	289,	289,	260)	(x;x;x;x;x;x;x;0;0;0;x;0;x;0;0)
108	(1,1,1,1,1,1,1,0,0,0,1,0,0,1,0)	(341,	303,	298,	298,	280,	280)	(x;x;x;x;x;x;x;0;0;0;x;0;0;x;0)
109	(1,1,1,1,0,1,1,1,0,1,1,0,0,0,0)	(122,	122,	122,	117,	117,	0)	(x;x;x;x;0;x;x;x;0;x;x;0;0;0;0)
110	(1,1,1,1,0,1,1,1,0,1,0,1,0,0,0)	(303,	303,	373,	285,	278,	258)	(x;x;x;x;0;x;x;x;0;x;0;x;0;0;0)
111	(1,1,1,1,0,1,1,1,0,1,0,0,0,0,1)	(306,	306,	283,	283,	366,	256)	(x;x;x;x;0;x;x;x;0;x;0;0;0;0;36)
112	(1,1,1,1,0,1,1,1,0,0,0,1,0,1,0)	(311,	311,	306,	306,	282,	284)	(x;x;x;x;0;x;x;x;0;0;0;56;0;56;0)
113	(1,1,1,1,0,1,1,0,1,1,0,0,0,0,1)	(322,	322,	286,	286,	292,	292)	(x;x;x;x;0;x;x;0;x;x;0;0;0;0;35)
114	(1,1,1,1,0,1,1,0,1,0,1,1,0,0,0)	(319,	319,	319,	281,	281,	281)	(x;x;x;x;0;x;x;0;x;0;x;x;0;0;0)
115	(1,1,1,1,0,1,1,0,1,0,1,0,1,0,0)	(295,	378,	290,	290,	288,	259)	(x;x;x;x;0;x;x;0;x;0;x;0;x;0;0)
116	(1,1,1,1,0,1,1,0,1,0,1,0,0,1,0)	(322,	322,	299,	299,	279,	279)	(x;x;x;x;0;x;x;0;x;0;x;0;0;x;0)
117	(1,1,1,1,0,1,1,0,1,0,1,0,0,0,1)	(311,	324,	293,	282,	304,	286)	(x;x;x;x;0;x;x;0;x;0;x;0;0;0;34)
118	(1,1,1,1,0,1,1,0,0,0,1,0,0,1,1)	(306,	295,	295,	308,	308,	288)	(x;x;x;x;0;x;x;0;0;0;x;0;0;36;26)
119	(1,1,1,1,0,1,1,0,0,0,0,1,0,1,1)	(316,	292,	298,	298,	287,	309)	(x;x;x;x;0;x;x;0;0;0;0;25;0;25;34)
120	(1,1,1,1,0,1,0,0,1,0,0,1,1,1,0)	(327,	291,	291,	304,	280,	307)	(x;x;x;x;0;x;0;0;x;0;0;x;x;25;0)
121	(1,1,1,1,0,1,0,0,1,0,0,1,0,1,1)	(323,	291,	291,	286,	286,	323)	(x;x;x;x;0;x;0;0;x;0;0;x;0;x;x)
122	(1,1,1,0,0,1,0,1,0,0,0,1,1,1,1)	(300,	300,	300,	300,	300,	300)	(x;x;26;0;0;x;0;16;0;0;0;15;x;x;x)
123	(1,1,1,0,0,0,0,1,1,0,1,1,1,1,0)	(300,	300,	300,	300,	300,	300)	(34;24;23;0;0;0;0;16;15;0;16;15;16;15;0)
124	(1,1,1,1,1,1,1,1,1,1,0,0,0,0,0)	(325,	325,	290,	290,	285,	285)	(x;x;x;x;x;x;x;x;x;x;0;0;0;0;0)
125	(1,1,1,1,1,1,1,1,0,1,1,0,0,0,0)	(382,	292,	292,	287,	287,	260)	(x;x;x;x;x;x;x;x;0;x;x;0;0;0;0)
126	(1,1,1,1,1,1,1,1,0,1,0,1,0,0,0)	(328,	308,	308,	290,	283,	283)	(x;x;x;x;x;x;x;x;0;x;0;x;0;0;0)
127	(1,1,1,1,1,1,1,1,0,1,0,0,0,0,1)	(331,	311,	288,	288,	301,	281)	(x;x;x;x;x;x;x;x;0;x;0;0;0;0;x)
128	(1,1,1,1,1,1,1,1,0,0,0,1,0,1,0)	(320,	313,	295,	295,	284,	293)	(x;x;x;x;x;x;x;x;0;0;0;x;0;x;0)
129	(1,1,1,1,1,1,1,0,0,0,1,0,0,1,1)	(315,	297,	297,	297,	297,	297)	(x;x;x;x;x;x;x;0;0;0;x;0;0;x;x)
130	(1,1,1,1,0,1,1,1,0,1,1,0,1,0,0)	(120,	120,	120,	120,	120,	0)	(x;x;x;x;0;x;x;x;0;x;x;0;x;0;0)
131	(1,1,1,1,0,1,1,1,0,1,1,0,0,1,0)	(293,	293,	293,	376,	286,	259)	(x;x;x;x;0;x;x;x;0;x;x;0;0;x;0)
132	(1,1,1,1,0,1,1,1,0,1,0,1,0,1,0)	(309,	309,	309,	309,	282,	282)	(x;x;x;x;0;x;x;x;0;x;0;x;0;x;0)
133	(1,1,1,1,0,1,1,1,0,1,0,1,0,0,1)	(301,	301,	314,	290,	307,	287)	(x;x;x;x;0;x;x;x;0;x;0;x;0;0;46)
134	(1,1,1,1,0,1,1,1,0,0,0,1,0,1,1)	(303,	303,	298,	298,	298,	300)	(x;x;x;x;0;x;x;x;0;0;0;45;0;35;34)
135	(1,1,1,1,0,1,1,0,1,0,1,1,1,0,0)	(301,	314,	314,	294,	294,	283)	(x;x;x;x;0;x;x;0;x;0;x;x;x;0;0)
136	(1,1,1,1,0,1,1,0,1,0,1,0,1,0,1)	(298,	315,	293,	293,	315,	286)	(x;x;x;x;0;x;x;0;x;0;x;0;x;0;x)
137	(1,1,1,1,0,1,1,0,1,0,1,0,0,1,1)	(306,	306,	295,	295,	299,	299)	(x;x;x;x;0;x;x;0;x;0;x;0;0;x;34)
138	(1,1,1,1,0,1,0,0,1,0,0,1,1,1,1)	(318,	291,	291,	291,	291,	318)	(x;x;x;x;0;x;0;0;x;0;0;x;x;x;x)
139	(1,1,1,1,1,1,1,1,1,1,1,0,0,0,0)	(317,	317,	297,	292,	292,	285)	
140	(1,1,1,1,1,1,1,1,1,1,0,0,0,0,1)	(320,	320,	290,	290,	290,	290)	
141	(1,1,1,1,1,1,1,1,0,1,1,0,1,0,0)	(380,	290,	290,	290,	290,	260)	
142	(1,1,1,1,1,1,1,1,0,1,1,0,0,1,0)	(318,	298,	298,	311,	291,	284)	
143	(1,1,1,1,1,1,1,1,0,1,0,1,0,0,1)	(310,	303,	303,	292,	296,	296)	
144	(1,1,1,1,1,1,1,1,0,0,0,1,0,1,1)	(307,	304,	295,	295,	295,	304)	
145	(1,1,1,1,0,1,1,1,0,1,1,0,0,1,1)	(296,	296,	296,	313,	313,	286)	
146	(1,1,1,1,0,1,1,1,0,1,0,1,0,1,1)	(301,	301,	301,	301,	298,	298)	
147	(1,1,1,1,0,1,1,0,1,0,1,1,1,1,0)	(301,	301,	305,	305,	294,	294)	
148	(1,1,1,1,1,1,1,1,1,1,1,1,0,0,0)	(306,	306,	306,	294,	294,	294)	
149	(1,1,1,1,1,1,1,1,1,1,1,0,1,0,0)	(315,	315,	295,	295,	295,	285)	
150	(1,1,1,1,1,1,1,1,1,1,1,0,0,1,0)	(307,	307,	300,	300,	293,	293)	
151	(1,1,1,1,1,1,1,1,0,1,1,0,0,1,1)	(305,	298,	298,	302,	302,	295)	
152	(1,1,1,1,0,1,1,0,1,0,1,1,1,1,1)	(300,	300,	300,	300,	300,	300)	
153	(1,1,1,1,1,1,1,1,1,1,1,1,1,0,0)	(304,	304,	304,	297,	297,	294)	
154	(1,1,1,1,1,1,1,1,1,1,1,0,0,1,1)	(302,	302,	299,	299,	299,	299)	
155	(1,1,1,1,1,1,1,1,1,1,1,1,1,1,0)	(301,	301,	301,	301,	298,	298)	
156	(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)	(300,	300,	300,	300,	300,	300)	

Table 5: Payoffs for game of equation (3) and associated links, used in the proof of lemma 4.1, part 3
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