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INFORMATIONALLY ROBUST EQUILIBRIA

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Informationally Robust Equilibria

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Abstract: Informationally Robust Equilibria (*IRE*) are introduced in *Robson* (1994) as a refinement of Nash equilibria for e.g. bimatrix games, i.e. mixed extensions of two person finite games. Similar to the concept of perfect equilibria, basically the idea is that an *IRE* is a limit of some sequence of equilibria of perturbed games. Here, the perturbation has to do with the hypothetical possibility that the action of one the players is revealed to his fellow player before the fellow player has to decide on his own action. Whereas Robson models these perturbations in extensive form and uses subgame perfection to solve these games, we model the perturbations in strategic form, thus remaining in the class of bimatrix games. Moreover, within the perturbations we impose two possible types of tie breaking rules, which leads to the notions optimistic and pessimistic *IRE*.
The paper provides motivation on *IRE* and its definition. Several properties will be discussed. In particular, we have that *IRE* is a strict concept, and that *IRE* components are faces of Nash components. Specific results from potential games and matrix games are obtained. Possibilities to extend the definition of *IRE* to more-person games are proposed.

Keywords: bimatrix game, equilibrium selection, leakage of information.

JEL Classification Number: C72.

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1 Introduction

Through the years, a vast stream of literature on refinements of the Nash equilibrium concept based on the notion of perfectness as introduced by *Selten* (1975) has been developed. It culminated into the work of *Kohlberg and Mertens* (1986), but not before notions of properness (*Myerson* (1978)), strictly perfectness (*Okada* (1984)) and many others had been introduced. An overview can be found in *Van Damme* (1991). The original underlying idea of these concepts is that players undergo a thought experiment in which all players make mistakes with small but positive probabilities. The current paper is following a refinement based on a similar but different type of thought experiment suggested by *Robson* (1994). Here, the idea is that there is a small but positive probability that one of the players' action is revealed ('leakage of information'). Let us elaborate on this idea by means of an example.

Example 1: Consider the bimatrix game:[4]

$$\begin{bmatrix} (1,1) & (0,0) \\ (1,0) & (0,2) \end{bmatrix}.$$

The row player has no direct influence on his payoff by his own action. He can however have the following line of thought:

> "If there would be a slight chance that my opponent can act upon my action, then I'd better play the top row; my opponent's best reply to this action is playing the left column. This leads to a benefit of 1."

It is not difficult to provide contexts in which leakage of information is relevant, e.g. in a poker game it is crucial to hide (the strength of) your hand, and in a 'battle-of-the-sexes'-game it is beneficial to be able to reveal your action.

Our approach is very similar to the one of Robson; two person games are considered in which the players act simultaneously. The main differences in the models are that we restrict ourselves to games in which the players have finitely many pure strategies and secondly we introduce two optional behavioral tie breaking rules, an optimistic and a pessimistic one.

The games are perturbed by allocating small percentages to two (disjoint) events. With large probability the original game is played. There is a small probability that one of the players acts first. If, say, player one acts first, player two receives the information of the decision of player one. If player one plays a mixed strategy, player two is informed about the outcome of the chance mechanism. Thereafter, he can base his decision on this information. Player one cannot distinguish between this case and the regular one, i.e., he does not know if he is revealing his action or not.

[4] We refer to section 2 for the definition of a bimatrix game.

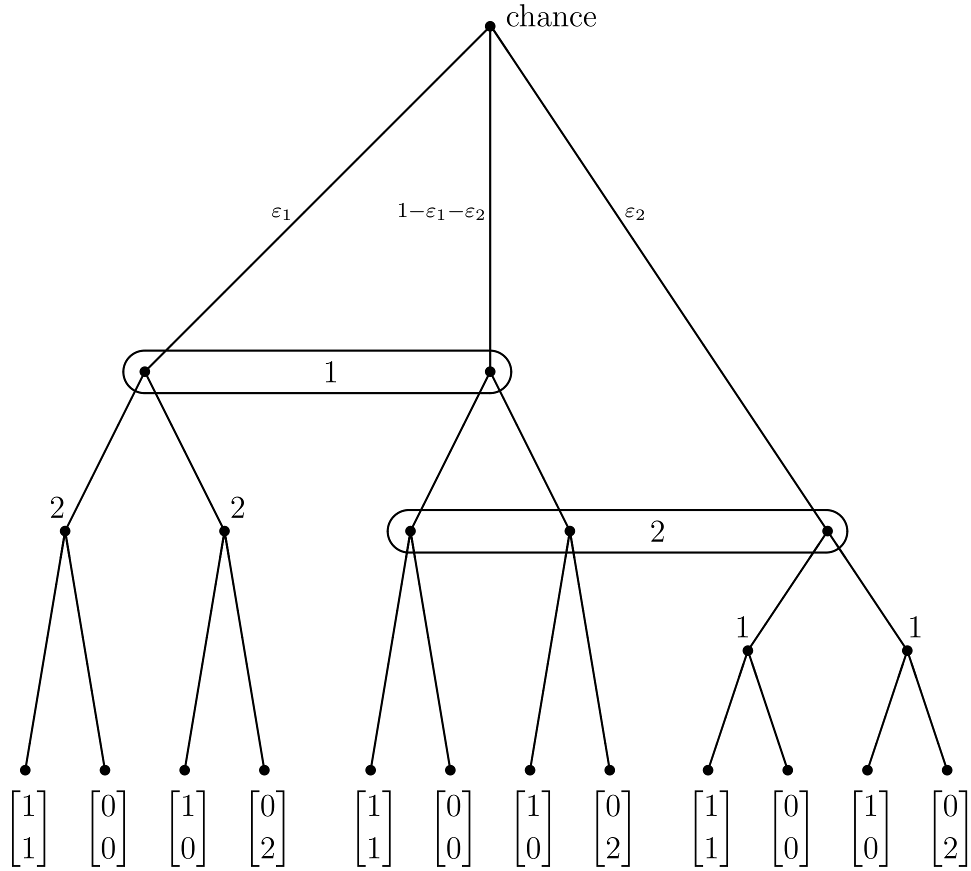

Figure 1: The perturbation of Example 1 in extensive form

Similarly, player two may act first (not knowing this himself) and player one can respond. The events *player one acts first* and *player two acts first* do not necessarily have the same probabilities.

To illustrate this setting, the extensive form of the perturbation of Example 1 is depicted in Figure 1, in which ε_i denotes the probability that player i's action is revealed to the other player ($i = 1, 2$).

Two-person games with finitely many pure strategies, or bimatrix games, in which players act simultaneously can be represented very efficiently in normal form. This is not the case however for the perturbed games; e.g. if the players initially both have 3 pure strategies, they both have 81 pure strategies in the perturbations. To avoid this exponential growth, we will put restrictions on the behavior of the players in such a way that the perturbed games have the same size as the original one. These restrictions are based on the following. In a subgame in which a player must act secondly, he has full information and the situation has the nature of a one-person game. We assume from a rational player that he chooses a strategy that maximizes utility. To diminish the strategy spaces, we delete all other strategies beforehand. In order to decrease the number

of options in such a subgame to just one, we will either assume that the player chooses a best strategy for himself that maximizes the payoff to the other one (the *optimistic* approach), or the player chooses a best strategy for himself that minimizes the payoff to the other (the *pessimistic* approach). If this setup still does not discriminate which strategy will be played, we take an arbitrary remaining one; for both players it is of no importance at all which it will be.
Now that we have established in this way that the perturbed games have the same size as the original game, the definition of an informationally robust equilibrium becomes straightforward. It is a profile that is the limit of a series of equilibria of perturbed games. We will formalize this concept in the next section.
Let us highlight the differences between the approaches of *Robson* (1994) and ours. Robson considers 'two person games with simultaneous moves'. The set of pure strategies of a players is assumed to be compact. The set of bimatrix games can be considered to be the subclass of this type of games at which the pure strategy spaces of the players are finite. Furthermore, Robson requires that the perturbed equilibria must be subgame perfect. This requirement, translated to our setting, boils down to a commitment that players must maximize their payoff in each subgame in which they have full information, even if the subgame is played with probability zero. Our behavioral approach is even more restrictive, since the players must choose a *particular* optimal strategy.

The paper has been organized as follows. Section 2 formalizes the ideas displayed in this introduction and settles notations. Furthermore, it provides an alternative way to describe informationally robustness (Lemma 4) and shows the non-emptiness and closedness of the set of *IRE* (Theorem 5). Section 3 questions whether one of the versions (optimistic vs. pessimistic) of *IRE* is superior to the other, section 4 defines the notion of strict *IRE* and shows that it is coincides with *IRE* itself. The sections 6 and 7 deal with the classes of potential and matrix games respectively. Section 8 discusses ways to generalize informationally robustness to n-person games.

2 *IRE*

Let us fix the notations that are used throughout the paper. A bimatrix game is the mixed extension of a finite two person noncooperative game. It is characterized by a pair (A, B) of real valued matrices of equal, finite, size. The players are called one and two. Player one chooses a row and player two chooses a column. We use m for the number of rows and n for the number of columns. The index sets of the rows and columns are denoted by M and N respectively:

$$M := \{1, \dots, m\} \qquad \text{and} \qquad N := \{1, \dots, n\}.$$

Typical characters to index rows are i and k, typical characters to index columns are j and ℓ. The spaces of mixed strategies are called Δ_m and Δ_n respectively. Furthermore,

$\Delta := \Delta_m \times \Delta_n$; the space of strategy profiles. The unit vectors of Δ_m and Δ_n (i.e. the pure strategies) are denoted by e_i ($i \in M$) and f_j ($j \in N$). A typical element of Δ_m will be denoted by p, a typical element of Δ_n by q. Players have a pure best reply correspondence:

$$PB_1(A,q) := \operatorname*{argmax}_{i \in M} \ e_i A q \qquad \text{and} \qquad PB_2(B,p) := \operatorname*{argmax}_{j \in N} \ pBf_j.$$

The correspondences are upper semi continuous in both coordinates, e.g. if (A_t, q_t) tends to (A, q), then $PB_1(A_t, q_t) \subseteq PB_1(A, q)$ for sufficiently large t.
The *carrier* $C(x)$ of a vector x is the set of its non-zero coordinates, i.e.:

$$C(x) := \{i \mid x_i \neq 0\}.$$

A Nash equilibrium (p, q) is a profile of mixed strategies such that $C(p) \subseteq PB_1(A, q)$ and $C(q) \subseteq PB_2(B, p)$. The set of all Nash equilibria of the game (A, B) is denoted by $E(A, B)$.
In principle, three extra parameters are needed to give the perturbations of A and B. The probability that the action of player one is revealed to player two is called ε_1. The probability that the action of player two is revealed to player one is called ε_2. Furthermore, if the optimistic approach is chosen, the perturbations are labelled with a $^+$. If the pessimistic approach is chosen, they are labelled with a $^-$. So, what will e.g. $A^+_{ij}(\varepsilon_1, \varepsilon_2)$ be? It is the payoff player one receives in an optimistically perturbed game when player one chooses strategy e_i and player two chooses f_j. With large probability $(1 - \varepsilon_1 - \varepsilon_2)$ he receives the original amount A_{ij}. With probability ε_1, player two can respond optimal to e_i. In the optimistic case he will play one of the strategies $f_\ell \in PB_2(B, e_i)$ that maximizes $A_{i\ell}$. With probability ε_2, player one can react optimally against strategy f_j of his opponent, resulting in $\max_{k \in M} A_{kj}$. This leads to the following definition:

Definition 2: Let (A, B) be an $m \times n$-bimatrix game and let ε_1 and ε_2 be positive real numbers satisfying $\varepsilon_1 + \varepsilon_2 < 1$. The optimistic perturbed game $(A^+(\varepsilon_1, \varepsilon_2), B^+(\varepsilon_1, \varepsilon_2))$ is the bimatrix game given by:

$$A^+_{ij}(\varepsilon_1, \varepsilon_2) := (1 - \varepsilon_1 - \varepsilon_2)A_{ij} + \varepsilon_1 \max_{\ell \in PB_2(B, e_i)} A_{i\ell} + \varepsilon_2 \max_{k \in M} A_{kj} \qquad \text{and}$$

$$B^+_{ij}(\varepsilon_1, \varepsilon_2) := (1 - \varepsilon_1 - \varepsilon_2)PB_{ij} + \varepsilon_1 \max_{\ell \in N} B_{i\ell} + \varepsilon_2 \max_{k \in PB_1(A, f_j)} B_{kj}.$$

Similarly, the *pessimistic perturbed game* $(A^-(\varepsilon_1, \varepsilon_2), B^-(\varepsilon_1, \varepsilon_2))$ is defined by:

$$A^-_{ij}(\varepsilon_1, \varepsilon_2) := (1 - \varepsilon_1 - \varepsilon_2)A_{ij} + \varepsilon_1 \min_{\ell \in PB_2(B, e_i)} A_{i\ell} + \varepsilon_2 \max_{k \in M} A_{kj} \qquad \text{and}$$

$$B^-_{ij}(\varepsilon_1, \varepsilon_2) := (1 - \varepsilon_1 - \varepsilon_2)B_{ij} + \varepsilon_1 \max_{\ell \in N} B_{i\ell} + \varepsilon_2 \min_{k \in PB_1(A, f_j)} B_{kj}.$$

We now have made all preparations to define informationally robust equilibria.

Definition 3: Let (A, B) be an $m \times n$-bimatrix game. A profile (p, q) is an optimistic informationally robust equilibrium or *IRE*$^+$ if there exist sequences $(\varepsilon_1^t)_{t \in \mathbb{N}}$ and $(\varepsilon_2^t)_{t \in \mathbb{N}}$

of positive real numbers converging to zero, and a sequence $(p^t, q^t)_{t\in\mathbb{N}}$ in Δ converging to (p, q) such that for all $t \in \mathbb{N}$:

$$(p^t, q^t) \in E(A^+(\varepsilon_1^t, \varepsilon_2^t), B^+(\varepsilon_1^t, \varepsilon_2^t)).$$

Similarly, a profile (p, q) is an pessimistic informationally robust equilibrium or *IRE*$^-$ if there exist sequences $(\varepsilon_1^t)_{t\in\mathbb{N}}$ and $(\varepsilon_2^t)_{t\in\mathbb{N}}$ of positive real numbers converging to zero, and a sequence $(p^t, q^t)_{t\in\mathbb{N}}$ in $\Delta_m \times \Delta_n$ converging to (p, q) such that for all $t \in \mathbb{N}$:

$$(p^t, q^t) \in E(A^-(\varepsilon_1^t, \varepsilon_2^t), B^-(\varepsilon_1^t, \varepsilon_2^t)).$$

The sets of optimistic and pessimistic informationally robust equilibria of (A, B) are denoted by *IRE*$^+(A, B)$ and *IRE*$^-(A, B)$ respectively.

Let us give an alternative, convenient characterization of *IRE*$^+$ and *IRE*$^-$ by means of best reply equivalent perturbed games. Two bimatrix games (A, B) and (C, D) of equal size are called *best reply equivalent* if their pure best reply functions coincide:

$$PB_1(A, \cdot) = PB_1(C, \cdot) \qquad \text{and} \qquad PB_2(B, \cdot) = PB_2(D, \cdot).$$

We will denote this type of equivalence by $(A, B) \equiv_b (C, D)$.

Fix an $m \times n$-bimatrix game (A, B). Let R^+ and $R^- \in \mathbb{R}^{m\times n}$ be defined by:

$$R^+_{ij} := \max_{\ell \in PB_2(B,e_i)} A_{i\ell} \qquad \text{and} \qquad R^-_{ij} := \min_{\ell \in PB_2(B,e_i)} A_{i\ell}.$$

So, rows of R^+ and R^- are constant. Similarly, define $S^+, S^- \in \mathbb{R}^{m\times n}$ by:

$$S^+_{ij} := \max_{e_k \in PB_1(A,f_j)} B_{kj} \qquad \text{and} \qquad S^-_{ij} := \min_{e_k \in PB_1(A,f_j)} B_{kj}.$$

The alternative perturbations of A and B will be:

$$\begin{array}{l} A^+(\varepsilon_1) := A + \varepsilon_1 R^+, \\ B^+(\varepsilon_2) := B + \varepsilon_2 S^+ \end{array} \qquad \text{and} \qquad \begin{array}{l} A^-(\varepsilon_1) := A + \varepsilon_1 R^-, \\ B^-(\varepsilon_2) := B + \varepsilon_2 S^-. \end{array}$$

Lemma 4: *Let (A, B) be an $m \times n$-bimatrix game. A profile (p, q) is IRE$^+$ if and only if there exist sequences $(\varepsilon_1^t)_{t\in\mathbb{N}}$ and $(\varepsilon_2^t)_{t\in\mathbb{N}}$ of positive real numbers converging to zero, and a sequence $(p^t, q^t)_{t\in\mathbb{N}}$ in Δ converging to (p, q) such that for all $t \in \mathbb{N}$:*

$$(p^t, q^t) \in E(A^+(\varepsilon_1^t), B^+(\varepsilon_2^t)).$$

Similarly, a profile (p, q) is IRE$^-$ if and only if there exist sequences $(\varepsilon_1^t)_{t\in\mathbb{N}}$ and $(\varepsilon_2^t)_{t\in\mathbb{N}}$ of positive real numbers converging to zero, and a sequence $(p^t, q^t)_{t\in\mathbb{N}}$ in Δ converging to (p, q) such that for all $t \in \mathbb{N}$:

$$(p^t, q^t) \in E(A^-(\varepsilon_1^t), B^-(\varepsilon_2^t)).$$

Proof: We will only prove this result for the optimistic case; the other case can be proved similarly. Let $\varepsilon_1,\ \varepsilon_2 > 0$. Best reply equivalent games have identical equilibrium sets. Since the definition of *IRE*$^+$ concerns equilibrium sets of perturbed games, we might as well use other perturbations as long as they are best reply equivalent. It is easy to verify that (A, B) and (tA, uB) are best reply equivalent for any positive real numbers t and u, and so are (A, B) and $(A + T, B + U)$ if T is a matrix with constant columns and U is a matrix with constant rows. When defining $T, U \in \mathbb{R}^{M\times N}$ by:

$$T_{ij} := \varepsilon_2 \max_{k\in M} A_{kj} \qquad \text{and} \qquad U_{ij} := \varepsilon_1 \max_{\ell\in N} B_{i\ell},$$

results in:

$$(A^+(\varepsilon_1^t, \varepsilon_2^t), B^+(\varepsilon_1^t, \varepsilon_2^t)) = ((1 - \varepsilon_1^t - \varepsilon_2^t)A + \varepsilon_1^t R^+ + T, (1 - \varepsilon_1^t - \varepsilon_2^t)B + \varepsilon_2^t S^+ + U) \equiv_b$$
$$(A + \frac{\varepsilon_1^t}{1 - \varepsilon_1^t - \varepsilon_2^t} R^+, B + \frac{\varepsilon_2^t}{1 - \varepsilon_1^t - \varepsilon_2^t} S^+) = (A^+(\frac{\varepsilon_1^t}{1 - \varepsilon_1^t - \varepsilon_2^t}), B^+(\frac{\varepsilon_2^t}{1 - \varepsilon_1^t - \varepsilon_2^t})).$$

Define for all $t \in \mathbb{N}$ and $i \in \{1, 2\}$: $\quad \epsilon_i^t := \dfrac{\varepsilon_i^t}{1 - \varepsilon_1^t - \varepsilon_2^t}$. Then one might as well use the sequences $(\epsilon_1^t)_{t\in\mathbb{N}}$ and $(\epsilon_2^t)_{t\in\mathbb{N}}$ in combination with perturbed games of the form:

$$(A + \epsilon_1^t R^+, B + \epsilon_2^t S^+). \qquad \square$$

Let us apply the previous lemma to Example 1 from the introduction. The game $(A^+(\varepsilon_1, \varepsilon_2), B^+(\varepsilon_1, \varepsilon_2))$ is given by:

$$(1 - \varepsilon_1 - \varepsilon_2) \begin{bmatrix} (1,1) & (0,0) \\ (1,0) & (0,2) \end{bmatrix} + \varepsilon_1 \begin{bmatrix} (1,1) & (1,1) \\ (0,2) & (0,2) \end{bmatrix} + \varepsilon_2 \begin{bmatrix} (1,1) & (0,2) \\ (1,1) & (0,2) \end{bmatrix} =$$

$$\begin{bmatrix} (1, 1) & (\varepsilon_1, \varepsilon_1 + 2\varepsilon_2) \\ (1 - \varepsilon_1, 2\varepsilon_1 + \varepsilon_2) & (0, 2) \end{bmatrix},$$

and the alternative perturbation $(A + \varepsilon_1 R^+, B + \varepsilon_2 S^+)$ is:

$$\begin{bmatrix} (1 + \varepsilon_1, 1 + \varepsilon_2) & (\varepsilon_1, 2\varepsilon_2) \\ (1, \varepsilon_2) & (0, 2 + 2\varepsilon_2) \end{bmatrix}.$$

The following theorem is a special case of Theorem 3 of *Robson* (1994). Because of our different approach and notations and its plainness, we include a proof.

Theorem 5: *Let (A, B) be a bimatrix game. Then IRE$^+(A, B)$ and IRE$^-(A, B)$ are non-empty and closed subsets of $E(A, B)$.*

Proof: Like the proof of Lemma 4, we only give the optimistic part of the proof. Firstly, we show the non-emptyness. Let $(\varepsilon_1^t)_{t\in\mathbb{N}}$ and $(\varepsilon_2^t)_{t\in\mathbb{N}}$ be rows converging to 0. Let for all t in $\mathbb{N}$: $(p^t, q^t) \in E(A^+(\varepsilon_1^t), B^+(\varepsilon_2^t))$. Because of the compactness of the strategy spaces, there exists a pair of subsequences converging to, say, $(p, q) \in \Delta$, which is thereby by definition 3 and Lemma 4 an element of *IRE*$^+(A, B)$.

To show that $(p,q) \in IRE^{+}(A,B)$ is an equilibrium, we have to show that $C(p) \subseteq PB_1(A,q)$ and $C(q) \subseteq PB_2(B,p)$. Obviously, it suffices to prove the first statement. Take sequences $(\varepsilon_1^t, \varepsilon_2^t)$ converging to $(0,0)$ and $(p^t, q^t) \in E(A^{+}(\varepsilon_1^t), B^{+}(\varepsilon_2^t))$ converging to (p,q). Let $i \in C(p)$. Then for sufficiently large t we have that $i \in C(p^t)$. Hence,

$$e_i A^{+}(\varepsilon_1^t) q^t \geqslant e_k A^{+}(\varepsilon_1^t) q^t \qquad \text{for all } k \in M.$$

Taking t to infinity, we find:

$$e_i A q \geqslant e_k A q \qquad \text{for all } k \in M.$$

Finally, we show the closedness of $IRE^{+}(A,B)$. Take a converging sequence $(p^t, q^t)_{t \in \mathbb{N}}$ in $IRE^{+}(A,B)$ with limit (p,q). For every t, there are sequences $(\varepsilon_1^{tk}, \varepsilon_2^{tk})_{k \in \mathbb{N}}$ converging to $(0,0)$ and $(p^{tk}, q^{tk})_{k \in \mathbb{N}}$ converging to (p^t, q^t) with:

$$(p^{tk}, q^{tk}) \in E(A^{+}(\varepsilon_1^{tk}), B^{+}(\varepsilon_2^{tk})).$$

Consider the sequences $(\varepsilon_1^{tt}, \varepsilon_2^{tt})_{t \in \mathbb{N}}$ and $(p^{tt}, q^{tt})_{t \in \mathbb{N}}$. They demonstrate that (p,q) is an IRE^{+}. $\square$

3 Optimistic or pessimistic?

In order to get more aquainted with *IRE* we give two examples. The first one shows that the optimistic and pessimistic approaches can lead to different outcomes. The optimistic approach selects the Pareto optimal equilibrium, while the pessimistic approach selects the (unique) perfect equilibrium.

Example 6: Let $(A,B) := \begin{bmatrix} (1,0) & (1,1) & (2,0) \\ (0,1) & (1,0) & (2,1) \end{bmatrix}$.

Take $\varepsilon_1 > 0, \varepsilon_2 > 0$ with $1 - \varepsilon_1 - \varepsilon_2 > 0$. Then:

$$(A^{+}(\varepsilon_1), B^{+}(\varepsilon_2)) = \begin{bmatrix} (1+\ \varepsilon_1, 0) & (1+\ \varepsilon_1, 1+\varepsilon_2) & (2+\ \varepsilon_1, \qquad \varepsilon_2) \\ (\qquad 2\varepsilon_1, 1) & (1+2\varepsilon_1, \qquad \varepsilon_2) & (2+2\varepsilon_1, 1+\varepsilon_2) \end{bmatrix}.$$

Regardless the values of ε_1 and ε_2, the first column is strictly dominated by the third column. Hence, in every optimistic perturbation (e_2, f_3) is the only equilibrium, so $IRE^{+}(A,B) = \{(e_2, f_3)\}$.
We have:

$$(A + \varepsilon_1 R^{-}, B + \varepsilon_2 S^{-}) = \begin{bmatrix} (1+\varepsilon_1, 0) & (1+\varepsilon_1, 1) & (2+\varepsilon_1, 0) \\ (0 \qquad, 1) & (1 \qquad, 0) & (2 \qquad, 1) \end{bmatrix}.$$

We find that $IRE(A,B)^{-} = \{(e_1, f_2)\}$.

The following example shows that two best reply equivalent games can have different *IRE*'s. As one might have observed at Example 1 in the introduction, best reply equivalence is in our opinion not so innocent.

Example 7: Consider the following best reply equivalent games:

$$(A, B) := \begin{bmatrix} (1,1) & (0,1) \\ (0,0) & (0,0) \end{bmatrix} \qquad \text{and} \qquad (A, B') := \begin{bmatrix} (1,0) & (0,0) \\ (0,1) & (0,1) \end{bmatrix}.$$

The equivalence follows by observing that $B' = B + \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$. The equilibrium set of both games equals $(\{e_1\} \times [f_1, f_2]) \cup ([e_1, e_2] \times \{f_2\})$. Since the payoff to player two is independent on his own decision, one might have the opinion that he can play arbitrarily and should not invest time or effort in his decisions.

However, if there is pre-play communication, in the game (A, B) he could try to convince player one that it is profitable to play the top row. A pessimistic player would therefore announce to play the left column. An optimistic player two would not bother; he would think that player one plays the top row anyhow.

In the game (A, B'), player two could try to convince player one that there is no use in playing the top row, because he (player two) plays the right column. If player two succeeds, player one becomes indifferent and can play anything. An optimistic player two would therefore announce to play the right column. A pessimistic player two would not bother.

If there is no pre-play communication, but player two takes a slight probability into account that player one can react on his decisions, he should play, in our opinion, f_1 in the game (A, B) and f_2 in the game (A, B').

The *IRE* concept rather closely comports with the above ideas. It is easy to verify that:

$$\begin{array}{lcl} IRE^+(A,B) = \{e_1\} \times [f_1, f_2], & & IRE^-(A,B) = \{(e_1, f_1)\}, \\ IRE^+(A,B') = \{(e_1, f_2)\} & \text{and} & IRE^-(A,B') = E(A,B'). \end{array}$$

In the game (A, B), the pessimistic approach selects in our opinion the most natural outcome. In the game (A, B') however, this is established by the optimistic approach.

Even though the approaches lead to different outcomes, most results in the paper hold for both the pessimistic and the optimistic version of *IRE*. Moreover, many proofs hardly rely on which of the approaches is chosen. If so, we omit the flag $^+$ or $^-$. Hence, whenever we use the notation *IRE*, one of the options IRE^+ or IRE^- is meant. Similarly, flags on matrices are omitted if it does not matter which option is treated.

4 Strict *IRE*

Like *Robson* (1994), we allowed the probabilities with which the respective players act first to be unequal. It will turn out that if we do require equal probabilities, this will not change the set of informationally robust equilibria. We can go even further; if there is *some* sequence of perturbed games making some profile an *IRE*, then *any* sequence of perturbed games converging to the original game supports this profile being an *IRE*. This section proves the above statement. Firstly, the notion of strict *IRE* is defined, analogously to the way *Okada* (1984) has refined perfectness to strict perfectness.

Definition 8: An equilibrium (p,q) of game (A,B) is called *strict IRE* if for *all* decreasing sequences $(\varepsilon_1^t,\varepsilon_2^t)_{t\in\mathbb{N}}$ converging to $(0,0)$ there is a sequence $(p^t,q^t)_{t\in\mathbb{N}}$ converging to (p,q) with $(p^t,q^t)\in NE(A(\varepsilon_1^t,\varepsilon_2^t),B(\varepsilon_1^t,\varepsilon_2^t))$ for all $t\in\mathbb{N}$.

Theorem 9: *For any bimatrix game (A,B) the sets of IRE and strict IRE coincide.*

Proof: Obviously strict *IRE* is a (weakly) stronger property than *IRE*, so it is suffices to show that any informationally robust equilibrium is strict. Let (p,q) be an *IRE*, with corresponding decreasing rows of perturbations $(\delta_1^t)_{t\in\mathbb{N}}$ and $(\delta_2^t)_{t\in\mathbb{N}}$. Let the row of corresponding equilibria be $(p^t,q^t)_{t\in\mathbb{N}}$. By using subsequences we can establish that $C(p)\subseteq C(p^t)=C(p^{t'})$ and $C(q)\subseteq C(q^t)=C(q^{t'})$ for all $t,t'\in\mathbb{N}$. Take an arbitrary decreasing sequence $(\varepsilon_1^t,\varepsilon_2^t)_{t\in\mathbb{N}}$ converging to $(0,0)$.
Fix $T\in\mathbb{N}$ (think of a large number) with $\delta_1^1>\varepsilon_1^T$ and $\delta_2^1>\varepsilon_2^T$. Choose a $\bar{t}\in\mathbb{N}$ such that:

$$\delta_1^1>\varepsilon_1^T>\delta_1^{\bar{t}} \qquad \text{and} \qquad \delta_2^1>\varepsilon_2^T>\delta_2^{\bar{t}}.$$

Take $\lambda,\mu\in(0,1)$ that are uniquely determined by:

$$\varepsilon_1^T=\lambda\delta_1^1+(1-\lambda)\delta_1^{\bar{t}} \qquad \text{and} \qquad \varepsilon_2^T=\mu\delta_2^1+(1-\mu)\delta_2^{\bar{t}}.$$

Define the profile $(\hat{p}^T,\hat{q}^T)$ by:

$$\hat{p}^T:=\mu p^1+(1-\mu)p^{\bar{t}} \qquad \text{and} \qquad \hat{q}^T:=\lambda q^1+(1-\lambda)q^{\bar{t}}.$$

Clearly, it suffices to show that $(\hat{p}^T,\hat{q}^T)\in E(A(\varepsilon_1^T),B(\varepsilon_2^T))$. Because of the similarity, we only show that $C(\hat{p}^T)\subseteq PB_1(A(\varepsilon_1^T),\hat{q}^T)$. Take $i\in C(\hat{p}^T)$.
Because $C(\hat{p}^T)=C(p^1)=C(p^{\bar{t}})$ and $(p^1,q^1)\in E(A(\delta_1^1),B(\delta_2^1))$, we have for all $k\in M$:

$$e_i(A+\delta_1^1R)q^1\geqslant e_k(A+\delta_1^1R)q^1.$$

Because the rows of R are constant, we can rewrite this to be:

$$e_iAq^1+\delta_1^1r_i\geqslant e_kAq^1+\delta_1^1r_k, \tag{1}$$

in which $r\in\mathbb{R}^m$ is any column of R. Similarly, for all $k\in M$:

$$e_iAq^{\bar{t}}+\delta_1^{\bar{t}}r_i\geqslant e_kAq^{\bar{t}}+\delta_1^{\bar{t}}r_k \tag{2}$$

Adding λ times inequality (1) and $(1-\lambda)$ times inequality (2) results in $(k\in M)$:

$$e_iA\hat{q}^T+\varepsilon_1^Tr_i\geqslant e_kA\hat{q}^T+\varepsilon_1^Tr_k,$$

which boils down to:

$$e_i(A+\varepsilon_1^TR)\hat{q}^T\geqslant e_k(A+\varepsilon_1^TR)\hat{q}^T$$

for all $k\in M$. Hence, $\hat{p}^T$ is a best response to $\hat{q}^T$ in the game $(A+\varepsilon_1^TR,B+\varepsilon_2^TS)$. $\square$

Because *IRE* and strict *IRE* coincide, one might as well only look at the perturbations: $(A(\varepsilon),B(\varepsilon))=(A+\varepsilon R,B+\varepsilon S)$:

Corollary 10: *$(p,q)\in IRE(A,B)$ if and only if it is the limit of some trajectory $(p_\varepsilon,q_\varepsilon)_{\varepsilon\downarrow 0}$ with $(p_\varepsilon,q_\varepsilon)\in E(A+\varepsilon R,B+\varepsilon S)$.*

5 The structure of *IRE*

In bimatrix games, the set of Nash equilibria is the union of finitely many Nash components (*Jansen* (1981)) . We show that *IRE*(A, B) can be decomposed in the same way. Extreme points of such a component are extreme points of some Nash component. Not all Nash components contain *IRE*'s and sometimes two *IRE* components are situated in the same Nash component.

Definition 11: Let (A, B) be a bimatrix game. A set G of profiles is called an *IRE* component if:

(i) G is a convex subset of *IRE*(A, B),

(ii) G is a product set, i.e. $G = G_1 \times G_2$ for some $G_1 \subseteq \Delta_m$, $G_2 \subseteq \Delta_n$.

(iii) G is maximal with respect to properties (i) and (ii),

If we replace *IRE*(A, B) by $E(A, B)$ in the previous definition, we obtain the definition of a Nash component. *Jansen* (1981) has shown that Nash components are polytopes and that there are finitely many of them. We will show the same for *IRE* components with the help of the following claim.

Claim 12: *Let (p, q) be an informationally robust equilibrium of the game (A, B). Let (p', q') be an element of the relative interior of the smallest face of the Nash component in which (p, q) is situated. Then $(p', q') \in$ IRE as well.*

Because *IRE* is a closed set, we might as well omit the presumption that (p', q') is situated in the *relative interior*. However, presuming it gives that the carriers and best reply sets concerning p and p' coincide (see e.g. *Jurg* (1993), §2.2).

Proof: Because a component is the cartesian product of two polytopes, (p', q) is situated on the same face as (p, q) and (p', q') do. Without loss of generality we assume that q equals q', since if we can prove that $(p', q) \in$ *IRE*, we can repeat the argument for (p', q'), given that $(p', q) \in$ *IRE*.
Inside the relative interior of the face of a Nash component the carrier $C(\cdot)$ and pure best reply correspondence $PB_2(B, \cdot)$ are constant. Hence, we have $C(p) = C(p')$ and $PB_2(B, p) = PB_2(B, p')$. Furthermore, there is a decreasing sequence $(\varepsilon^t)_{t \in \mathbb{N}}$ with limit 0 and a series of profiles $(p^t, q^t)_{t \in \mathbb{N}}$ converging to (p, q) such that (p^t, q^t) is an equilibrium of the game $(A(\varepsilon^t), B(\varepsilon^t))$. For all t, define:

$$\hat{p}^t := p' - p + p^t.$$

Then $\hat{p}^t$ converges to p'. For large t, $\hat{p}^t$ is a strategy of player 1, because:

$$\sum_{i \in M} \hat{p}_i^t = \sum_{i \in M} p_i' - \sum_{i \in M} p_i + \sum_{i \in M} p_i^t = 1$$

and if $\hat{p}_i^t < 0$, then $i \in C(p) = C(p')$, so $p'_i > 0$. Hence, increasing t sufficiently will lead to a positive value of $\hat{p}_i^t$.
The proof is complete when we can show that $(\hat{p}^t, q^t) \in E(A(\varepsilon^t), B(\varepsilon^t))$. We have:

$$C(\hat{p}^t) \subseteq C(p') \cup C(p) \cup C(p^t) = C(p^t) \subseteq PB_1(A(\varepsilon^t), q^t).$$

Let j be an element of $C(q^t)$ and let ℓ be in $PB_2(B(\varepsilon^t), \hat{p}^t)$. Since $C(q^t) \subseteq PB_2(B(\varepsilon^t), p^t)$, we have:

$$p^t(B + \varepsilon^t S)f_\ell \leqslant p^t(B + \varepsilon^t S)f_j. \tag{3}$$

Because pure best reply correspondences are upper semi continuous (see section 2), for t sufficiently large we obtain:

$$C(q^t) \subseteq PB_2(B(\varepsilon^t), p^t) \subseteq PB_2(B, p) \quad \text{and} \quad PB_2(B(\varepsilon^t), \hat{p}^t) \subseteq PB_2(B, p').$$

Combining these statements gives:

$$\{j, \ell\} \subseteq PB_2(B, p) = PB_2(B, p').$$

This implies that:

$$pBf_\ell = pBf_j \text{ and } -p'Bf_\ell = -p'Bf_j. \tag{4}$$

Because the columns of S are constant, we have:

$$p(\varepsilon^t S)f_\ell = p^t(\varepsilon^t S)f_\ell \text{ and } p(\varepsilon^t S)f_j = p^t(\varepsilon^t S)f_j. \tag{5}$$

The observations in (3), (4) and (5) together lead to:

$$(p - p' + p^t)(B + \varepsilon^t S)f_\ell \leqslant (p - p' + p^t)(B + \varepsilon^t S)f_j.$$

Hence, like ℓ, the strategy j is an element of $PB_2(B(\varepsilon^t), \hat{p}^t)$. We conclude that $(\hat{p}^t, q^t)$ is an element of $E(A(\varepsilon^t), B(\varepsilon^t))$. □

Claim 12 and the fact that *IRE*(A, B) is a closed set lead to the observation that *IRE* components behave like Nash components. We have established the following result:

Theorem 13: *The IRE components of a bimatrix game (A, B) are faces of its Nash components. They are thereby polytopes and there are finitely many of them.*

6 Potential games

Potential games have been introduced by *Monderer and Shapley* (1996). There are many economic situations that can be modelled by potential games. For an overview we refer to *Voorneveld* (1999). The main virtue of having a potential function for a finite game is that it implies the existence of an (easily traceable) Nash equilibrium in pure strategies.

Perhaps the most natural definition of a potential is the *cardinal* (or exact) potential function. On the other hand, the *ordinal* potential generalizes this concept to a much wider class of games and can still be used to obtain the result of this section. Therefore, we give the definition of the latter type of potential:

Definition 14: A bimatrix game (A, B) is an *ordinal potential game* if there exists a function $P : \Delta \longrightarrow \mathbb{R}$ such that for all $p, p' \in \Delta_m$ and $q, q' \in \Delta_n$:

$$\begin{array}{lll} pAq > p'Aq & \text{if and only if} & P(p,q) > P(p',q) \\ pBq > pBq' & \text{if and only if} & P(p,q) > P(p,q'). \end{array} \quad \text{and}$$

The function P is called an *(ordinal) potential* of the game (A, B).

It turns out that *IRE* and the set of strategy pairs at which the potential is maximal always have a profile in common.

Theorem 15: *Let (A, B) be a bimatrix game with ordinal potential P. Then there exists a pure informationally robust equilibrium that maximizes the potential.*

Proof: Define the $m \times n$-matrix $\bar{P}$ as the restriction of P to the pure strategy profiles of (A, B):

$$\bar{P}_{ij} := P(e_i, f_j). \qquad (i \in M,\ j \in N)$$

By the definition of a potential, for all $i, k \in M$ and all $j, \ell \in N$:

$$\begin{array}{l} A_{ij} > A_{kj} \iff \bar{P}_{ij} > \bar{P}_{kj} \\ B_{ij} > B_{i\ell} \iff \bar{P}_{ij} > \bar{P}_{i\ell}. \end{array} \tag{6}$$

Let us call a matrix satisfying (6) a *potential matrix*. Firstly, we show that the perturbation $(A + \varepsilon R, B + \varepsilon S)$ has potential matrix $\bar{P} + \varepsilon(R + S)$ if ε is chosen sufficiently small. Let $i, k \in M$ and $j \in N$. If $A_{ij} = A_{kj}$, then $\bar{P}_{ij} = \bar{P}_{kj}$ and therefore:

$$(A + \varepsilon R)_{ij} > (A + \varepsilon R)_{kj} \iff (\bar{P} + \varepsilon R)_{ij} > (\bar{P} + \varepsilon R)_{kj}. \tag{7}$$

If $A_{ij} > A_{kj}$, then $\bar{P}_{ij} > \bar{P}_{kj}$ and we can choose ε sufficiently small to obtain the validity of the statements $(A+\varepsilon R)_{ij} > (A+\varepsilon R)_{kj}$ and $(\bar{P}+\varepsilon R)_{ij} > (\bar{P}+\varepsilon R)_{kj}$ in (7). Similarly, (7) holds when $A_{ij} < A_{kj}$ and ε is sufficiently small (switch the roles of i and k). Because S has constant columns we have $S_{ij} = S_{kj}$, making (7) equivalent with:

$$(A + \varepsilon R)_{ij} > (A + \varepsilon R)_{kj} \iff (\bar{P} + \varepsilon R + \varepsilon S)_{ij} > (\bar{P} + \varepsilon R + \varepsilon S)_{kj}.$$

Similarly, for all $i \in M$ and all $j, k \in N$ and sufficiently small ε:

$$(B + \varepsilon S)_{ij} > (B + \varepsilon S)_{i\ell} \iff (\bar{P} + \varepsilon R + \varepsilon S)_{ij} > (\bar{P} + \varepsilon R + \varepsilon S)_{i\ell}.$$

Hence, the perturbations have potential matrices as well. It is easy to infer that a pure strategy profile maximizing a potential matrix is a Nash equilibrium. There are finitely many pure profiles, so for any sequence of perturbed games converging to (A,B), there exists a subsequence of it and a pure profile (e_i, f_j) such that (e_i, f_j) is a 'potential matrix maximizer' in all games in the subsequence. Since the potential matrices of the perturbed games converge to $\bar{P}$, (e_i, f_j) is a pure IRE maximizing the potential P. $\square$

Remark 16: A function $P : \Delta \longrightarrow \mathbb{R}$ is called a *cardinal* or *exact* potential of (A,B) if for all $p, p' \in \Delta_m$ and all $q, q' \in \Delta_n$ we have:

$$pAq - p'Aq = P(p,q) - P(p',q) \qquad \text{and} \qquad pBq - pBq' = P(p,q) - P(p,q').$$

In the case that P is a cardinal potential, then P is the multilinear extension of $\bar{P}$. Along the lines of the proof of Theorem 15 it can be shown that the multilinear extension of $(\bar{P} + \varepsilon(R+S))$ is a cardinal potential of $(A+\varepsilon R, B+\varepsilon S)$.

In general, not all potential maximizers survive. In the following cardinal potential game, the set of potential maximizers is the union of two line segments. In the pessimistic version of *IRE*, the three pure equilibria survive:

Example 17: Consider the game $(A,B) = \begin{bmatrix} (2,2) & (1,2) \\ (2,1) & (0,0) \end{bmatrix}$ with cardinal potential (matrix) $P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Then $E(A,B) = ([e_1, e_2] \times \{f_1\}) \cup (\{e_1\} \times [f_1, f_2])$. All equilibria maximize P. The perturbed game $(A^-(\varepsilon), B^-(\varepsilon)) = \begin{bmatrix} (2+\ \varepsilon, 2+\ \varepsilon) & (1+\ \varepsilon, 2+2\varepsilon) \\ (2+2\varepsilon, 1+\ \varepsilon) & (\quad 2\varepsilon, \quad 2\varepsilon) \end{bmatrix}$ has potential matrix $P + \varepsilon(R^- + S^-) = \begin{bmatrix} 1+2\varepsilon & 1+3\varepsilon \\ 1+3\varepsilon & 4\varepsilon \end{bmatrix}$. It has three Nash equilibria: (e_1, f_2), (e_2, f_1) and $((1-\varepsilon)e_1 + \varepsilon e_2, (1-\varepsilon)f_1 + \varepsilon f_2)$.

7 Matrix games

A *matrix game* is a bimatrix game (A,B) with $B = -A$. It is customary to denote such a game by A. Two-person zero-sum games can be considered to be the utmost noncooperative games, since any action of an agent motivated by an increase of utility, automatically leads to a corresponding decrease of utility to the other one. There appears to be no room at all for negotiation which profile (equilibrium) should be played. Still, there is support to play Nash equilibria, without introducing prior beliefs concerning the opponents plans. The equilibrium set of a matrix game has a product structure, i.e., if (p,q) and (p',q') are equilibria, then so are (p',q) and (p,q'). This justifies speaking about an equilibrium strategy rather than an equilibrium profile. This section shows that *IRE*(A) is, like the Nash set, a product set, so we can consider informationally robust strategies. It is, again like the Nash set, a polytope, and an element of it can be found in polynomial time.

In the zero-sum case, the optimistic and the pessimistic view coincide, because playing a best response determines the payoff to the other player. $A^+(\varepsilon_1, \varepsilon_2)$ and $A^-(\varepsilon_1, \varepsilon_2)$ both equal the zero-sum game defined by $(i \in M,\ j \in N)$:

$$A(\varepsilon_1, \varepsilon_2)_{ij} := (1 - \varepsilon_1 - \varepsilon_2)A_{ij} + \varepsilon_1 \min_{\ell \in N} A_{i\ell} + \varepsilon_2 \max_{k \in M} A_{kj}.$$

The matrices R^+ and R^- also coincide in a zero-sum game. We denote:

$$R_{ij} := \max_{\ell \in PB_2(-A, e_i)} A_{i\ell} = \min\nolimits_{\ell \in N} A_{i\ell} \qquad (i \in M)$$

Similarly:

$$S_{ij} := \max_{e_k \in PB_1(A, f_j)} - A_{kj} = \min_{k \in M} -A_{kj} = -\max_{k \in M} A_{kj} \qquad (j \in N)$$

By Lemma 4 one might as well consider the disturbed game:

$$(A + \varepsilon_1 R, -A + \varepsilon_2 S).$$

This game is strategically equivalent with the zero-sum game:

$$(A + \varepsilon_1 R - \varepsilon_2 S).$$

Finally, because *IRE* and strict *IRE* coincide, one might as well consider the disturbed game:

$$A + \varepsilon(R - S).$$

Let $r \in \mathbb{R}^M$ be any column of R (they are identical) and let $s \in \mathbb{R}^N$ be any row of S.

Theorem 18: *Let A be a matrix representing a zero-sum game. Let $O(A)_1$ and $O(A)_2$ be the polytopes of optimal strategies of players one and two respectively. Then IRE(A) is a product set, i.e. it can be decomposed: IRE$(A) = IO(A)_1 \times IO(A)_2$. IRE$(A)_1$ is the face of $O(A)_1$ at which the linear function:*

$$O(A)_1 \longrightarrow \mathbb{R}, \quad p \mapsto \langle p, r \rangle$$

is maximized.
Similarly, $IO(A)_2$ is the face of $O(A)_2$ at which the linear function:

$$O(A)_2 \longrightarrow \mathbb{R}, \quad q \mapsto \langle -s, q \rangle$$

is minimized.

The proof is based on the following idea. We have seen that *IRE*$(A) \subseteq E(A)$. It appears that the primal concern of a player is to play an optimal strategy of the original game A. The term εR is of secondary concern to player one. Hence, he should, within his Nash polytope, maximize this term. The term $-\varepsilon S$ has no strategic influence to player one since the columns of S are constant.

Because of its technical nature the proof has been postponed to the appendix. It requires acquaintance with the Simplex method (e.g. *Nemhauser and Wolsey* (1988)).

$$\begin{array}{ll}
\underline{\text{Keeper}} \diagdown \underline{\text{Forward}} & \begin{array}{ccc} \text{dive to the left} & \text{dive to the right} & \text{stand still} \end{array} \\
\begin{array}{l} \text{aim at left corner:} \\ \text{aim at right corner:} \\ \text{just kick firmly:} \end{array} & \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}
\end{array}$$

Figure 2: A penalty shot

The nature of zero-sum games supports the refinement of informationally robustness. For instance, it reduces the harm 'not having a poker face' can have, or the disutility that occurs if it is possible to be 'cheaten' with small probabilities. Let us give as an illustration a situation in which *IRE* selects in our opinion the profile that fits best with the context.

Example 19: Consider a situation in which a penalty shot has been assigned to a soccer team. Let us give the forward who has to shoot three options; he can aim at the left corner, the right corner or he can just give a firm kick. If the forward is skilled, it is obvious that the best thing to do is aim at a corner. If his aiming is poor however and he faces an excellent keeper, he'd better shoot firmly and hope for the best. The keeper has three pure strategies as well: dive to the left (from the perspective of the forward), dive to the right or stand still and react on the shot.

In our example, depicted in Figure 2, the forward is moderate and we have designed the figures such that he has various optimal strategies. Because the forward cannot aim perfectly, the figures in the matrix do not represent certain outcomes, but expectations. The keeper has one optimal strategy: $q := \frac{1}{2}(f_1 + f_2)$. The forward has two extreme optimal strategies: $p^1 := \frac{1}{2}(e_1 + e_2)$ and $p^2 := \frac{1}{6}(e_1 + e_2 + 4e_3)$. Which one is better? In spite of the fact that p^2 is weakly dominated by p^1, the concept of *IRE* recommends p^2. In the spirit of the concept, p^2 should be played according to the following lines of thought of the forward:

> "Suppose the keeper can see which corner I am aiming at. Then my chances reduce. On the other hand, if the keeper can see I go for the firm kick, this information is of less value to him."

By using Theorem 18 it is easy to infer that (p^2, q) is indeed the only informationally robust equilibrium; any row r of R equals $(-1, -1, -\frac{1}{2})$ and $-1 = \langle r, p^1 \rangle < \langle r, p^2 \rangle = -\frac{2}{3}$.

8 The n-person case

It is not straightforward how to generalize the concept of information robustness to games with more than two players, i.e. n-matrix games. We give three options:

(i) Each player, but at most one at a time, hears with a small possibility the strategies of all of his opponents. The player can adapt his decision to a (specific) best response.

(ii) Each player, but at most one at a time, reveals with a small possibility his strategy to all of his opponents. The others play an (n-1)-person game thereafter.

(iii) For each ordered pair of players (i, j), there is a slight chance that i finds out the action of player j.

The practical advantage of the first option is that the perturbed games are n-matrix games as well, with the size of the original game. Furthermore, we can again distinguish between an optimistic and a pessimistic approach. It takes too far to elaborate the n-person case in this paper. It is an interesting subject for future research. We will restrict ourselves to one example in which we choose the first of the three options. The optimistic approach is able to make a strict selection of the equilibrium set.

Example 20: There are three players, each of them takes either 1 or 2 coins in his hand. If a player chooses differently from the others, he receives the number he has chosen.

The story can be depicted by the following scheme:

	$(1,1)$	$(1,2)$ or $(2,1)$	$(2,2)$
1 coin	0	0	1
2 coins	2	0	0

The rows represent the two actions of a player (1 coin or 2 coins). The columns represent the combined actions of the opponents. There are seven 'Nash components'. Six of them are segments, one is a singleton:

$$\begin{array}{lll} [(e_1, f_1, g_2), (e_2, f_1, g_2)], & [(e_1, f_1, g_2), (e_1, f_2, g_2)], & [(e_1, f_2, g_1), (e_1, f_2, g_2)], \\ [(e_1, f_2, g_1), (e_2, f_2, g_1)], & [(e_2, f_1, g_1), (e_2, f_2, g_1)], & [(e_2, f_1, g_1), (e_2, f_1, g_2)] \end{array}$$

and $\{(p, p, p)\}$, in which $p = \lambda e_1 + (1 - \lambda)e_2$ and λ is the root in $[0, 1]$ of the expression $x^2 + 2x - 1 = 0$, i.e. $\lambda = \sqrt{2} - 1$.

Let us assume that each agent hears the strategies of the other players with a probability ε, and with probability $1-3\varepsilon$ nobody hears anything. What will be the optimistic payoff scheme? If $A \in \mathbb{R}^{2\times2\times2}$ is the trimatrix of player one, his perturbed payoff trimatrix will be given by $(i, j, k \in \{1, 2\})$:

$$(1 - 3\varepsilon)A_{ijk} + \varepsilon(\max_{\ell\in\{1,2\}} A_{\ell jk}) + \varepsilon(\max_{\ell\in PB_2(B,e_i,g_k)} A_{i\ell k}) + \varepsilon(\max_{\ell\in PB_3(e_i,f_j)} A_{ij\ell}).$$

The optimistic payoff scheme becomes:

	$(1,1)$	$(1,2)$ or $(2,1)$	$(2,2)$
1 coin	2ε	ε	1
2 coins	2	2ε	ε

Four equilibria survive: $(e_1, f_2, g_2), (e_2, f_1, g_2), (e_2, f_2, g_1)$ and (p, p, p).
The pessimistic payoff scheme will be:

	$(1,1)$	$(1,2)$ or $(2,1)$	$(2,2)$
1 coin	2ε	0	$1-2\varepsilon$
2 coins	$2-4\varepsilon$	0	ε

All equilibria survive.

Appendix

In order to prove Theorem 18, a result is needed from Linear Algebra, providing sufficient conditions for convergence of solution sets of perturbed systems of linear equations:

Claim 21: *Let D be an $m \times n$-matrix and let $(d^t)_{t\in\mathbb{N}}$ be a sequence in $\mathbb{R}^m$ converging to d. Let for all t in $\mathbb{N}$, $F^t \subset \mathbb{R}^n$ be the set of feasible points of the system of equations $\{x \in \mathbb{R}^n_+ \mid Dx = d^t\}$. Let F be the set of feasible points of $\{x \in \mathbb{R}^n_+ \mid Dx = d\}$. Suppose there exists a uniform bound $M \in \mathbb{N}$, i.e. $\|x\| \leqslant M$ for all $x \in \bigcup F^t$. If all solution sets F^t are non-empty, then F^t converges to F in the sense that:*

(i) *if $\hat{x}^t \in F^t$ for all $t \in \mathbb{N}$ and $\lim\limits_{t\to\infty} \hat{x}^t = \hat{x}$, then $\hat{x} \in F$,*

(ii) *for all $\hat{x} \in F$ there exists a sequence $(\hat{x}^t)_{t\in\mathbb{N}}$ in $(F^t)_{t\in\mathbb{N}}$ converging to $\hat{x}$.*

Proof: It is easy to infer statement (i) by a continuity argument. The difficult part is to show that any element of F is the limit of some sequence in $(F^t)_{t\in\mathbb{N}}$. The proof will be by induction to n; the number of columns. The case $n = 1$ is left to the reader.
We distinguish between two cases:

Case I: *There exists a strictly positive element $s \in \mathbb{R}^n_{++}$ of F.*

Linear operations like adding rows to others, or multiplying a row with a non-zero number will not change the solutions sets, nor the feature that the constraint vectors converge. Hence, without loss of generality, D has an echelon form: $D = \begin{bmatrix} I_r & M \\ \bar{0} & \bar{0} \end{bmatrix}$, in which r is the rank of D, I_r is an identity matrix, M is some matrix with r rows and $n-r$ columns and the zeros represent zero matrices.
Because $F^t \neq \phi$ for all $t \in \mathbb{N}$, we have that $d_i = d_i^t = 0$ for all $t \in \mathbb{N}$ and all $i > r$. Hence, we might as well remove the $m-r$ zero-rows of D, which boils down to assuming that D is of full rank: $r = m$.

Let $q := (d_1, \dots, d_m, 0, \dots, 0) \in \mathbb{R}^n$. Then $Dq = d$. Similarly, let $q^t := (d^t, \bar{0}) \in \mathbb{R}^n$, so $Dq^t = d^t$. Define $s^t := s + q^t - q$. Let $\delta > 0$ be such that $s_i > \delta$ for all $i \leqslant n$. Then $s_i^t > \frac{1}{2}\delta$ for large t and $i \leqslant n$.

Let $\hat{x}$ be any element of F. Define $\hat{x}^t := \hat{x} + q^t - q$. Then $D\hat{x}^t = d^t$ and $\hat{x}^t \longrightarrow \hat{x}$. Let $\lambda^t := \min\{\lambda \in [0,1] \mid \lambda s^t + (1-\lambda)\hat{x}^t \geqslant \bar{0}\}$ and define $\tilde{x}^t := \lambda^t s^t + (1-\lambda^t)\hat{x}^t \in F^t$.

Let $\varepsilon > 0$. Choose t so large that $\hat{x}_i^t > -\varepsilon$ for all i. If $\hat{x}^t \notin \mathbb{R}_+^n$, then $\lambda^t = \max\limits_{i \leqslant n} \dfrac{-\hat{x}_i^t}{s_i^t - \hat{x}_i^t} \leqslant \dfrac{\varepsilon}{\frac{1}{2}\delta}$. Since δ is fixed and ε can be chosen to be as small as desired, λ^t tends to 0. Hence, $\|\hat{x} - \tilde{x}^t\| \underset{t\to\infty}{\longrightarrow} 0$. This ends Case I.

Case II: *For some $i \leqslant n$, $x_i = 0$ for all $x \in F$.*

Without loss of generality, choose $i = n$. Let $\delta^t := \min\limits_{x \in F^t} x_n$. Let δ be an accumulation point of $(\delta^t)_{t\in\mathbb{N}}$. Because of the uniform bound M, there exists a row $x^{t_1}, x^{t_2}, x^{t_3}, \dots$ converging to, say, x with $x_n^{t_k} = \delta^{t_k}$ and $\lim\limits_{k\to\infty} \delta^{t_k} = \delta$. By continuity, $x \in F$ and $\lim\limits_{k\to\infty} \delta^{t_k} = x_n = 0$. Hence, 0 is the only accumulation point; $\lim\limits_{t\to\infty} \delta^t = 0$.
Substitute, for all $t \in \mathbb{N}$, $x_n = \delta^t$ in the equation set $Dx = d^t$. The solution sets may become smaller, but remain non-empty. By now, the right column can be removed from all sets of equations and obtain a setting with one dimension less. Hence, we can apply the induction hypothesis. For an arbitrary element $\hat{x} = (\hat{x}_1, \dots, \hat{x}_{n-1}, 0)$ of F, we can give an element $(\hat{x}_1^t, \dots, \hat{x}_{n-1}^t, \delta^t)$ in F^t close to $\hat{x}$. $\square$

Notice that if the constraint matrix D is perturbed as well, convergence is not guaranteed. E.g. if $D^t := \begin{bmatrix} 1-\frac{1}{t} & 1+\frac{1}{t} \\ 1+\frac{1}{t} & 1-\frac{1}{t} \end{bmatrix}$ and $d^t := \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, the solution sets of F^t all equal $\{(1,1)\}$, while the solution set of F equals $\{(x, 2-x) : x \in [0,2]\}$.

Proof of Theorem 18: Because for every $\varepsilon > 0$, $E(A + \varepsilon(R - S))$ is a product set and a polytope and because $IRE(A)$ coincides with strict $IRE(A)$ (Theorem 9), $IRE(A)$ is a product set and a polytope as well, say $IRE(A) = IO(A)_1 \times IO(A)_2 \subseteq \Delta_m \times \Delta_n$. The assertions concerning $IO(A)_1$ and $IO(A)_2$ are so similar that we suffice with the proof of the latter. Assume without loss of generality that $A > 0$. Then R is as well a strictly positive matrix and S is a strictly negative matrix. Furthermore, $v(A)$, the value of the game, is strictly positive. Let $(\varepsilon^t)_{t\in\mathbb{N}}$ be a decreasing row with limit 0.
$O(A + \varepsilon^t(R - S))_2$ is the set of optimal solutions of the linear program:

$$\begin{array}{c} \text{minimize } v \text{ subject to:} \\ v \in \mathbb{R}_+,\ q \in \mathbb{R}_+^N \end{array} \quad \begin{bmatrix} 0 & 1 & \cdots & \cdots & 1 \\ 1 & & & & \\ \vdots & & -A + \varepsilon^t(S - R) & & \\ 1 & & & & \end{bmatrix} \begin{bmatrix} v \\ q_1 \\ \vdots \\ q_n \end{bmatrix} \geqslant \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The left column will be referred to as column v, the top row as row 0 and each other row by its corresponding pure strategy: row i ($i \in M$).
If we would like to apply the Simplex method, for each row a slack variable has to be added, except for row 0, since $\sum_{j \in N} q_j$ has to equal 1. We get:

$$\begin{array}{c} \text{minimize } \langle e_v, \begin{bmatrix} v \\ q \\ p \end{bmatrix} \rangle \text{ s.t.:} \\ v \in \mathbb{R}_+,\ q \in \mathbb{R}_+^N,\ p \in \mathbb{R}_+^M \end{array} \quad \begin{bmatrix} 0 & 1 & \cdots & \cdots & 1 & 0 \cdots 0 \\ 1 & & & & & \\ \vdots & & -A + \varepsilon^t (S - R) & & -I_m & \\ 1 & & & & & \end{bmatrix} \begin{bmatrix} v \\ q \\ p \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Here, e_v denotes the unit vector of $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^M$ corresponding to v and I_m denotes the $m \times m$ identity matrix. By adding row 0 of the table $\varepsilon^t r_i$ times to row i ($i \in M$), the table becomes independent of the matrix R, except for the constraint vector. The resulting table will be denoted by LP^t:

$$\begin{array}{c} \text{minimize } \langle e_v, \begin{bmatrix} v \\ q \\ p \end{bmatrix} \rangle \text{ s.t.:} \\ v, q, p \geqslant 0 \end{array} \quad \begin{bmatrix} 0 & 1 & \cdots & 1 & 0 \cdots 0 \\ 1 & & & & \\ \vdots & & -A + \varepsilon^t S & & -I_m \\ 1 & & & & \end{bmatrix} \begin{bmatrix} v \\ q \\ p \end{bmatrix} = \begin{bmatrix} 1 \\ \varepsilon^t r_1 \\ \vdots \\ \varepsilon^t r_m \end{bmatrix} \tag{8}$$

Denote the constraint matrix in the program LP^t by D^t. The program and constraint matrix obtained by substituting $\varepsilon^t := 0$, will be called LP and D respectively. They correspond to the non-perturbed game A.
After having performed the Simplex method, the table has become of the following form:[5]

$$\begin{array}{c} \text{minimize } \langle a^t, \left[v, q, p\right]^\intercal \rangle \text{ s.t.:} \qquad B^t \left[v, q, p\right]^\intercal = b^t. \\ v, q, p \geqslant 0 \end{array} \tag{9}$$

Let us recall the features of the Simplex method that are important for our purpose. The final object vector $a^t \in \mathbb{R}_+ \times \mathbb{R}_+^N \times \mathbb{R}_+^M$ is nonnegative and the sum of the original object vector e_v and some linear combination of the rows of LP^t. The main principle of the Simplex method is, that one might as well optimize the final object vector, because for any row $D^t_{i\cdot}$, the inner product $\langle D^t_{i\cdot}, x\rangle$ is independent on x (as long as x is chosen feasible). The set of optimal points consists of all feasible points with inner product zero with the final object vector.
Because the tables consists of linear equations, we can normalize them such that for all $t \in \mathbb{N}$, all numbers in B^t, b^t and a^t are in some compact segment, e.g. $[-1, 1]$. Hence, by taking a suitable subsequence of the row $(\varepsilon^t)_{t \in \mathbb{N}}$, we can accomplish that B^t, b^t and a^t converge to, say, B, b and a respectively. This limit (minimize $\langle a, x\rangle$ s.t. $Bx = b$)

[5] The $^\intercal$ denotes that the vector is transposed

is a table for the original game and could have been obtained by applying the Simplex method on LP. Hence, a equals e_v plus some linear combination of the rows of LP:

$$a = e_v + \sum_{i=0}^{m} c_i D_{i\cdot} \text{ for some } c \in \mathbb{R} \times \mathbb{R}^M. \tag{10}$$

Because $v(A)$ is strictly positive, we have that $x_v = v(A) > 0$ for all optimal points, so $a_v = 0$. Focussing at the first column of LP, equation (10) gives:

$$0 = a_v = (e_v)_v + \sum_{i=0}^{m} c_i D_{iv} = 1 + \sum_{i=1}^{m} c_i. \tag{11}$$

We have that $a_i^t > 0$ for large t and all $i \in C(a)$. Hence, all variables corresponding to elements of $C(a)$ have value 0 in any optimal point and all corresponding columns can be removed[6] from the tables LP^t and LP without changing optimal sets: columns in $C(a) \cap M$ correspond to pure strategies on which player one can put some weight while playing optimal in the original game A and columns in $C(a) \cap N$ correspond to pure strategies on which player two does not put positive weight in any equilibrium of A.
Denote the complement of the carrier of a by $Z(a)$ (the 'zero part' of a):

$$Z(a) := \{i : a_i = 0\}.$$

Denote the matrices D^t and D of which the redundant columns have been deleted by $\bar{D}^t$ and $\bar{D}$ respectively. Similarly, let $\bar{e}_v := (1, 0, \ldots, 0) \in \mathbb{R}^{Z(a)}$ be the first unit vector of $\mathbb{R}^{Z(a)}$, let $\bar{s} \in \mathbb{R}^{Z(a)}$ be the restriction of the vector $(0, s, 0, \ldots, 0) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^M$ and let $\bar{a}^t$ be the restriction of a^t (so $\bar{a}^t = \bar{0}$ for all t). We can omit these columns as well in equation (10):

$$\bar{0} = \bar{a} = \bar{e}_v + \sum_{i=0}^{m} c_i \bar{D}_{i\cdot}$$

Adding the rows of $\bar{D}^t$ to $\bar{e}_v$, weighted by the same combination c, results in:

$$\bar{e}_v + \sum_{i=0}^{m} c_i \bar{D}^t_{i\cdot} = \sum_{i=0}^{m} c_i \left(\bar{D}^t_{i\cdot} - \bar{D}_{i\cdot}\right) = \sum_{i=1}^{m} c_i \varepsilon^t \bar{s} = (-1) \cdot \varepsilon^t \bar{s}.$$

To infer the second equality, consider program LP^t, (table (8)): the difference between row i of LP^t and row i of LP is ε^t times the vector $(0, s, 0, \ldots, 0)$ for all $i \geqslant 1$. For the right equality we refer to (11). Hence, for all $t \in \mathbb{N}$, in stead of minimizing $\langle \bar{e}_v, x \rangle$, we might as well minimize $\langle -\varepsilon^t \bar{s}, x \rangle$, or $\langle -\bar{s}, x \rangle$.
Call the alternative optimization problem ALP^t:

$$\begin{array}{ll} \text{minimize } \langle -\bar{s}, x \rangle \text{ s.t.:} & \bar{D}^t x = \left[1, \varepsilon^t r_1, \ldots, \varepsilon^t r_m\right]^\mathsf{T}. \\ x \in \mathbb{R}_+^{Z(a)} & \end{array}$$

[6]the removed variables of course still have to be stored and are set to be zero

Let us repeat the results so far. For all $t \in \mathbb{N}$, the set $O(A+\varepsilon^t(R-S))_2$ is described by ALP^t in the sense that for all $q \in \Delta_n$, the following statements are equivalent:

(i) $q \in O(A+\varepsilon^t(R-S))_2$ and

(ii) $q_j = 0$ for all $j \in N \cap C(a)$ and $q_j = x_j$ for all $j \in N \cap Z(a)$ and some optimal solution $x \in \mathbb{R}_+^{Z(a)}$ of ALP^t.

Consequently, the program obtained by substituting $\varepsilon^t := 0$ in ALP^t will be called ALP. The set of feasible points of ALP corresponds to $O(A)_2$ in the sense that for all $q \in \Delta_n$: $q \in O(A)_2$ if and only if $q_j = 0$ for all $j \in N \cap C(a)$ and $q_j = x_j$ for all $j \in N \cap Z(a)$ and some *feasible* point $x \in \mathbb{R}_+^{Z(a)}$ of ALP. The optimal set of ALP corresponds to the face of $O(A)_2$ of which Theorem 18 claims that it coincides with $IO(A)_2$. Hence, we are done if we can show that the optimal set of ALP^t converges to the optimal set of ALP.

After having performed the Simplex method on table ALP^t, we get again a table of the form:

$$\begin{array}{ll}\text{minimize } \langle h^t, x\rangle \text{ s.t.:} & G^t x = g^t. \\ \qquad x \geqslant 0 & \end{array}$$

Here, $h^t \in \mathbb{R}_+^{Z(a)}$. The following lines of argumentation copies the one just after table (9), so details are omitted. Assume that h^t converges to h. Then:

$$h = -\bar{s} + \sum_{i=0}^{m} \bar{c}_i \bar{D}_{i\cdot} \text{ for some } \bar{c} \in \mathbb{R} \times \mathbb{R}^M. \tag{12}$$

We have that $h_i^t > 0$ for large t and all $i \in C(h)$. Columns corresponding to elements of $C(h)$ are removed from the tables ALP^t and ALP without changing optimal sets.

Denote the matrices $\bar{D}^t$ and $\bar{D}$ of which the redundant columns have been deleted by $\hat{D}^t$ and $\hat{D}$ respectively. Similarly, let $\hat{e}_v \in \mathbb{R}^{Z(h)}$ be the first unit vector of $\mathbb{R}^{Z(h)}$, let $\hat{s} \in \mathbb{R}^{Z(h)}$ be the restriction of $\bar{s}$. Omit the redundant columns in equation (12):

$$\bar{0} = -\hat{s} + \sum_{i=0}^{m} \bar{c}_i \hat{D}_{i\cdot}$$

If we add the rows of $\hat{D}^t$ to $-\hat{s}$, weighted by combination $\bar{c}$, we obtain:

$$-\hat{s} + \sum_{i=0}^{m} \bar{c}_i \hat{D}^t_{i\cdot} = \sum_{i=0}^{m} \bar{c}_i \left(\hat{D}^t_{i\cdot} - \hat{D}_{i\cdot}\right) = \sum_{i=1}^{m} \bar{c}_i \varepsilon^t \hat{s}.$$

The object vector $-\hat{s}$ manifests to be a linear combination of the rows of $\hat{D}^t$. Hence, the linear function $\langle -\hat{s}, \cdot\rangle$ is constant on the polytope $F^t := \{x \in \mathbb{R}_+^{Z(h)} \mid \hat{D}^t x = \left[1, \varepsilon^t r\right]^\intercal\}$, say $k^t := \langle -\hat{s}, x\rangle$ for all $x \in F^t$. Add to all rows of ALP^t but the first, the equation $\varepsilon^t\langle -s, x\rangle = \varepsilon^t k^t$, to obtain:

$$F^t = \left\{x \in \mathbb{R}_+^{Z(h)} \;\middle|\; \hat{D}x = [1, \varepsilon^t(r_1 + k^t), \ldots, \varepsilon^t(r_m + k^t)]^\intercal\right\}.$$

Observe that the constraint matrix of this description is no longer dependent on t. Conclusion: we have found a description of the form ($\hat{D}x = d^t$, $x \geqslant 0$) of the optimal set of ALP^t and a description ($\hat{D}x = (1, 0, \ldots, 0)$, $x \geqslant 0$) of the optimal set of ALP. Apply Claim 21 and conclude the validity of Theorem 18. □

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