CentER

Discussion Paper

No. 2004–84

GENERALIZED PROBABILITY-PROBABILITY PLOTS

By N.A. Mushkudiani, J.H.J. Einmahl

September 2004

ISSN 0924-7815

Tilburg University

Generalized Probability-Probability Plots *

Nino A. Mushkudiani
Erasmus MC, Rotterdam

John H.J. Einmahl
Tilburg University

20th September 2004

Abstract

We introduce generalized Probability-Probability (P-P) plots in order to study the one-sample goodness-of-fit problem and the two-sample problem, for real valued data. These plots, that are constructed by indexing with the class of closed intervals, globally preserve the properties of classical P-P plots and are distribution-free under the null hypothesis. We also define the generalized P-P plot process and the corresponding, consistent tests. The behaviour of the tests under contiguous alternatives is studied in detail; in particular, limit theorems for the generalized P-P plot processes are presented. By their structure, the tests perform very well for spike (or pulse) alternatives. We also study the finite sample properties of the tests through a simulation study.

AMS 2000 subject classification. Primary 62G10, 62G20, 62G30; secondary 60F17.

Key words and phrases. Contiguous alternative, generalized P-P plot, goodness-of-fit, limit theorem, two-sample problem.

1 Introduction

Graphical methods in nonparametric statistics have a long history and are nowadays commonly used for analyzing data. Recent developments in computer science and its interaction with nonparametric statistics made the practical applications of the graphical methods even more possible. As a result of this interaction, highly developed statistical packages are used in almost all scientific fields that deal with large quantities of raw, empirical data

*Research partially performed at Eindhoven University of Technology.

and graphical methods are used to visualize the performed analysis. These methods are generally applied while investigating location, scale, skewness, kurtosis or other differences in two-sample problems, symmetry or goodness-of-fit problems, analysis of covariance, and k-sample or other multivariate procedures. For an extensive review and bibliography of graphical methods in nonparametric statistics, see, e.g., Doksum [1977], Gnanadesikan [1977], Fisher [1983], Sawitzki [1994], and Polonik [1999].

Most of the graphical methods are based on diagnostic plots. These plots are often used for detecting validity of the model or analyzing data in an already defined model. Therefore fitting diagnostic or other plots is a necessary step during the data analysis. Citing Fisher [1983]: "in nonparametric statistics probably the most powerful and useful graphical methods are those based on comparison of the sample distribution functions". The most prominent examples of those methods are based on probability-probability (P-P) plots (see, e.g., Beirlant and Deheuvels [1990], Deheuvels and Einmahl [1992], Hsieh and Turnbull [1996], Girling [2000]) and related techniques, like quantile-quantile (Q-Q) plots, pair charts, receiver operating characteristic (ROC) curves, proportional hazards plots, etc.

In this paper we introduce *generalized* P-P plots in order to study the one- and two-sample problem for one-dimensional data. These plots, that are constructed by indexing with the class of closed intervals, globally preserve the properties of the classical P-P plots and are distribution-free under the null hypothesis. Next, based on the generalized P-P plot we define the generalized P-P plot process and use it to define our test statistics. Consequently, these test statistics are distribution-free under the null hypothesis as well and the corresponding tests are consistent against all fixed alternatives. We also study in detail the behaviour under contiguous alternatives. Since the proposed test statistics resemble the classical scan statistic by their structure, so-called spike (or pulse) alternatives are natural to consider.

The paper is organized as follows. In Sections 2 and 3 we deal with the one- and two-sample problem, respectively, considering separately fixed and contiguous alternatives. In Section 4 a simulation study is presented and Section 5 contains the proofs of the main results.

2 One-sample problem

Let $X_1, X_2, \ldots,$ be a sequence of i.i.d. one-dimensional random variables, defined on a probability space $(\Omega, \mathcal{F}, I\!P)$. Let $\mathcal{B}$ be the σ-algebra of Borel sets on $I\!R$. Denote the unknown common distribution with P and let P_n be the empirical probability measure of the sample $X_1, \ldots, X_n$:

$$P_n(B) = \frac{1}{n} \sum_{i=1}^{n} I_B(X_i), \quad B \in \mathcal{B}.$$

Suppose we want to compare P with a given distribution P_0, with continuous distribution function (df) F_0. Now in order to compare the two distributions, based on $X_1, \ldots, X_n$, we need to introduce the class of sets on which we compare these measures. Define the generalized P-P plot as

$$m_n(t) := \sup\{P_n(A) \,:\, P_0(A) \le t,\, A \in \mathcal{A}\}, \quad t \in [0,1],$$

where $\mathcal{A} \subset \mathcal{B}$ is the class of all closed or half-open intervals: $A = [x, y]$, $(-\infty, y]$ or $[x, \infty)$, with $x, y \in I\!R$. Although the indexing class is the class of intervals, thus allowing the detection of only one spike, the procedure can be generalized to the indexing class of unions of at most k ($\in I\!N$) intervals. Observe that when $\mathcal{A}$ would be the class $\{(-\infty, y] \,:\, y \in I\!R\}$, m_n would be the classical P-P plot.

Clearly when $H_0 : P = P_0$ holds, we obtain that the theoretical version of the generalized P-P plot

$$\sup\{P(A) \,:\, P_0(A) \le t,\, A \in \mathcal{A}\}$$

is equal to t, for $t \in [0,1]$. Hence in order to see if P deviates from P_0, we compare m_n with the diagonal, but note that under H_0 typically m_n lies above the diagonal, due to the randomness (see the first plot in Figure 1). Under the alternative we obtain a much more substantial deviation from the diagonal. A shift of the distribution or a decrease of scale leads to a generalized P-P plot far above the diagonal, whereas an increase of scale yields a plot which lies above the diagonal for small values and below for larger values. A spike alternative yields a plot which lies above the diagonal for small values, but where the deviation fades out for larger values. (See again Figure 1.)

Now for testing purposes, define the generalized P-P plot process by

$$M_n(t) := \sqrt{n}\big(m_n(t) - t\big), \quad t \in [0,1].$$

Based on this process we construct, for $c \in (0,1]$, the test statistic

$$T_{n,c} := \sup_{t \in [0,c]} M_n(t).$$

We will reject H_0 when $T_{n,c}$ is large (see Section 4). Below we will study the one-sample problem for fixed and contiguous alternatives and show that $T_{n,c}$ is distribution-free under the null hypothesis and that the corresponding test is consistent. We also derive the limiting distribution of the generalized P-P plot process for contiguous alternatives.

2.1 Null hypothesis and fixed alternatives

Consider the testing problem $H_0 : P = P_0$ against $H_1 : P \neq P_0$. To study this problem we will use the generalized P-P plot, the P-P plot process and the test statistic $T_{n,c}$.

Let us first investigate the behavior of the generalized P-P plot process under the null hypothesis. For fixed $n \ge 1$, we have

$$\begin{aligned} M_n(t) &= \sup\{\sqrt{n}(P_n(A) - P_0(A)) \,:\, P_0(A) = t,\, A \in \mathcal{A}\} \\ &\stackrel{d}{=} \sup\{\Gamma_n(A) \,:\, V(A) = t,\, A \in \mathcal{A}_{[0,1]}\}, \end{aligned}$$

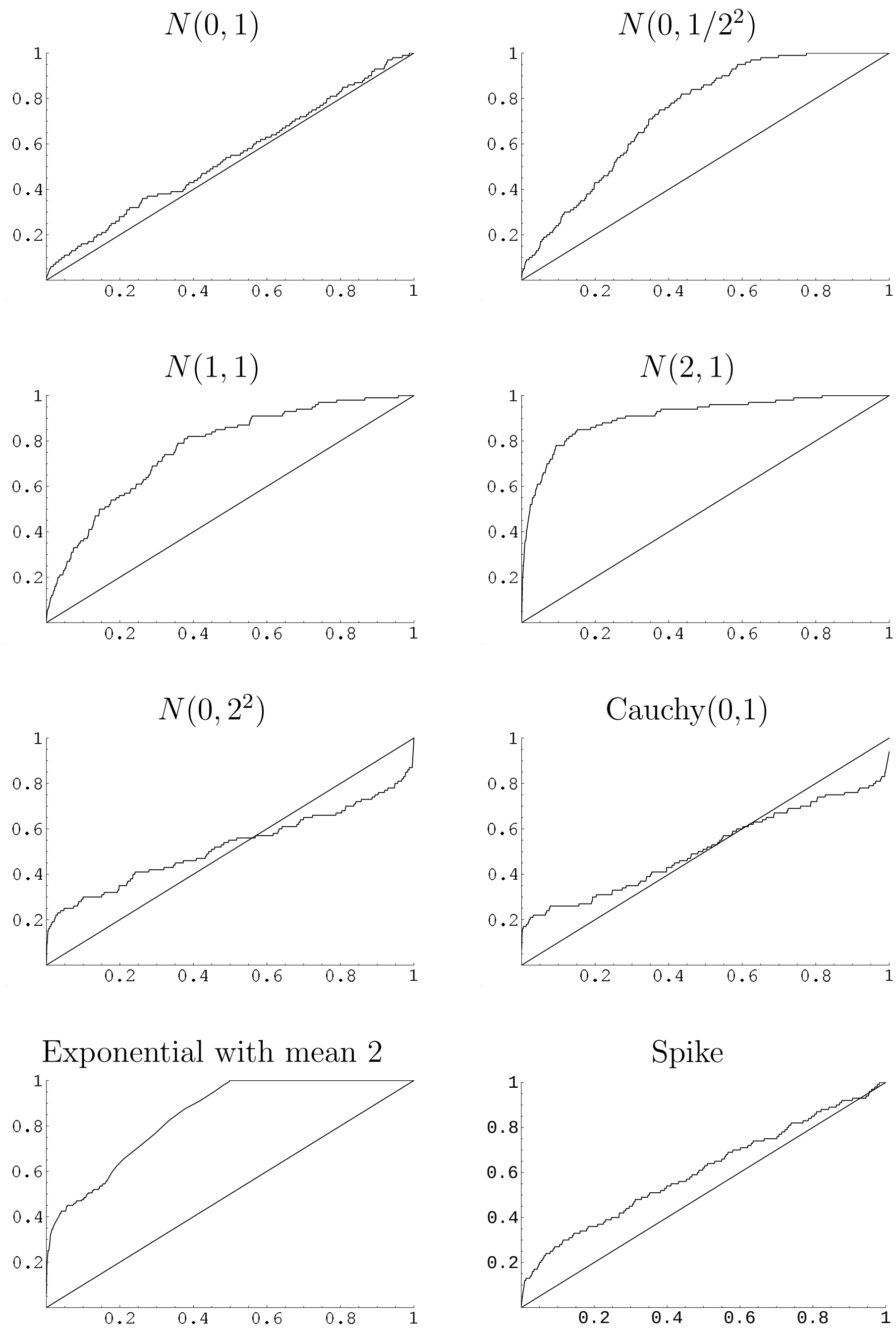

Figure 1: Generalized P-P plots m_n for 100 observations. For the first 7, P_0 is the standard normal distribution and for the last one it is the Uniform$(0,1)$ distribution. The distributions from which the samples are drawn are indicated in the plots; 'Spike' means that the distribution is based on g_2 of Section 4.

where $\Gamma_n(A) := \Gamma_n(v) - \Gamma_n(u-)$, $A = [u, v]$, is a uniform empirical process indexed by the class $\mathcal{A}_{[0,1]}$, that is the restriction of $\mathcal{A}$ to $[0, 1]$, and where $V(A)$ denotes the Lebesgue measure of A. Hence the process M_n, $n \geq 1$, is *distribution-free* under the null hypothesis. To formulate the limiting results for our processes, let C be the class of all continuous functions on $[0, 1]$, endowed with the supremum norm metric and let $\mathcal{C}$ be the Borel σ-algebra generated by the open sets from C. Similarly by $(D, \mathcal{D})$ denote the class of all right-continuous functions having left-hand limits at each point that are defined on $[0, 1]$, with the supremum norm metric and the σ-field generated by the open balls in D. It is easy to show that the process M_n takes values in D. Convergence in distribution of our processes will be meant to take place on $(D, \mathcal{D})$. Using the convergence in distribution of Γ_n and the Skorokhod construction, it is rather trivial to obtain the limiting distribution of M_n.

Theorem 1 *When $P = P_0$, we have as $n \to \infty$, that*

$$M_n \xrightarrow{d} M_0, \tag{1}$$

where

$$M_0(t) = \sup_{\substack{V(A)=t \\ A\in\mathcal{A}_{[0,1]}}} B(A) := \sup_{0\leq u\leq 1-t} (B(u + t) - B(u))$$

and B is a Brownian bridge, a mean zero Gaussian process with continuous sample paths on $[0, 1]$, and covariance $s \wedge t - st$, for $0 \leq s, t \leq 1$.

Hence by the continuous mapping theorem for any functional $\psi : D \to I\!\!R$, that is $(\mathcal{D}, \mathcal{B})$-measurable and continuous on C with respect to the supremum metric, we have that

$$\psi(M_n) \xrightarrow{d} \psi(M_0), \quad \text{as} \quad n \to \infty \tag{2}$$

(see, e.g., Shorack and Wellner [1986]). From (2) one obtains the limiting distributions for various statistics. Then, under H_0, for the test statistic defined above, we have

$$T_{n,c} \xrightarrow{d} \sup_{\substack{0\leq u<v\leq 1 \\ v-u\leq c}} (B(v) - B(u)), \quad \text{as} \quad n \to \infty. \tag{3}$$

We now show that the test based on $T_{n,c}$ is consistent. Write $\alpha_n = \sqrt{n}(P_n - P)$. Observe that when $P \neq P_0$, then there exists a $t_0 \in (0, c]$, such that for some $A^* \in \mathcal{A}$, $P_0(A^*) = t_0$ and $P(A^*) = t_0 + \varepsilon$, for some $\varepsilon > 0$. Then trivially for n large enough, almost surely,

$$\begin{aligned} M_n(t_0) = &\sup_{\substack{P_0(A)=t_0 \\ A\in\mathcal{A}}} \left(\alpha_n(A) + \sqrt{n}(P(A) - P_0(A))\right) \\ &\geq \alpha_n(A^*) + \sqrt{n}\varepsilon, \end{aligned}$$

and hence

$$T_{n,c} = \sup_{t\in[0,c]} M_n(t) \xrightarrow{I\!\!P} \infty, \quad n \to \infty.$$

For the intuitive understanding of $T_{n,c}$ we note that, when $c = 1$, it can be compared with the Kolmogorov-Smirnov (KS) and the Kuiper (K) statistic, since KS $\leq T_{n,1} \leq$ K, and as a consequence, the same relation holds for the limiting distributions under H_0 of KS, $T_{n,1}$, and K, which are respectively

$$\sup_{0\leq u\leq 1} |B(u)| \leq \sup_{0\leq u<v\leq 1} (B(v) - B(u)) \leq \sup_{0\leq u<v\leq 1} |B(v) - B(u)|.$$

2.2 Contiguous alternatives

Suppose that under the null hypothesis each $X_i, 1 \leq i \leq n$, has a known distribution P_0, with continuous df F_0, whereas under the alternative each $X_i, 1 \leq i \leq n$, has distribution $P^{(n)}$ defined by

$$\left(\frac{dP^{(n)}}{dP_0}(x)\right)^{1/2} = 1 + \frac{1}{2\sqrt{n}} h_n(x). \tag{4}$$

Here the functions h_n, $n \geq 1$, satisfy the following necessary and sufficient conditions for contiguity of the distribution of $(X_1, \ldots, X_n)$ under $P^{(n)}$ to the distribution under P_0:

(i) $$\overline{\lim_{n\to\infty}} \int_{I\!R} h_n^2(x) dP_0(x) < \infty,$$

(ii) $$n I\!P^{(n)}\left\{\frac{dP^{(n)}}{dP_0}(X_i) > K_n\right\} \to 0, \quad \text{for any sequence} \quad K_n \to \infty$$

(see, e.g., Oosterhoff and van Zwet [1979]), where $I\!P^{(n)}$ denotes the probability measure on $(\Omega, \mathcal{F})$, when $P = P^{(n)}$. Clearly $P^{(n)}$, $n \geq 1$, is absolutely continuous with respect to P_0. Note that for $h_n \equiv 0$, conditions (i) and (ii) remain true. Hence $P^{(n)}$ satisfying (4), (i) and (ii), includes the null hypothesis. Therefore when dealing with the testing procedures, we throughout assume that under the alternative $h_n \not\equiv 0$. Let us also introduce the notation

$$H_n(A) := \int_A h_n(x) dP_0(x)$$

and

$$\left\|h_n\right\|_A := \left[\int_A h_n^2(x) dP_0(x)\right]^{\frac{1}{2}}, \quad A \in \mathcal{A} \cup \{I\!R\}.$$

The functions H_n are often called shift functions. Also, write d_0 for the pseudo-metric on $\mathcal{B}$, defined by

$$d_0(B_1, B_2) = P_0(B_1 \triangle B_2), \quad \text{for } B_1, B_2 \in \mathcal{B}.$$

For convenient presentation Theorem 2, Corollary 2, and Theorem 4 are presented in an *approximation* setting (with the D-valued random elements involved, defined on *one* probability space), via the Skorokhod construction. So the random elements (like M_n) in these results are only equal in distribution to the original ones, but we do not add the usual tildes to the notation.

Theorem 2 *When (i) and (ii) hold, we have that*

$$(5) \qquad M_n - M_{1n} \xrightarrow{d} 0 \quad as \quad n \to \infty,$$

where $M_{1n}(t) = \sup\left\{B_{P_0}(A) + H_n(A) : P_0(A) = t,\ A \in \mathcal{A}\right\}$ *and* B_{P_0} *is a* P_0*-Brownian bridge: i.e., a bounded, mean zero Gaussian process, uniformly continuous on* $(\mathcal{A}, d_0)$*, with covariance* $P_0(A_1 \cap A_2) - P_0(A_1)P_0(A_2)$*,* $A_1, A_2 \in \mathcal{A}$*.*

Note that, by choosing $h_n \equiv 0$, Theorem 2 implies Theorem 1.

In the literature often a stronger condition than (i) and (ii) is considered: there exists a function h such that

$$\text{(iii)} \qquad 0 < \int_{\mathbb{R}} h^2(x)dP_0(x) < \infty \quad \text{and} \quad \int_{\mathbb{R}} (h_n(x) - h(x))^2 dP_0(x) \to 0 \quad \text{as } n \to \infty.$$

It is easy to see that condition (iii) implies (i) and (ii) and hence the following corollary to Theorem 2 holds true.

Corollary 1 *When condition (iii) holds, we have that*

$$(6) \qquad M_n \xrightarrow{d} M \quad as \quad n \to \infty,$$

where $M(t) = \sup\left\{B_{P_0}(A) + H(A) : P_0(A) = t, A \in \mathcal{A}\right\}$*, with* $H(A) := \int_A h(x)dP_0(x)$*.*

From this we immediately obtain

$$(7) \qquad T_{n,c} \xrightarrow{d} \sup_{t \in [0,c]} M(t) \quad \text{as} \quad n \to \infty.$$

In the second corollary to Theorem 2 we deal with the case of random sample sizes, which occurs often in practice, and includes the Poisson process situation. Let N_n, $n \geq 1$, be a sequence of random variables, taking values in $\mathbb{N}$. Suppose also that the N_n, $n \geq 1$, are independent of $X_1, X_2, \ldots$, and that

$$N_n \xrightarrow{\mathbb{P}} \infty \quad \text{as} \quad n \to \infty.$$

Let $X_1, \ldots, X_{N_n}$ be our data.

Corollary 2 *Suppose conditions (i) and (ii) hold, then*

$$M_{N_n} - M_{1N_n} \xrightarrow{d} 0 \quad as \quad n \to \infty,$$

where $M_{N_n}(t) := \sqrt{N_n}\big(\sup\{P_{N_n}(A) : P_0(A) \leq t,\ A \in \mathcal{A}\} - t\big)$*,* $t \in [0,1]$*, and* $M_{1N_n}(t) := \sup\left\{B_{P_0}(A) + H_{N_n}(A) : P_0(A) = t,\ A \in \mathcal{A}\right\}$*,* $t \in [0,1]$*.*

In the following remarks we discuss and compare our generalized P-P plots and corresponding tests.

Remark 1 *Scan statistics.* Generally, the scan statistic (see Glaz et al. [2001]) is defined in terms of scanning with a window of one fixed length. Since the scan statistic searches for the maximum mass it can be used for testing for uniformity (see, e.g., Dijkstra et al. [1984]). The test statistic $T_{n,c}$ is an analogue of the scan statistic, though the length of its scanning window varies and this makes it possible to detect clusters of small, unknown size.

Remark 2 *Chimeric alternatives.* In Khmaladze [1998], goodness-of-fit problems are studied for so-called chimeric, contiguous alternatives. The nature of these alternatives is that they can not be detected unless the window of the test statistic is in agreement with their range and convergence rate. In principle, our procedure can be adapted to deal with chimeric alternatives, but our test statistics as they stand are not suitable for dealing with these alternatives, since they essentially deal with *fixed-length* intervals of various lengths, but not depending on n.

Remark 3 On $\mathbb{R}^k$, $k \geq 2$, we could define the generalized P-P plot and the generalized P-P plot process as above, based on an indexing class $\mathcal{G}$. When $\mathcal{G}$ is P_0-Donsker we will have that Theorem 1 remains true. Hence, under H_0,

$$M_n \xrightarrow{d} \sup_{\substack{P_0(G)=t \\ G\in\mathcal{G}}} B_{P_0}(G).$$

Clearly M_n is asymptotically distribution-free when $\mathcal{G}$ is the class of level sets of the density corresponding to P_0. However this class is not large enough in the sense that the tests do not have good power properties: certain contiguous alternatives satisfying condition (iii) will lead to the same limiting distribution for M_n as the one under H_0. Taking a substantially larger class $\mathcal{G}$ can improve the power of the tests and lead to tests which have similar power properties as in the one-dimensional case, but then the (asymptotic) distribution-freeness under H_0 will be lost.

3 Two-sample problem

In this section we consider the two-sample problem, i.e., we define a generalized P-P plot and corresponding testing procedure for comparing two independent random samples. From a statistical point of view this section is maybe more important than the previous one, since the two-sample problem occurs more often in practice, than testing goodness-of-fit with a simple null hypothesis.

Let $X_{11}, X_{12}, \ldots$, and $X_{21}, X_{22}, \ldots$, be two independent sequences of i.i.d. one-dimensional random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, from unknown probability measures P_1 and P_2, respectively. Let $\mathcal{B}, \mathcal{A}$ be as in the previous section and let P_{jn_j} denote the empirical distribution of the samples $X_{j1}, \ldots, X_{jn_j}$, $j = 1, 2$. Define the generalized P-P plot as follows

$$m_{n_1 n_2}(t) := \sup\{P_{1n_1}(A) \, : \, P_{2n_2}(A) \leq t, \, A \in \mathcal{A}\}, \quad t \in [0,1].$$

(Observe that when $\mathcal{A}$ would be $\{(-\infty, y] \, : \, y \in I\!R\}$, here as well we would get the classical P-P plot.) Then for each $t \in [0,1]$ and $n_1, n_2 \geq 1$, with $n = n_1 + n_2$, define the generalized P-P plot process as

$$M_{n_1 n_2}(t) := \sqrt{\frac{n_1 n_2}{n}} \Big(m_{n_1 n_2}(t) - t \Big). \tag{8}$$

Note that the generalized P-P plot and consequently the generalized P-P plot process are not symmetrical with respect to interchanging the samples. This can be exploited when deciding which distribution is P_1 and which P_2. We now study the two-sample problem using the generalized P-P plot process $M_{n_1 n_2}$.

3.1 Null hypothesis and fixed alternatives

Consider $H_0 : P_1 = P_2$ against $H_1 : P_1 \neq P_2$, where P_1, P_2 have continuous df's. It is easy to show that for fixed $n_1, n_2 \geq 1$, $M_{n_1 n_2}$ is distribution-free under H_0. The following result provides the limiting distribution of the generalized P-P plot process. Indeed, we have the same limit as in the one-sample case. Let $n = n_1 + n_2$, such that $n_1 = n_1(n)$ and $n_1 \to \infty$ if $n \to \infty$, and $n_2 = n_2(n)$ and $n_2 \to \infty$ if $n \to \infty$.

Theorem 3 *When $P_1 = P_2$ we have as $n \to \infty$, that*

$$M_{n_1 n_2} \xrightarrow{d} M_0 \; . \tag{9}$$

Define the test statistics $T_{n_1 n_2, c} := \sup_{t \in [0,c]} M_{n_1 n_2}(t)$. Then trivially, under H_0,

$$T_{n_1 n_2, c} \xrightarrow{d} \sup_{\substack{A \in \mathcal{A}_{[0,1]} \\ V(A) \leq c}} B(A).$$

It can also be shown, similarly as for the one-sample case, that the test based on $T_{n_1 n_2, c}$ is consistent.

3.2 Contiguous alternatives

Suppose that $P_1^{(n)}$ and $P_2^{(n)}$ are the distributions of X_{1i}, $1 \leq i \leq n_1$, and X_{2i}, $1 \leq i \leq n_2$, respectively, and that under the null hypothesis $P_1^{(n)} = P_2^{(n)} = P_0$, where P_0 is a given probability measure, with continuous df F_0. Under the alternative $P_1^{(n)} \neq P_2^{(n)}$, we have

$$(10)\qquad \left(\frac{dP_1^{(n)}}{dP_0}(x)\right)^{\frac{1}{2}} = 1 + \frac{1}{2\sqrt{n_1}} h_{1n}(x),$$

$$(11)\qquad \left(\frac{dP_2^{(n)}}{dP_0}(x)\right)^{\frac{1}{2}} = 1 + \frac{1}{2\sqrt{n_2}} h_{2n}(x).$$

We assume, similarly as for the one-sample problem, the following necessary and sufficient conditions for contiguity of the distribution of the samples under $P_1^{(n)}$ and $P_2^{(n)}$ to the distribution under P_0:

$$\text{(iv)}\qquad \overline{\lim_{n\to\infty}} \int_{I\!R} h_{jn}^2(x) dP_0(x) < \infty, \quad \text{for} \;\; j = 1, 2,$$

$$\text{(v)}\qquad n_j I\!P^{(n)} \left\{ \frac{dP_j^{(n)}}{dP_0}(X_{ji}) > K_n \right\} \to 0, \;\text{for}\; j = 1, 2 \text{ and any sequence } K_n \to \infty,$$

where $I\!P^{(n)}$ denotes the probability measure on $(\Omega, \mathcal{F})$, when X_{ji} is distributed according to $P_j^{(n)}$, $j = 1, 2$. For each $n \geq 1$, define the sequence of shift functions

$$(12)\qquad H_{n_1 n_2}(A) := \sqrt{\frac{n_2}{n}} \int_A h_{1n}(x) dP_0(x) - \sqrt{\frac{n_1}{n}} \int_A h_{2n}(x) dP_0(x), \quad A \in \mathcal{A}.$$

Our interest lies in obtaining the limiting distribution of $M_{n_1 n_2}$ under these alternatives. With substantially more effort we will establish the analogue of Theorem 2. Again the result is presented in an approximation setting.

Theorem 4 *Assume that the probability measures $P_1^{(n)}$ and $P_2^{(n)}$ defined by (10) and (11) satisfy conditions (iv) and (v), then*

$$(13)\qquad M_{n_1 n_2} - M_{12n} \xrightarrow{d} 0 \quad as \quad n \to \infty,$$

where for each $n_1, n_2 \geq 1$, $M_{12n}(t) := \sup\{B_{P_0}^{(n)}(A) + H_{n_1 n_2}(A) : P_0(A) = t, \; A \in \mathcal{A}\}$ and the P_0- Brownian bridge $B_{P_0}^{(n)}(A) := \sqrt{\frac{n_2}{n}} B_{1P_0}(A) - \sqrt{\frac{n_1}{n}} B_{2P_0}(A)$, $A \in \mathcal{A}$, with B_{1P_0} and B_{2P_0} two independent P_0-Brownian bridges.

Clearly Theorem 4 implies Theorem 3. It also easily yields the following result.

Corollary 3 *Assume for some function $H : \mathcal{A} \to I\!R$,*

$$(14)\qquad \sup_{A \in \mathcal{A}} |H_{n_1 n_2}(A) - H(A)| \to 0 \quad as \quad n \to \infty,$$

then (with M as in Corollary 1)

$$(15)\qquad M_{n_1 n_2} \xrightarrow{d} M \quad as \quad n \to \infty.$$

Note that condition (14) is satisfied if $n_1/n \to p \in [0,1]$, and for functions h_1 and h_2

$$0 < \int_{I\!R} h_j^2(x)dP_0(x) < \infty \quad \text{and} \quad \int_{I\!R} (h_{jn}(x) - h_j(x))^2 dP_0(x) \to 0 \quad \text{as } n \to \infty \ (j = 1, 2).$$

4 Simulation study

In this section we present simulation results in order to study the small sample behaviour of our tests statistics. First we give a brief description of an algorithm for computing the test statistic $T_{n,c}$, of Section 3. Rewrite m_n as

$$m_n(t) = \sup_{\substack{v-u=t \\ 0 \le u < v \le 1}} \overline{P}_n([u,v]),$$

where $\overline{P}_n([u,v])$ is the empirical measure of the interval $[u,v]$, based on the transformed sample $F_0(X_1), \dots, F_0(X_n)$. It is easy to see that m_n is a right-continuous step-function taking values $1/n, 2/n, \dots, 1$. Since each observation can be covered by a closed interval of length 0,

$$m_n(t) = 1/n \quad \text{for} \quad 0 \le t < \min_{1 \le i \le n-1} \{Y_{(i+1)} - Y_{(i)}\},$$

where the $Y_{(i)}$ are the order statistics of $Y_i = F_0(X_i)$, $1 \le i \le n$. Similarly, for $0 \le k \le n-1$,

$$m_n(t) = \frac{k+1}{n}, \quad W_k \le t < W_{k+1},$$

with $W_0 = 0$ and where $W_k = \min_{1 \le i \le n-k} \{Y_{(i+k)} - Y_{(i)}\}$, $1 \le k \le n-1$, are the jump points of m_n. Now computing $T_{n,c}$ is trivial. Each simulation below consists of 10,000 replications.

In Table 1 the simulated critical values, corresponding to $\alpha = 0.05$, for the test statistic $T_{n,c}$, for $c = 0.05$ and $c = 1$, respectively, and for the Kolmogorov-Smirnov and Kuiper statistics are given. We indeed see, as observed in Section 2.1, that the critical values of $T_{n,1}$ are between those of KS and K.

n	10	20	50	100	300	500	∞
$T_{n,0.05}$	1.13	0.98	0.95	0.91	0.87	0.86	0.82
$T_{n,1}$	1.58	1.60	1.59	1.60	1.62	1.63	1.64
KS	1.29	1.31	1.34	1.34	1.35	1.35	1.36
K	1.62	1.66	1.69	1.71	1.72	1.73	1.75

Table 1: Critical values for $T_{n,0.05}$, $T_{n,1}$, KS and K, for $\alpha = 0.05$.

In Table 2 simulated powers of $T_{n,c}$ are presented for the following testing problems:

(a) alternative density $f(x) = \frac{1}{2\sqrt{x}}$, $x \in [0,1]$, against null distribution Uniform$(0,1)$;

(b) alternative Normal(1,1) against null distribution Normal(0,1);

(c) alternative Beta(2,1) against null distribution Normal($\frac{2}{3}, (3\sqrt{2})^{-2}$).

Note that in case (c) the parameters of the Normal distribution have the same mean and variance as the Beta(2,1)-distribution. As mentioned before the test statistics $T_{n,c}$ resemble the scan statistic somewhat and hence can be used for testing uniformity against spike alternatives (case (a)). Indeed, Table 2 shows that the tests have high power for case (a). In addition, the shift of case (b) and the - difficult - shape change of case (c) are detected (very) well.

case	(a)				(b)				(c)			
n	10	20	50	100	10	20	50	100	10	20	50	100
$T_{n,0.05}$	.20	.39	.82	.99	.36	.68	.97	1.00	.10	.16	.37	.68
$T_{n,1}$	.19	.40	.87	1.00	.54	.90	1.00	1.00	.09	.15	.42	.76

Table 2: Power of $T_{n,0.05}$ and $T_{n,1}$ for fixed alternatives.

Now we consider contiguous alternatives. Consider three examples of the function $g = h_n + \frac{h_n^2}{4\sqrt{n}}$ (see (4)):

(d) $g_1(x) = -I_{[0,\frac{1}{2})}(x) + 9\, I_{[\frac{1}{2},\frac{3}{5}]}(x) - I_{(\frac{3}{5},1]}(x)$, for $x \in [0,1]$;

(e) $g_2(x) = -I_{[0,\frac{1}{2})}(x) + 99\, I_{[\frac{1}{2},\frac{51}{100}]}(x) - I_{(\frac{51}{100},1]}(x)$, for $x \in [0,1]$;

(f) $g_3(x) = -2\, I_{[0,\frac{1}{2}]}(x) + 2\, I_{(\frac{1}{2},1]}(x)$, for $x \in [0,1]$.

In Table 3 simulated powers when testing uniformity against these contiguous alternatives are presented for $T_{n,c}$, KS and K. They show that often our test statistics are outperforming KS and K. In case of an extreme spike $T_{n,0.05}$ is better than $T_{n,1}$ and K and KS perform much worse. For a more moderate spike $T_{n,0.05}$ and $T_{n,1}$ give almost the same results, whereas KS and K again perform worse. For a standard-type contiguous alternative $T_{n,1}$, KS and K behave similarly, although KS is slightly better. $T_{n,0.05}$ which only looks at small intervals, does worse here.

In summary our test statistics behave very well to excellent. In particular, when some indication of a spike-type alternative is available, our test procedures clearly outperform competing procedures.

case	(d)				(e)				(f)			
n	10	20	50	100	10	20	50	100	10	20	50	100
$T_{n,0.05}$	.31	.32	.35	.36	.54	.67	.75	.82	.13	.13	.12	.12
$T_{n,1}$	.31	.32	.37	.38	.35	.38	.46	.47	.37	.36	.39	.38
KS	.10	.13	.14	.15	.08	.15	.17	.18	.47	.45	.44	.43
K	.28	.28	.29	.28	.32	.34	.36	.35	.33	.34	.33	.33

Table 3: Power of $T_{n,0.05}$, $T_{n,1}$, KS and K for contiguous alternatives.

5 Proofs

Proof of Theorem 2 Since $\Gamma_n \xrightarrow{d} B$, a Skorokhod construction yields the existence of a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{I\!P})$, carrying $\widetilde{\Gamma}_1, \widetilde{\Gamma}_2, \ldots$ and $\widetilde{B}$, with

$$\mathcal{L}(\widetilde{B}) = \mathcal{L}(B), \quad \mathcal{L}(\widetilde{\Gamma}_n) = \mathcal{L}(\Gamma_n), \quad \text{for } n \geq 1, \tag{16}$$
$$\text{and}$$
$$\sup_{t\in[0,1]} |\widetilde{\Gamma}_n(t) - \widetilde{B}(t)| \to 0 \text{ a.s.}, \quad \text{as } n \to \infty.$$

Let $F^{(n)}$ denote the distribution function corresponding to $P^{(n)}$. Define processes $\widetilde{\alpha}_n$, $n \geq 1$, $\widetilde{B}_{P^{(n)}}$, $n \geq 1$, and $\widetilde{B}_{P_0}$, all indexed by the class $\mathcal{A}$, by

$$\begin{aligned}
\widetilde{\alpha}_n(A) &:= \widetilde{\Gamma}_n(F^{(n)}(y)) - \widetilde{\Gamma}_n(F^{(n)}(x-)),\\
\widetilde{B}_{P^{(n)}}(A) &:= \widetilde{B}(F^{(n)}(y)) - \widetilde{B}(F^{(n)}(x)),\\
\widetilde{B}_{P_0}(A) &:= \widetilde{B}(F_0(y)) - \widetilde{B}(F_0(x)), \quad \text{for } A = [x, y] \in \mathcal{A}.
\end{aligned}$$

We have that $\widetilde{\alpha}_n \stackrel{d}{=} \alpha_n$ and that $\widetilde{B}_{P_0}$ and $\widetilde{B}_{P^{(n)}}$ are P_0- and $P^{(n)}$-Brownian bridges, indexed by $\mathcal{A}$. Note that, since $P^{(n)}$ is absolutely continuous with respect to P_0, the process $\widetilde{B}_{P^{(n)}}$ will be uniformly continuous with respect to d_0 on $\mathcal{A}$. Then (16) implies that

$$\sup_{A\in\mathcal{A}} |\widetilde{\alpha}_n(A) - \widetilde{B}_{P^{(n)}}(A)| \to 0 \text{ a.s.}, \quad n \to \infty. \tag{17}$$

Henceforth, for convenience, we will drop the tildes from the notation.

We have, for $A \in \mathcal{A}$,

$$\begin{aligned}
P^{(n)}(A) &= P_0(A) + \frac{1}{\sqrt{n}} \int_A h_n(x) dP_0(x) + \frac{1}{4n} \int_A h_n^2(x) dP_0(x)\\
&= P_0(A) + \frac{1}{\sqrt{n}} H_n(A) + \frac{1}{4n} \|h_n\|_A^2.
\end{aligned}$$

By the continuity of F_0

$$\begin{aligned}M_n(t) &= \sqrt{n}\sup\{P_n(A) - t \;:\; P_0(A) = t,\; A \in \mathcal{A}\}\\ &= \sup\Big\{\sqrt{n}\big(P_n(A) - P^{(n)}(A)\big) + \sqrt{n}\big(P^{(n)}(A) - P_0(A)\big) : P_0(A) = t,\; A \in \mathcal{A}\Big\}\\ &= \sup\Big\{\alpha_n(A) + H_n(A) + \frac{1}{4\sqrt{n}}\big\|h_n\big\|_A^2 \;:\; P_0(A) = t,\; A \in \mathcal{A}\Big\},\end{aligned}$$

which yields

$$\begin{aligned}&\sup_{t\in[0,1]}\Big|M_n(t) - \sup_{\substack{P_0(A)=t\\ A\in\mathcal{A}}}\big(B_{P_0}(A) + H_n(A)\big)\Big|\\ &= \sup_{t\in[0,1]}\Big|\sup_{\substack{P_0(A)=t\\ A\in\mathcal{A}}}\big(\alpha_n(A) + H_n(A) + \frac{1}{4\sqrt{n}}\|h_n\|_A^2\big) - \sup_{\substack{P_0(A)=t\\ A\in\mathcal{A}}}\big(B_{P_0}(A) + H_n(A)\big)\Big|\\ &\le \sup_{t\in[0,1]}\Bigg[\sup_{\substack{P_0(A)=t\\ A\in\mathcal{A}}}\big|\alpha_n(A) - B_{P^{(n)}}(A)\big| + \sup_{\substack{P_0(A)=t\\ A\in\mathcal{A}}}\big|B_{P^{(n)}}(A) - B_{P_0}(A)\big| + \sup_{\substack{P_0(A)=t\\ A\in\mathcal{A}}}\frac{1}{4\sqrt{n}}\big\|h_n\big\|_A^2\Bigg]\end{aligned}$$

$$\le \sup_{A\in\mathcal{A}}\big|\alpha_n(A) - B_{P^{(n)}}(A)\big| + \sup_{A\in\mathcal{A}}\big|B_{P^{(n)}}(A) - B_{P_0}(A)\big| + \frac{1}{4\sqrt{n}}\big\|h_n\big\|_{I\!R}^2\,. \tag{18}$$

To complete our proof we will show that each term in (18) converges to zero, almost surely. By (17) and condition (i) it remains to show that

$$\sup_{A\in\mathcal{A}}|B_{P^{(n)}}(A) - B_{P_0}(A)| \to 0 \;\text{ a.s.},\quad n\to\infty. \tag{19}$$

By the uniform continuity of B this follows from

$$\sup_{A\in\mathcal{A}}|P^{(n)}(A) - P_0(A)| \to 0 \qquad n\to\infty.$$

However, this is equivalent to

$$\sup_{A\in\mathcal{A}}\Big|\frac{1}{\sqrt{n}}H_n(A) + \frac{1}{4n}\big\|h_n\big\|_A^2\Big| \to 0 \qquad n\to\infty. \tag{20}$$

Observe that

$$\begin{aligned}&\sup_{A\in\mathcal{A}}\Big|\frac{1}{\sqrt{n}}\int_A h_n(x)dP_0(x) + \frac{1}{4n}\int_A h_n^2(x)dP_0(x)\Big|\\ &\qquad\le \frac{1}{\sqrt{n}}\int_{I\!R}|h_n(x)|dP_0(x) + \frac{1}{4n}\int_{I\!R}h_n^2(x)dP_0(x)\\ &\qquad\le \frac{1}{\sqrt{n}}\sqrt{\int_{I\!R}h_n^2(x)dP_0(x)} + \frac{1}{4n}\int_{I\!R}h_n^2(x)dP_0(x),\end{aligned}$$

by the Cauchy-Schwarz inequality. Now (20), and hence (19), follows from (i). $\square$

Proof of Theorem 4 Consider two independent samples $U_{j1}, \dots, U_{jn_j}$, $n_j \geq 1$, for $j = 1, 2$, of i.i.d. uniform random variables defined on some probability space $(\Omega', \mathcal{F}', I\!\!P')$ with values in $[0, 1]$. Let Γ_{jn_j} be the uniform empirical process based on $U_{j1}, \dots, U_{jn_j}$, $n_j \geq 1$, for $j = 1, 2$. The process Γ_{jn_j} converges in distribution to a Brownian bridge B_j on $(D, \mathcal{D})$ and B_1 and B_2 are independent. Then by a Skorokhod construction there exists a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{I\!\!P})$ carrying, for $j = 1, 2$, processes $\widetilde{\Gamma}_{j1}, \widetilde{\Gamma}_{j2}, \dots$ on $(D, \mathcal{D})$, with $\{\widetilde{\Gamma}_{1n}\}_{n \in I\!\!N}$ and $\{\widetilde{\Gamma}_{2n}\}_{n \in I\!\!N}$ independent, and independent processes $\widetilde{B}_j$ on $(C, \mathcal{C})$ such that

$$\widetilde{B}_j \stackrel{d}{=} B_j, \ \widetilde{\Gamma}_{jn_j} \stackrel{d}{=} \Gamma_{jn_j}, \quad n_j \geq 1,$$

and

$$\sup_{t \in [0,1]} |\widetilde{\Gamma}_{jn_j}(t) - \widetilde{B}_j(t)| \to 0 \ \text{ a.s.}, \quad n \to \infty. \tag{21}$$

For $j = 1, 2$, define the processes $\widetilde{\alpha}_{jn_j}$, $n_j \geq 1$, $\widetilde{B}_{jP_j}$, $n_j \geq 1$, and $\widetilde{B}_{jP_0}$ indexed with the class $\mathcal{A}$ by

$$\begin{aligned}
\widetilde{\alpha}_{jn_j}(A) &:= \widetilde{\Gamma}_{jn_j}(F_j^{(n)}(y)) - \widetilde{\Gamma}_{jn_j}(F_j^{(n)}(x-)), \\
\widetilde{B}_{jP_j}(A) &:= \widetilde{B}_j(F_j^{(n)}(y)) - \widetilde{B}_j(F_j^{(n)}(x)), \\
\widetilde{B}_{jP_0}(A) &:= \widetilde{B}_j(F_0(y)) - \widetilde{B}_j(F_0(x)), \quad A = [x, y] \in \mathcal{A},
\end{aligned}$$

where $F_j^{(n)}$ is the distribution function corresponding to $P_j^{(n)}$, for $j = 1, 2$. Note that

$$\widetilde{\alpha}_{jn_j}(A) = \sqrt{n_j}\big(\widetilde{P}_{jn_j}(A) - P_j^{(n)}(A)\big), \ \text{for } j = 1, 2, \tag{22}$$

where $\widetilde{P}_{jn_j}$ is the empirical measure of the random variables $(F_j^{(n)})^{-1}(\widetilde{U}_{ji}), i = 1, \dots, n_j$. Then (21) yields

$$\sup_{A \in \mathcal{A}} \big|\widetilde{\alpha}_{jn_j}(A) - \widetilde{B}_{jP_j}(A)\big| \to 0 \ \text{ a.s.}, \quad n \to \infty, \quad j = 1, 2. \tag{23}$$

The processes $\widetilde{B}_{jP_j}$ and $\widetilde{B}_{jP_0}$ are $P_j^{(n)}$- and P_0-Brownian bridges, respectively, and $\widetilde{B}_{1P_1}$ and $\widetilde{B}_{2P_2}$ are independent as are $\widetilde{B}_{1P_0}$ and $\widetilde{B}_{2P_0}$. Observe that for all $n_1, n_2 \geq 1$, the process

$$\widetilde{B}_{P_0}^{(n)}(A) := \sqrt{\frac{n_2}{n}}\widetilde{B}_{1P_0}(A) - \sqrt{\frac{n_1}{n}}\widetilde{B}_{2P_0}(A), \quad A \in \mathcal{A}, \tag{24}$$

is a P_0-Brownian bridge. From now on we will drop the tildes, for notational convenience.

By (10) and (11) we obtain that

$$\begin{aligned}
&\sqrt{\frac{n_1 n_2}{n}}\big(P_1^{(n)}(A) - P_2^{(n)}(A)\big) \\
&\qquad = H_{n_1 n_2}(A) + \sqrt{\frac{n_2}{16 n_1 n}}\big\|h_{1n}\big\|_A^2 - \sqrt{\frac{n_1}{16 n_2 n}}\big\|h_{2n}\big\|_A^2.
\end{aligned} \tag{25}$$

Rewrite $M_{n_1n_2}$ as follows,

$$M_{n_1n_2}(t) \stackrel{\text{a.s.}}{=} \sqrt{\frac{n_1n_2}{n}} \sup\Big\{ \big(P_{1n_1}(A) - t\big) \ : \ P_{2n_2}(A) = \frac{\lfloor n_2 t\rfloor}{n_2},\ A \in \mathcal{A}\Big\}.$$

Using (25) and (22) we obtain that

$$\begin{aligned} M_{n_1n_2}(t) &\stackrel{\text{a.s.}}{=} \sup_{\substack{P_{2n_2}(A)=\bar t\\ A\in\mathcal{A}}} \Bigg[\sqrt{\frac{n_2}{n}}\alpha_{1n_1}(A) - \sqrt{\frac{n_1}{n}}\alpha_{2n_2}(A) + H_{n_1n_2}(A) \\ &+ \sqrt{\frac{n_2}{16n_1n}}\big\|h_{1n}\big\|_A^2 - \sqrt{\frac{n_1}{16n_2n}}\big\|h_{2n}\big\|_A^2 + \sqrt{\frac{n_1n_2}{n}}\big(\bar t - t\big)\Bigg], \end{aligned} \tag{26}$$

with $\bar t := \frac{\lfloor n_2 t\rfloor}{n_2}$. Set

$$W_t^{(n_1n_2)}(A) := \sqrt{\frac{n_2}{16n_1n}}\big\|h_{1n}\big\|_A^2 - \sqrt{\frac{n_1}{16n_2n}}\big\|h_{2n}\big\|_A^2 + \sqrt{\frac{n_1n_2}{n}}\big(\bar t - t\big).$$

Then (26) implies that, almost surely,

$$\begin{aligned} &\sup_{t\in[0,1]} \Bigg| M_{n_1n_2}(t) - \sup_{\substack{P_0(A)=t\\ A\in\mathcal{A}}} \big(B_{P_0}^{(n)}(A) + H_{n_1n_2}(A)\big)\Bigg| \\ &= \sup_{t\in[0,1]} \Bigg| \sup_{\substack{P_{2n_2}(A)=\bar t\\ A\in\mathcal{A}}} \Bigg[\sqrt{\frac{n_2}{n}}\alpha_{1n_1}(A) - \sqrt{\frac{n_1}{n}}\alpha_{2n_2}(A) + H_{n_1n_2}(A) + W_t^{(n_1n_2)}(A)\Bigg] \\ &\qquad - \sup_{\substack{P_0(A)=t\\ A\in\mathcal{A}}} \big(B_{P_0}^{(n)}(A) + H_{n_1n_2}(A)\big)\Bigg|. \end{aligned} \tag{27}$$

Write $\mathcal{A}_{0t} := \{A \in \mathcal{A} \ : \ P_0(A) = t\}$, $\mathcal{A}_{2t}^{(n)} := \{A \in \mathcal{A} \ : \ P_{2n_2}(A) = \bar t\}$ and consider first

$$\begin{aligned} \sup_{t\in[0,1]} \Bigg\{ &\sup_{A\in\mathcal{A}_{2t}^{(n)}} \Bigg[\sqrt{\frac{n_2}{n}}\alpha_{1n_1}(A) - \sqrt{\frac{n_1}{n}}\alpha_{2n_2}(A) + H_{n_1n_2}(A) + W_t^{(n_1n_2)}(A)\Bigg] \\ &- \sup_{A\in\mathcal{A}_{0t}} \big(B_{P_0}^{(n)}(A) + H_{n_1n_2}(A)\big)\Bigg\}, \end{aligned}$$

which, by (24), is equal to

$$\begin{aligned} &\sup_{t\in[0,1]} \sup_{A_2\in\mathcal{A}_{2t}^{(n)}} \inf_{A_0\in\mathcal{A}_{0t}} \Bigg\{ \sqrt{\frac{n_2}{n}}\big(\alpha_{1n_1}(A_2) - B_{1P_1}(A_2)\big) \\ &+ \sqrt{\frac{n_1}{n}}\big(B_{2P_2}(A_2) - \alpha_{2n_2}(A_2)\big) + \sqrt{\frac{n_2}{n}}\big(B_{1P_1}(A_2) - B_{1P_0}(A_2)\big) \\ &+ \sqrt{\frac{n_1}{n}}\big(B_{2P_0}(A_2) - B_{2P_2}(A_2)\big) + \big(B_{P_0}^{(n)}(A_2) + H_{n_1n_2}(A_2)\big) \\ &+ W_t^{(n_1n_2)}(A_2) - \big(B_{P_0}^{(n)}(A_0) + H_{n_1n_2}(A_0)\big)\Bigg\}. \end{aligned} \tag{28}$$

Observe that this is bounded from above by

(29)

$$\sup_{A\in\mathcal{A}}\sqrt{\frac{n_2}{n}}\Big|\alpha_{1n_1}(A)-B_{1P_1}(A)\Big|+\sup_{A\in\mathcal{A}}\sqrt{\frac{n_1}{n}}\Big|B_{2P_2}(A)-\alpha_{2n_2}(A)\Big|$$
$$+\sup_{A\in\mathcal{A}}\sqrt{\frac{n_2}{n}}\Big|B_{1P_1}(A)-B_{1P_0}(A)\Big|+\sup_{A\in\mathcal{A}}\sqrt{\frac{n_1}{n}}\Big|B_{2P_2}(A)-B_{2P_0}(A)\Big|+\sup_{A\in\mathcal{A}}\Big|W_t^{(n_1n_2)}(A)\Big|$$
$$+\sup_{t\in[0,1]}\left|\sup_{A\in\mathcal{A}_{2t}^{(n)}}\Big(B_{P_0}^{(n)}(A)+H_{n_1n_2}(A)\Big)-\sup_{A\in\mathcal{A}_{0t}}\Big(B_{P_0}^{(n)}(A)+H_{n_1n_2}(A)\Big)\right|.$$

Similarly it can be shown that the 'negative' part of the absolute value in (27) is also bounded by the expression in (29). Using similar arguments as for (19) and the uniform continuity of B_{jP_j} and B_{jP_0}, respectively, for $j=1,2$, we obtain

$$\sup_{A\in\mathcal{A}}|B_{jP_j}(A)-B_{jP_0}(A)|\to 0 \ \text{ a.s.}, \quad n\to\infty, \quad \text{for} \ \ j=1,2. \tag{30}$$

Hence by (23) and condition (iv) it remains to show that almost surely, as $n\to\infty$,

$$\sup_{t\in[0,1]}\left|\sup_{A\in\mathcal{A}_{2t}^{(n)}}\Big(B_{P_0}^{(n)}(A)+H_{n_1n_2}(A)\Big)-\sup_{A\in\mathcal{A}_{0t}}\Big(B_{P_0}^{(n)}(A)+H_{n_1n_2}(A)\Big)\right|\to 0. \tag{31}$$

On the other hand, since $\{B_{P_0}^{(n)}+H_{n_1n_2}\}_{n\in\mathbb{N}}$ is d_0-uniformly equicontinuous almost surely (see Lemma 1 below), using a similar argument as for (28) we have to show that

$$\sup_{t\in[0,1]}\sup_{A_2\in\mathcal{A}_{2t}^{(n)}}\inf_{A_0\in\mathcal{A}_{0t}}d_0(A_2,A_0)\to 0 \ \text{ a.s.}, \quad n\to\infty \tag{32}$$

and

$$\sup_{t\in[0,1]}\sup_{A_0\in\mathcal{A}_{0t}}\inf_{A_2\in\mathcal{A}_{2t}^{(n)}}d_0(A_2,A_0)\to 0 \quad \text{a.s.}, \quad n\to\infty. \tag{33}$$

We can also state (32) as follows: for every $\varepsilon>0$ we can choose $N_\varepsilon\geq 1$ such that for $n\geq N_\varepsilon$ and for all $t\in[0,1]$, $A_2\in\mathcal{A}_{2t}^{(n)}$ there exists an $A_0=A_0(A_2,\varepsilon,t)\in\mathcal{A}_{0t}$ and $d_0(A_2,A_0)<\varepsilon$ a.s. Take an arbitrary $\varepsilon>0$. Observe that there exists $N_\varepsilon^{(1)}\geq 1$ such that for $n\geq N_\varepsilon^{(1)}$ and for all $t\in[0,1]$ and all $A_2\in\mathcal{A}_{2t}^{(n)}$, $|P_2^{(n)}(A_2)-t|<\frac{\varepsilon}{2}$ a.s. Next choose $N_\varepsilon^{(2)}\geq 1$ such that for $n\geq N_\varepsilon^{(2)}$ and for all $A_2\in\mathcal{A}_{2t}^{(n)}$, $|P_0(A_2)-P_2^{(n)}(A_2)|<\frac{\varepsilon}{2}$. Let $N_\varepsilon:=\max(N_\varepsilon^{(1)},N_\varepsilon^{(2)})$. Then trivially for $n\geq N_\varepsilon$ and for all $A_2\in\mathcal{A}_{2t}^{(n)}$, we have $|P_0(A_2)-t|<\varepsilon$ a.s. So since F_0 is continuous there exists a set A_0, with $P_0(A_0)=t$ and $A_0\subset A_2$ or $A_0\supset A_2$ and hence $d_0(A_2,A_0)<\varepsilon$ a.s. Note that (33) can be treated similarly. Hence (31) holds true and thus the proof of the theorem is completed. □

A collection of functions $\mathcal{F}$ from some metric space (S,e) into another metric space (X,d) is d-uniformly equicontinuous if for every $\varepsilon>0$ there exists a $\delta>0$ such that $e(x,y)<\delta$ implies $d(f(x),f(y))<\varepsilon$, for all x and y in S and all f in $\mathcal{F}$.

Lemma 1 *The collection of functions* $\{B^{(n)}_{P_0} + H_{n_1 n_2}\}_{n\in I\!N}$ *is* d_0*-uniformly equicontinuous, almost surely.*

Proof We prove the statement using a well-known fact on the modulus of continuity of a standard Brownian bridge B (see, e.g., Shorack and Wellner [1986]):

$$\lim_{a\downarrow 0} \frac{\sup_{|t-s|\le a} |B(t) - B(s)|}{\sqrt{2a\log(1/a)}} = 1 \quad \text{a.s.}$$

Then by a simple transformation we have for a P_0-Brownian bridge that

$$\lim_{a\downarrow 0} \sup_{\substack{d_0(A_1,A_2)\le a\\ A_1,A_2\in\mathcal{A}}} |B_{P_0}(A_1) - B_{P_0}(A_2)| = 0 \quad \text{a.s.}$$

Using (24), we obtain that for any $\varepsilon > 0$ there exists a small $a > 0$ such that for all $n \ge 1$

$$\sup_{\substack{d_0(A_1,A_2)\le a\\ A_1,A_2\in\mathcal{A}}} |B^{(n)}_{P_0}(A_1) - B^{(n)}_{P_0}(A_2)| < \varepsilon \quad \text{a.s.}$$

and this implies that $\{B^{(n)}_{P_0} \,:\, n \in I\!N\}$ is d_0-uniformly equicontinuous, almost surely.

Let $A_1, A_2 \in \mathcal{A}$. We have

$$\begin{aligned}
&\Big|H_{n_1n_2}(A_1) - H_{n_1n_2}(A_2)\Big| \\
&\le \sqrt{\frac{n_2}{n}} \int_{I\!R} I_{A_1\triangle A_2}(x)\big|h_{1n}(x)\big|dP_0(x) + \sqrt{\frac{n_1}{n}} \int_{I\!R} I_{A_1\triangle A_2}(x)\big|h_{2n}(x)\big|dP_0(x).
\end{aligned}$$

By the Cauchy-Schwarz inequality, for $j = 1, 2$,

$$\int_{I\!R} I_{A_1\triangle A_2}(x)\big|h_{jn}(x)\big|dP_0(x) \le \|h_{jn}\|_{I\!R}\, \sqrt{d_0(A_1, A_2)}.$$

However, by condition (iv) the sequence $\|h_{jn}\|_{I\!R}$, $n \ge 1$, is bounded, hence for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $n_1, n_2 \in I\!N$ and any $A_1, A_2 \in \mathcal{A}$, with $d_0(A_1, A_2) < \delta$, we will have that

$$|H_{n_1n_2}(A_1) - H_{n_1n_2}(A_2)| < \varepsilon.$$

Thus $H_{n_1n_2}$ is d_0-uniformly equicontinuous as well. □

References

J. Beirlant and P. Deheuvels. On the approximation of P-P and Q-Q plot processes by Brownian bridges. *Statist. Probab. Lett.*, **9**:241–251, 1990.

P. Deheuvels and J. H. J. Einmahl. Approximations and two-sample tests based on P-P and Q-Q plots of the Kaplan-Meier estimators of lifetime distributions. *J. Multivariate Anal.*, **43**:200–217, 1992.

J. B. Dijkstra, T. J. M. Rietjens, and F. W. Steutel. A simple test for uniformity. *Statist. Neerlandica*, **38**:33–44, 1984.

K. Doksum. Some graphical methods in statistics. A review and some extensions. *Statist. Neerlandica*, **31**:53–68, 1977.

N. I. Fisher. Graphical methods in nonparametric statistics: a review and annotated bibliography. *Internat. Statist. Rev.*, **51**:25–58, 1983.

A. J. Girling. Rank statistics expressible as integrals under P-P-plots and receiver operating characteristic curves. *J. R. Stat. Soc. Ser.*, **B 62**:367–382, 2000.

J. Glaz, J. Naus, and S. Wallenstein. *Scan statistics*. Springer, New York, 2001.

R. Gnanadesikan. *Methods for Statistical Data Analysis of Multivariate Observations*. Wiley, New York, 1977.

F. Hsieh and B. W. Turnbull. Nonparametric and semiparametric estimation of the receiver operating characteristic curve. *Ann. Statist.*, **24**:25–40, 1996.

E. V. Khmaladze. Goodness of fit tests for "Chimeric" alternatives. *Statist. Neerlandica*, **52**:90–111, 1998.

J. Oosterhoff and W. R. van Zwet. A note on contiguity and Hellinger distance. In J. Jureckova, editor, *Contributions to Statistics, Jaroslav Hajek Memorial Volume*. Reidel, Dordrecht, 1979.

W. Polonik. Concentration and goodness-of-fit in higher dimensions: (Asymptotically) distribution-free methods. *Ann. Statist.*, **27**:1210–1229, 1999.

G. Sawitzki. Diagnostic plots for one-dimensional data. In P. Dirschedl and R. Ostermann, editors, *Papers collected on the Occasion of the 25th Conf. on Statistical Computing at Schloss Reisensburg*. Physica, Heidelberg, 1994.

G. R. Shorack and J. A. Wellner. *Empirical Processes with Applications to Statistics*. Wiley, New York, 1986.

Dept. of Public Health
Erasmus MC
P.O. Box 1738
3000 DR Rotterdam
The Netherlands
Email: n.mushkudiani@erasmusmc.nl

Dept. of Econometrics & OR
Tilburg University
P.O. Box 90153
5000 LE Tilburg
The Netherlands
Email: j.h.j.einmahl@uvt.nl